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Spatially Constrained Output Injection for State Estimation with Banded Closed-Loop Dynamics

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I. INTRODUCTION

For large-scale system models, such as models obtained from the discretization of partial differential equations, *data assimilation* refers to the use of measurements to obtain estimates of states that cannot be directly measured. Applications of data assimilation include terrestrial weather forecasting, space weather and ocean climate predication [1–3].

Although various techniques, such as 3Dvar, and 4Dvar [4] have been developed for data assimilation, the Kalman filter provides the most rigorous approach in accounting for measurement and process noise. The main drawback of the Kalman filter in data assimilation applications, however, is the need to propagate the error covariance, which has $O(n^3)$ complexity for a system with n states. One approach to this problem is to replace the error covariance propagation equation with a collection of stochastically driven state estimates, which are used to estimate the error covariance. The resulting *ensemble Kalman filter* [5, 6] can be viewed as a special case of *particle* Kalman filters [7, 8]. For the case of the ensemble Kalman filter, the size of the required ensemble determines the computation burden, and a first attempt to determine the required size is given in [9].

For a traditional error covariance propagation approach, the computational burden can be reduced by assuming that the error covariance is sparse, for example, banded. In some data assimilation implementations of the Kalman filter, only the banded portion of the covariance is propagated [10]. Physical arguments are typically used to argue that error covariance entries are small due to weak spatial correlations. In [11] we derived bounds on the magnitude of the error covariance entries located outside of a given band when the dynamics matrix is banded, which typically occurs in finite volume discretization of partial differential equations. These bounds can be used to estimate the error incurred when the non-banded entries of the covariance matrix are neglected.

The physical arguments used to justify the use of a sparse approximation of the state error covariance as well as the bounds given in [11], however, apply to the propagation of the *open-loop* system, that is, the system without data injection. In fact, the injection of data within the system's dynamics usually destroys the banded structure of the dynamics, and can potentially strengthen correlations among states that, without the use of data injection, may be weakly correlated.

The goal of the present paper is thus to develop an extension of the classical Kalman filter in which the data injection gain is structured so as to preserve the banded structure of the system's dynamics. While the restriction of the filter gain to a structured form leads to suboptimal estimates, the spatially localized nature of the data injection is often consistent with spatial correlations. In addition, since the banded structure of the dynamics is preserved, the use of a sparse approximation of the state covariance matrix is justified. The results in the present paper extend the results of [12], where the data injection is localized, but with less structure than the formulation of the present paper.

II. SPATIALLY LOCALIZED KALMAN FILTER

We consider the discrete-time dynamical system

$$x_{k+1} = A_k x_k + B_k u_k + w_k, \quad k \geq 0, \tag{2.1}$$

with outputs

$$y_{i,k} = C_{i,k} x_k + v_{i,k}, \quad i = 1, \ldots, p, \tag{2.2}$$

where $x_k \in \mathbb{R}^{n_k}$, $u_k \in \mathbb{R}^{m_k}$, for all $i = 1, \ldots, p$, $y_{i,k} \in \mathbb{R}^{l_{i,k}}$, and $A_k, B_k, C_{1,k}, \ldots, C_{p,k}$ are known real matrices of appropriate size. The input u_k and outputs $y_{1,k}, \ldots, y_{p,k}$ are assumed to be measured, and $w_k \in \mathbb{R}^{n_{k+1}}$ and $v_k \triangleq \begin{bmatrix} v_{1,k}^{\rm T} & \cdots & v_{p,k}^{\rm T} \end{bmatrix}^{\rm T}$ are white zero-mean noise process with variance and correlation

$$\mathcal{E}[w_k w_j^{\rm T}] = Q_k \delta_{kj}, \ \mathcal{E}[w_k v_{i,j}^{\rm T}] = S_{i,k} \delta_{kj}, \ \mathcal{E}[v_k v_j^{\rm T}] = R_k \delta_{kj}, \tag{2.3}$$

where δ_{kj} is the Kronecker delta, and $\mathcal{E}[\cdot]$ denotes expected value. Furthermore, for all $i, j = 1, \ldots, p$, let $R_{i,j,k} \in \mathbb{R}^{l_{i,k} \times l_{j,k}}$ denote the (i, j) block entry of R_k. The initial state x_0 is assumed to be uncorrelated with w_k and v_k. Note that the dimension n_k of the state x_k can be time varying, and thus $A_k \in \mathbb{R}^{n_{k+1} \times n_k}$ is not necessarily square.

For the system (2.1) and (2.2), we consider a state estimator of the form

$$\hat{x}_{k+1} = A_k \hat{x}_k + B_k u_k + \sum_{i=1}^{p} \Gamma_{i,k} K_{i,k} (y_{i,k} - \hat{y}_{i,k}), \tag{2.4}$$

with outputs

$$\hat{y}_{i,k} = C_{i,k} \hat{x}_k, \quad i = 1, \ldots, p, \tag{2.5}$$

where $\hat{x}_k \in \mathbb{R}^{n_k}$, and for all $i = 1, \ldots, p$, $\hat{y}_{i,k} \in \mathbb{R}^{l_{i,k}}$, $\Gamma_{i,k} \in \mathbb{R}^{n_{k+1} \times m_{i,k}}$, and $K_{i,k} \in \mathbb{R}^{m_{i,k} \times l_{i,k}}$. The nontraditional feature of (2.4) is the presence of the terms $\Gamma_{i,k}$. In the classical case $n_k = n$, $p = 1$ and $\Gamma_{1,k} = I_n$. Here, for all $i = 1, \ldots, p$, $\Gamma_{i,k}$ determines the component of the

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innovation $y_{i,k} - \hat{y}_{i,k}$ that directly affects specific estimator states. For example, $\Gamma_{i,k}$ can have the form

$$\Gamma_{i,k} = \begin{bmatrix} 0 & I_{l_k} & 0 \end{bmatrix}^{\mathrm{T}}, \tag{2.6}$$

where I_r denotes the $r \times r$ identity matrix. For all $i = 1, \ldots, p$ and $k \geqslant 0$, we assume that $\Gamma_{i,k}$ has full column rank.

Next, define the state estimation error

$$e_k \triangleq x_k - \hat{x}_k, \tag{2.7}$$

which satisfies

$$e_{k+1} = \tilde{A}_k e_k + \tilde{w}_k, \quad k \geq 0, \tag{2.8}$$

where

$$\tilde{A}_k \triangleq A_k - \sum_{i=1}^{p} \Gamma_{i,k} K_{i,k} C_{i,k}, \ \tilde{w}_k \triangleq w_k - \sum_{i=1}^{p} \Gamma_{i,k} K_{i,k} v_{i,k}. \tag{2.9}$$

Now define the state estimation error

$$J_k(K_{1,k}, \ldots, K_{p,k}) \triangleq \mathcal{E}[(L_k e_{k+1})^{\mathrm{T}} L_k e_{k+1}], \tag{2.10}$$

where $L_k \in \mathbb{R}^{q_k \times n_{k+1}}$ determines the weighted error components. Then,

$$J_k(K_{1,k}, \ldots, K_{p,k}) = \operatorname{tr}\left[P_{k+1} M_k\right], \tag{2.11}$$

where the error covariance $P_k \in \mathbb{R}^{n_k \times n_k}$ is defined by $P_k \triangleq \mathcal{E}[e_k e_k^{\mathrm{T}}]$ and $M_k \triangleq L_k^{\mathrm{T}} L_k \in \mathbb{R}^{n_{k+1} \times n_{k+1}}$. We assume that M_k is positive definite for all $k \geqslant 0$. The following lemma will be useful.

Lemma II.1. The error (2.7) satisfies

$$\mathcal{E}[e_k \tilde{w}_k^{\mathrm{T}}] = 0. \tag{2.12}$$

It follows from (2.8) that

$$\begin{aligned} \mathcal{E}[e_{k+1} e_{k+1}^{\mathrm{T}}] = \tilde{A}_k \mathcal{E}[e_k e_k^{\mathrm{T}}] \tilde{A}_k^{\mathrm{T}} + \tilde{A}_k \mathcal{E}[e_k \tilde{w}_k^{\mathrm{T}}] \\ + \mathcal{E}[\tilde{w}_k e_k^{\mathrm{T}}] \tilde{A}_k^{\mathrm{T}} + \mathcal{E}[\tilde{w}_k \tilde{w}_k^{\mathrm{T}}]. \end{aligned} \tag{2.13}$$

For notational convenience, we define

$$S_k \triangleq [S_{1,k} \ \cdots \ S_{p,k}], \ C_k \triangleq [C_{1,k}^{\mathrm{T}} \ \cdots \ C_{p,k}^{\mathrm{T}}]^{\mathrm{T}}. \tag{2.14}$$

Note that (2.3), (2.9) and (2.14) imply that

$$\mathcal{E}[\tilde{w}_k \tilde{w}_k^{\mathrm{T}}] = Q_k - S_k K_k^{\mathrm{T}} \Gamma_k^{\mathrm{T}} - \Gamma_k K_k S_k^{\mathrm{T}} + \Gamma_k K_k R_k K_k^{\mathrm{T}} \Gamma_k^{\mathrm{T}}, \tag{2.15}$$

where $\Gamma_k \in \mathbb{R}^{n_{k+1} \times m_k}$ and $K_k \in \mathbb{R}^{m_k \times l_k}$ are defined by

$$\Gamma_k \triangleq [\Gamma_{1,k} \ \cdots \ \Gamma_{p,k}], \ K_k \triangleq \begin{bmatrix} K_{1,k} & 0 & \cdots & 0 \\ 0 & K_{2,k} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & K_{p,k} \end{bmatrix}, \tag{2.16}$$

and $m_k \triangleq m_{1,k} + \cdots + m_{p,k}$, $l_k \triangleq l_{1,k} + \cdots + l_{p,k}$. It follows from (2.12), (2.13), and (2.15) that P_k satisfies

$$P_{k+1} = \tilde{A}_k P_k \tilde{A}_k^{\mathrm{T}} + \tilde{Q}_k, \tag{2.17}$$

where

$$\tilde{Q}_k \triangleq Q_k - S_k K_k^{\mathrm{T}} \Gamma_k^{\mathrm{T}} - \Gamma_k K_k S_k^{\mathrm{T}} + \Gamma_k K_k R_k K_k^{\mathrm{T}} \Gamma_k^{\mathrm{T}}. \tag{2.18}$$

Therefore,

$$J_k(K_{1,k}, \ldots, K_{p,k}) = \operatorname{tr}[(\tilde{A}_k P_k \tilde{A}_k^{\mathrm{T}} + \tilde{Q}_k) M_k]. \tag{2.19}$$

III. ONE-STEP SPATIALLY CONSTRAINED KALMAN FILER

In this section we derive a one-step spatially constrained Kalman filter that minimizes the cost (2.19). For convenience, for all $i = 1, \ldots, p$, we define

$$\hat{S}_{i,k} \triangleq A_k P_k C_{i,k}^{\mathrm{T}} + S_{i,k}, \quad \hat{R}_{i,j,k} \triangleq R_{i,j,k} + C_{i,k} P_k C_{j,k}^{\mathrm{T}}, \tag{3.1}$$

and $\pi_{i,k} \in \mathbb{R}^{n_{k+1} \times n_{k+1}}$ by

$$\pi_{i,k} \triangleq \Gamma_{i,k} \left[\Gamma_{i,k}^{\mathrm{T}} M_k \Gamma_{i,k}\right]^{-1} \Gamma_{i,k}^{\mathrm{T}} M_k. \tag{3.2}$$

Note that $\pi_{i,k}$ is an oblique projector, that is, $\pi_{i,k}^2 = \pi_{i,k}$, but is not necessarily symmetric. Next, for all $i = 1, \ldots, p$, define the complementary oblique projector $\pi_{i,k\perp}$, and $\tilde{M}_i \in \mathbb{R}^{m_{i,k} \times m_{i,k}}$ by

$$\pi_{i,k\perp} \triangleq I_{n_{k+1}} - \pi_{i,k}, \ \tilde{M}_{i,k} \triangleq \Gamma_{i,k}^{\mathrm{T}} M_k \Gamma_{i,k}. \tag{3.3}$$

Proposition III.1. *For all $i = 1, \ldots, p$, the gain $K_{i,k}$ that minimizes the cost $J_k(K_{1,k}, \ldots, K_{p,k})$ given by* (2.19) *satisfies*

$$K_{i,k} = \tilde{M}_{i,k}^{-1} \Gamma_{i,k}^{\mathrm{T}} M_k \left(\hat{S}_{i,k} - \sum_{j=\{1,\ldots,p\}-i} \Gamma_{j,k} K_{j,k} \hat{R}_{i,j,k}^{\mathrm{T}} \right) \hat{R}_{i,i,k}^{-1}, \tag{3.4}$$

where the error covariance P_k is updated using

$$P_{k+1} = \tilde{A}_k P_k \tilde{A}_k^{\mathrm{T}} + \tilde{Q}_k. \tag{3.5}$$

Proof. Setting $\frac{\partial}{\partial K_{i,k}} J_k(K_{1,k}, \ldots, K_{p,k}) = 0$ for $i = 1, \ldots, p$, yields (3.4). It follows from [14, p. 283] that, for all $A \in \mathbb{R}^{n \times m}$ and positive-definite $B \in \mathbb{R}^{m \times m}$, $A \to ABA^{\mathrm{T}}$ is convex. Using an extension of this fact, it can be shown that $J_k(K_{1,k}, \ldots, K_{p,k})$ in (2.19) is convex, and hence $K_{i,k}$'s that satisfy (3.4) are a global minimizer of $J_k(K_{1,k}, \ldots, K_{p,k})$. □

Left-multiplication and right-multiplication of (3.4) by $\tilde{M}_{i,k}$ and $\hat{R}_{i,i,k}$, respectively, yields for all $i = 1, \ldots, p$,

$$\tilde{M}_{i,k} K_{i,k} \hat{R}_{i,i,k} + \sum_{j=\{1,\ldots,p\}-i} \Gamma_{i,k}^{\mathrm{T}} M_k \Gamma_{j,k} K_{j,k} \hat{R}_{i,j,k}^{\mathrm{T}} = \Gamma_{i,k}^{\mathrm{T}} M_k \hat{S}_{i,k}. \tag{3.6}$$

Taking the vec of both sides of (3.6) yields

$$\mathcal{R}_k \operatorname{vec}(K_k) = \mathcal{S}_k, \tag{3.7}$$

where for all $i, j = 1, \ldots, p$, the (i,j)th block entry of $\mathcal{R}_k \in \mathbb{R}^{m_k l_k \times m_k l_k}$ is given by

$$(\mathcal{R}_k)_{i,j} = \hat{R}_{i,j,k} \otimes (\Gamma_{i,k}^{\mathrm{T}} M_k \Gamma_{j,k}), \tag{3.8}$$

$\mathcal{S}_k \in \mathbb{R}^{m_k l_k}$ is defined by

$$\mathcal{S}_k \triangleq \operatorname{vec}\begin{bmatrix} \Gamma_{1,k}^{\mathrm{T}} M_k \hat{S}_{1,k} & \cdots & \Gamma_{p,k}^{\mathrm{T}} M_k \hat{S}_{p,k} \end{bmatrix}, \tag{3.9}$$

and $\otimes$ denotes the Kronecker product. Note that $\mathcal{R}_k$ can be expressed as

$$\mathcal{R}_k = \hat{R}_k * \hat{\Gamma}_k, \tag{3.10}$$

where the (i,j) block entry of $\hat{\Gamma}_k$ and $\hat{R}_k$ are given by

$$(\hat{\Gamma}_k)_{i,j} = \Gamma_{i,k}^{\mathrm{T}} M_k \Gamma_{j,k} \tag{3.11}$$

and

$$(\hat{R}_k)_{i,j} = \hat{R}_{i,j,k}, \tag{3.12}$$

and $*$ denotes the Khatri-Rao product (see [15]) or block-

Kronecker product of two matrices. Since $\hat{R}_k$ is positive definite, $\hat{\Gamma}_k$ is positive semi-definite, and all the block-diagonal entries of $\hat{\Gamma}_k$ are positive definite, it follows from [15] that $\mathcal{R}_k$ is also positive-definite. Hence,

$$\mathrm{vec}(K_k) = (\mathcal{R}_k)^{-1}\mathcal{S}_k. \tag{3.13}$$

If $p = 1$, then (2.16) implies that $\Gamma_k = \Gamma_{1,k}$ and $K_k = K_{1,k}$. Furthermore $C_k = C_{1,k}$, $\hat{S}_k = \hat{S}_{1,k}$, $R_k = R_{1,1,k}$ and $\hat{R}_k = \hat{R}_{1,1,k}$. Hence, it follows from (3.6) that

$$K_k = \left(\Gamma_k^{\mathrm{T}} M_k \Gamma_k\right)^{-1} \Gamma_k^{\mathrm{T}} M_k \hat{S}_k \hat{R}_k^{-1}. \tag{3.14}$$

Substituting (3.14) into (3.5) yields

$$P_{k+1} = A_k P_k A_k^{\mathrm{T}} + \pi_{k\perp}\hat{S}_k\hat{R}_k^{-1}\hat{S}_k^{\mathrm{T}}\pi_{k\perp}^{\mathrm{T}} - \hat{S}_k\hat{R}_k^{-1}\hat{S}_k^{\mathrm{T}} + Q_k, \tag{3.15}$$

where $\pi_{k\perp} \triangleq \pi_{1,k\perp}$. In the classical case, $n_k = n$, $p = 1$ and $\Gamma_{1,k} = I_n$, which implies that $\pi_{1,k} = I$ and $\pi_{1,k\perp} = 0$, and hence it follows from (3.14) that the Kalman filter gain K_k is given by

$$K_k = (A_k P_k C_k^{\mathrm{T}} + S_k)(R_k + C_k P_k C_k^{\mathrm{T}})^{-1}, \tag{3.16}$$

and, (3.15) and (3.16) imply that the covariance update equation is

$$P_{k+1} = A_k P_k A_k^{\mathrm{T}} + Q_k - \hat{S}_k \hat{R}_k^{-1} \hat{S}_k^{\mathrm{T}}. \tag{3.17}$$

Finally, the estimator equation is

$$\hat{x}_{k+1} = A_k \hat{x}_k + B_k u_k + K_k(y_k - \hat{y}_k), \tag{3.18}$$

where $y_k \triangleq y_{1,k}$ $\hat{y}_k \triangleq \hat{y}_{1,k}$. In the classical case, $n_k = n$ for all $k \geq 0$.

IV. Preserving Banded Dynamics

Next, let $A_k \in \mathbb{R}^{n\times n}$ have scalar entries $a_{i,j,k}$, $i, j = 1, \ldots, n$. We assume that A_k is banded diagonal with semi-width b so that for all $k \geqslant 0$ and $i = 1, \ldots, n$,

$$a_{i,j,k} = 0, \text{ if } |i - j| > b. \tag{4.1}$$

Let x_k , w_k, and $\hat{x}_k$ have scalar components $x_{i,k}$, $w_{i,k}$ and $\hat{x}_{i,k}$, $i = 1, \ldots, n_k$, respectively. It then follows from (2.1) and (4.1) that

$$x_{i,k+1} = \overline{a}_{i,k}\overline{x}_{i,k} + w_{i,k}, \tag{4.2}$$

where for all $i = 1, \ldots, n$,

$$\overline{a}_{i,k} \triangleq [a_{i,\mathrm{L}(i),k} \ \cdots \ a_{i,\mathrm{R}(i),k}],\ \overline{x}_{i,k} \triangleq [x_{\mathrm{L}(i),k} \ \cdots \ x_{\mathrm{R}(i),k}]^{\mathrm{T}} \tag{4.3}$$

and

$$\mathrm{L}(i) = \max\{1, i - b\}, \ \ \mathrm{R}(i) = \min\{n, i + b\}. \tag{4.4}$$

As a special case, assume that for all $i = 1, \ldots, p$, $y_{i,k} \in \mathbb{R}$ is given by

$$y_{i,k} = x_{q_i,k} + v_{i,k}, \tag{4.5}$$

where q_i is an integer that denotes the state component being measured. The objective is to inject data $y_{1,k}, \ldots, y_{p,k}$ so that measurement $y_{i,k}$ of the state $x_{q_i,k}$ directly affects only the state estimates $\hat{x}_{\mathrm{L}(q_i),k}, \ldots, \hat{x}_{\mathrm{R}(q_i),k}$.

Next, we present a procedure for choosing $\Gamma_{i,k}$ so that the closed loop error dynamics $A_k - \sum_{i=1}^{p} \Gamma_{i,k} K_{i,k} C_{i,k}$ is also banded with semi-width b. For all $i = 1, \ldots, p$, let $\Gamma_{i,k}$ have the following structure

$$\Gamma_{i,k} = \left[\ 0_{\mathrm{L}(q_i)-1\times m_{i,k}} \quad \Upsilon_{i,k}^{\mathrm{T}} \quad 0\ \right]^{\mathrm{T}}, \tag{4.6}$$

where $\Upsilon_{i,k} \in \mathbb{R}^{m_{i,k}\times m_{i,k}}$. We choose $m_{i,k} = \mathrm{R}(q_i) - \mathrm{L}(q_i) + 1$ so that the only non-zero rows of $\Gamma_{i,k}$ are the rows with indices $\mathrm{L}(q_i), \ldots, \mathrm{R}(q_i)$. Next, for $i = 1, \ldots, p$, define $\Phi_{i,k} \in \mathbb{R}^n$ by

$$\Phi_{i,k} \triangleq \Gamma_{i,k} K_{i,k}(y_{i,k} - \hat{y}_{i,k}), \tag{4.7}$$

and for $j = 1, \ldots, n$, let $\phi_{i,j,k} \in \mathbb{R}$ be the scalar entries of $\Phi_{i,k}$. Since $K_{i,k}$ is a column vector for all $i = 1, \ldots, p$, it follows from (4.6) and (4.7) that for all $i = 1, \ldots, p$, and $j = 1, \ldots, n$,

$$\phi_{i,j,k} = 0, \text{ if } j \notin \{\mathrm{L}(q_i), \ldots, \mathrm{R}(q_i)\}. \tag{4.8}$$

Substituting (4.7) into (2.4) yields

$$\hat{x}_{k+1} = A_k \hat{x}_k + B_k u_k + \sum_{i=1}^{p} \Phi_{i,k}. \tag{4.9}$$

Hence, it follows from (4.1), (4.2), (4.5), (4.7) and (4.8) that for all $i = 1, \ldots, n$,

$$\hat{x}_{i,k+1} = \overline{a}_{i,k}\hat{\overline{x}}_{i,k} + \sum_{j=1}^{p} \phi_{j,i,k}(x_{q_j,k} - \hat{x}_{q_j,k} + v_{j,k}), \tag{4.10}$$

where

$$\hat{\overline{x}}_{i,k} \triangleq \left[\ \hat{x}_{\mathrm{L}(i),k} \quad \cdots \quad \hat{x}_{\mathrm{R}(i),k}\ \right]^{\mathrm{T}}. \tag{4.11}$$

Furthermore, it follows from (4.8) that (4.10) can be expressed as

$$\hat{x}_{i,k+1} = \tilde{a}_{i,k}\hat{\overline{x}}_{i,k} + \overline{u}_{i,k}, \tag{4.12}$$

where $\tilde{a}_{i,k} \in \mathbb{R}^{1\times \mathrm{R}(q_i)-\mathrm{L}(q_i)+1}$ and for all $i = 1, \ldots, n$,

$$\overline{u}_{i,k} \triangleq \sum_{j=1}^{p} \phi_{j,i,k}(x_{q_j,k} + v_{j,k}). \tag{4.13}$$

Hence, it follows from (4.11) and (4.12) that the closed-loop dynamics is also banded with semi-width b. Note that a similar procedure can be used to ensure that the closed-loop dynamics is banded with any arbitrary semi-width and not necessarily the same semi-width as the open-loop dynamics.

V. Two-Step Spatially Constrained Kalman Filter

We consider a two-step state estimator with spatially constrained output injection. The *forecast step*, which can be viewed as a physics update, is given by

$$x_{k+1}^{\mathrm{f}} = A_k x_k^{\mathrm{da}} + B_k u_k + \overline{w}_k, \tag{5.1}$$

where x_k^{f} is the *forecast estimate* and $x_k^{\mathrm{da}} \in \mathbb{R}^{n_k}$ is the *data assimilation estimate*. The *data assimilation step* is given by the data update

$$x_k^{\mathrm{da}} = x_k^{\mathrm{f}} + \sum_{i=1}^{p} \Gamma_{i,k} K_{i,k}(y_{i,k} - y_{i,k}^{\mathrm{f}}), \tag{5.2}$$

where for all $i = 1, \ldots, p$, $y_{i,k}^{\mathrm{f}} \triangleq C x_k^{\mathrm{f}}$, and C_k is defined in (2.16).

Proposition V.1. *The optimal $\overline{w}_k$ that minimizes $\mathcal{E}[(w_k -$*

$\overline{w}_k)(w_k-\overline{w}_k)^{\rm T}]$ *is given by*

$$\overline{w}_k = D_k(y_k - y_k^{\rm f}), \tag{5.3}$$

where $D_k \in \mathbb{R}^{n_{k+1}\times l_k}$ *is defined by*

$$D_k \triangleq S_k(C_kP_k^{\rm f}C_k^{\rm T}+R_k)^{-1}. \tag{5.4}$$

Proof. It follows from [13, pg. 67] that

$$\overline{w}_k = \mathcal{E}[w_k] + P_{{\rm wy}k}P_{{\rm yy}k}^{-1}(y_k-y_k^{\rm f}), \tag{5.5}$$

where

$$P_{{\rm wy}k} \triangleq \mathcal{E}[w_k(y_k-y_k^{\rm f})^{\rm T}],\ P_{{\rm yy}k}\triangleq\mathcal{E}[(y_k-y_k^{\rm f})(y_k-y_k^{\rm f})^{\rm T}]. \tag{5.6}$$

Note that

$$P_{{\rm wy}k}=S_k,\ \ P_{{\rm yy}k}=C_kP_k^{\rm f}C_k^{\rm T}+R_k, \tag{5.7}$$

where the *forecast error covariance* $P_k^{\rm f}$ is defined by

$$P_k^{\rm f}\triangleq\mathcal{E}[e_k^{\rm f}(e_k^{\rm f})^{\rm T}] \tag{5.8}$$

and the *forecast state error* $e_k^{\rm f}$ is defined by

$$e_k^{\rm f}\triangleq x_k-x_k^{\rm f}. \tag{5.9}$$

Since $\mathcal{E}[w_k]=0$, substituting (5.6) and (5.7) into (5.5) yields (5.3). □

It follows from (2.1) and (5.1) that

$$e_{k+1}^{\rm f}=A_ke_k^{\rm da}+w_k-\overline{w}_k,\quad k\ge 0, \tag{5.10}$$

where the *data assimilation state error* $e_k^{\rm da}$ is defined by

$$e_k^{\rm da}\triangleq x_k-x_k^{\rm da}. \tag{5.11}$$

Now, define the state estimation error

$$J_k(K_{1,k},\ldots,K_{p,k})\triangleq\mathcal{E}[(L_ke_k^{\rm da})^{\rm T}L_ke_k^{\rm da}], \tag{5.12}$$

where $L_k\in\mathbb{R}^{q_k\times n_{k+1}}$ determines the weighted error components. Then,

$$J_k(K_{1,k},\ldots,K_{p,k})={\rm tr}\left[P_k^{\rm da}M_k\right], \tag{5.13}$$

where the *data assimilation error covariance* $P_k^{\rm da}\in\mathbb{R}^{n_k\times n_k}$ is defined by $P_k^{\rm da}\triangleq\mathcal{E}[e_k^{\rm da}(e_k^{\rm da})^{\rm T}]$. It follows from (5.2), (5.9), and (5.11) that

$$e_k^{\rm da}=(I-\sum_{i=1}^p\Gamma_{i,k}K_{i,k}C_{i,k})e_k^{\rm f}-\sum_{i=1}^p\Gamma_{i,k}K_{i,k}v_{i,k}. \tag{5.14}$$

Hence, it follows from (2.16) that

$$e_k^{\rm da}=\tilde{K}_ke_k^{\rm f}-\tilde{\Gamma}_kv_k, \tag{5.15}$$

where $v_k\triangleq\left[\begin{array}{ccc} v_{1,k}^{\rm T} & \cdots & v_{p,k}^{\rm T}\end{array}\right]^{\rm T}$ and

$$\tilde{K}_k\triangleq I-\Gamma_kK_kC_k. \tag{5.16}$$

Note that substituting (5.3) and (5.15) into (5.10) yields

$$e_{k+1}^{\rm f}=(A_k\tilde{K}_k-D_kC_k)e_k^{\rm f}+w_k-(A_k\Gamma_kK_k+D_k)v_k. \tag{5.17}$$

Lemma V.1. The forecast error in (5.10) satisfies

$$\mathcal{E}[e_k^{\rm f}v_k^{\rm T}]=0. \tag{5.18}$$

Next, define

$$\hat{S}_{i,k}\triangleq P_k^{\rm f}C_{i,k}^{\rm T},\ \ \hat{R}_{i,j,k}\triangleq R_{i,j,k}+C_{i,k}P_k^{\rm f}C_{j,k}^{\rm T}. \tag{5.19}$$

Furthermore, define $\hat{Q}_k$ by

$$\hat{Q}_k\triangleq Q_k-A_kP_k^{\rm f}C_kD_k^{\rm T}-D_kC_k^{\rm T}P_k^{\rm f}A_k^{\rm T}-D_k\hat{R}_kD_k^{\rm T} \tag{5.20}$$

Proposition V.2. *The gain* $K_{i,k}$ *that minimizes the cost* $J_k(K_{1,k},\ldots,K_{p,k})$ *given by* (5.13) *satisfies*

$$K_{i,k}=\tilde{M}_{i,k}^{-1}\Gamma_{i,k}^{\rm T}M_k\left(\hat{S}_{i,k}-\sum_{j=\{1,\ldots,p\}-i}\Gamma_{j,k}K_{j,k}\hat{R}_{i,j,k}\right)\hat{R}_{i,i,k}^{-1}, \tag{5.21}$$

where $P_k^{\rm f}$ *and* $P_k^{\rm da}$ *are given by*

$$P_{k+1}^{\rm f}=A_kP_k^{\rm da}A_k^{\rm T}+\hat{Q}_k \tag{5.22}$$

and

$$P_k^{\rm da}=\tilde{K}_kP_k^{\rm f}\tilde{K}_k^{\rm T}+\Gamma_kK_kR_kK_k^{\rm T}\Gamma_k^{\rm T}. \tag{5.23}$$

Proof. Using (5.15), $P_k^{\rm da}$ satisfies

$$P_k^{\rm da}=\tilde{K}_kP_k^{\rm f}\tilde{K}_k^{\rm T}-\tilde{K}_k\mathcal{E}[e_k^{\rm f}v_k^{\rm T}]K_k^{\rm T}\Gamma_k^{\rm T} -\Gamma_kK_k\mathcal{E}[v_k(e_k^{\rm f})^{\rm T}]\tilde{K}_k^{\rm T}+\Gamma_kK_kR_kK_k^{\rm T}\Gamma_k^{\rm T}. \tag{5.24}$$

Substituting (5.18) into (5.24) yields (5.23). Furthermore, substituting (5.23) into (5.13) yields

$$J_k(K_k)={\rm tr}[(\tilde{K}_kP_k^{\rm f}\tilde{K}_k^{\rm T}+\Gamma_kK_kR_kK_k^{\rm T}\Gamma_k^{\rm T})M_k]. \tag{5.25}$$

To obtain the optimal gain $K_{i,k}$, we set $\frac{\partial}{\partial K_{i,k}}J_k(K_{1,k},\ldots,K_{p,k})=0$, which yields (5.21). It follows from [14, p. 283] that, for all $A\in\mathbb{R}^{n\times m}$ and positive-definite $B\in\mathbb{R}^{m\times m}$, $A\to ABA^{\rm T}$ is convex. Using an extension of this fact, it can be shown that $J_k(K_{1,k},\ldots,K_{p,k})$ in (5.25) is convex, and hence $K_{i,k}$'s that satisfy (5.21) are a global minimizer of $J_k(K_{1,k},\ldots,K_{p,k})$.

To update the forecast error covariance, we substitute (5.3) into (5.10) so that

$$e_{k+1}^{\rm f}=A_ke_k^{\rm da}-D_kC_ke_k^{\rm f}+w_k-D_kv_k. \tag{5.26}$$

Hence, it follows from (5.4), (5.15) and (5.18) that the forecast error covariance update is given by (5.22). □

If $p=1$, then $\Gamma_k=\Gamma_{1,k}$, $K_k=K_{1,k}$, $\hat{S}_k=\hat{S}_{1,k}$ and $\hat{R}_k=\hat{R}_{1,1,k}$. Hence, (5.21) implies that

$$K_k=\left(\Gamma_k^{\rm T}M_k\Gamma_k\right)^{-1}\Gamma_k^{\rm T}M_k\hat{S}_k\hat{R}_k^{-1}. \tag{5.27}$$

The forecast covariance update equation is given by (5.22), and substituting (5.27) into (5.23) yields the data assimilation covariance update equations

$$P_k^{\rm da}=P_k^{\rm f}+\pi_{k\perp}\hat{S}_k\hat{R}_k^{-1}\hat{S}_k^{\rm T}\pi_{k\perp}^{\rm T}-\hat{S}_k\hat{R}_k^{-1}\hat{S}_k^{\rm T}, \tag{5.28}$$

where $\pi_{k\perp}=\pi_{1,k\perp}$. Next, in the classical case, $n_k=n$, $p=1$ and $\Gamma_k=I_n$. Substituting $\Gamma_k=I_n$ in (5.27) and substituting the resulting expression into (5.1) and (5.2) yields the classical one-step Kalman filter equation.

$$x_{k+1}^{\rm f}=A_kx_k^{\rm f}+B_ku_k+(A_kP_k^{\rm f}C_k^{\rm T}+S_k)\hat{R}_k^{-1}(y_k-y_k^{\rm f}). \tag{5.29}$$

Furthermore, in the classical case, the data assimilation covariance update equation is given by

$$P_k^{\rm da}=P_k^{\rm f}-P_k^{\rm f}C_k^{\rm T}\hat{R}_k^{-1}C_kP_k^{\rm f}. \tag{5.30}$$

Substituting (5.30) into (5.22) yields

$$P_{k+1}^{\rm f}=A_kP_k^{\rm f}A_k^{\rm T}-(A_kP_k^{\rm f}C_k+S_k)\hat{R}_k^{-1}(C_kP_k^{\rm f}A_k^{\rm T}+S_k^{\rm T})+Q_k. \tag{5.31}$$

Note that if $n_k=n$, $p=1$ and $\Gamma_{1,k}=I$, then the one-step estimator in (3.16)-(3.18) and the two-step estimator in (5.29) and (5.31) are equivalent. However, if $\Gamma_{1,k}\ne I$, then the one-step estimator in (3.14) and (3.15) and the the two-step filter in (5.27) and (5.28) are not equivalent. However, simulations suggest that the performance of the one-step estimator and the two-step estimators are similar with neither performing consistently better than the other.

VI. Example : One-Dimensional Heat Conduction

We consider heat conduction in a one-dimensional bar. The temperature distribution along the bar is governed by the following partial differential equation.

$$\frac{\partial}{\partial t}T(x,t) = \alpha\frac{\partial^2}{\partial x^2}T(x,t) + u(x,t), \tag{6.1}$$

where $T(x,t)$ is the temperature at position $x \in [0, L]$ and time t, $u(x,t)$ represents external heat sources acting on the bar, and α is the thermal diffusivity. We use numerical discretization methods to directly obtain a discrete-time state space model.

Assume that the bar consists of n cells of width Δ_x and for $i = 1, \ldots, n$, let T_i be the temperature at the center of the ith cell. For all $k \geqslant 0$, let $u_1(k)$ and $u_2(k)$ denote the boundary conditions of the first and last cell, respectively, so that $u_{1,k} = T_1(k),\ \ u_{2,k} = T_n(k)$. Next, define the state vector $X_k \in \mathbb{R}^{n-2}$ by $X_k \triangleq \begin{bmatrix} T_2(k) & \cdots & T_{n-1}(k) \end{bmatrix}^{\mathrm{T}}$ and the input vector $u_k \in \mathbb{R}^2$ by $u_k \triangleq \begin{bmatrix} u_{1,k} & u_{2,k} \end{bmatrix}^{\mathrm{T}}$. A central difference discretization with time step Δ_t yields

$$X_{k+1} = AX_k + Bu_k + w_k, \tag{6.2}$$

where $A \in \mathbb{R}^{n-2\times n-2}$ and $B \in \mathbb{R}^{n-2\times 2}$ are defined by

$$A \triangleq \begin{bmatrix} 1-2\delta & \delta & 0 & \cdots & 0 \\ \delta & 1-2\delta & \delta & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & \delta & 1-2\delta & \delta \\ 0 & \cdots & \cdots & \delta & 1-2\delta \end{bmatrix}, B \triangleq \begin{bmatrix} \delta & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & \delta \end{bmatrix},$$

and $\delta \triangleq \alpha\frac{\Delta_t}{\Delta_x^2}$. Note that to guarantee asymptotic stability of A, Δ_t and Δ_x must be chosen so that $\delta < 0.5$. Since A is tridiagonal, the semi-width $b = 1$. We assume that $w_k \in \mathbb{R}^{n-2}$ is zero-mean white Gaussian process noise with covariance matrix $Q \in \mathbb{R}^{(n-2)\times(n-2)}$, where $Q = \mathrm{diag}(Q_2, \ldots, Q_{n-1})$ and for all $i = 2, \ldots, n-1$, $Q_i \in \mathbb{R}$ is defined by

$$Q_i = \begin{cases} 5, & \text{if } i \in \mathcal{I}_w, \\ 0, & \text{else.} \end{cases} \tag{6.3}$$

It follows from (6.2), and (6.3) that the temperature in only the cells with indices given by $\mathcal{I}_w$ are directly affected by w_k. Next, we assume that p measurements of the temperature at the various cells are available so that, for all $i = 1, \ldots, p$,

$$y_{i,k} = C_iX_k + v_{i,k}, \tag{6.4}$$

where $C_i \triangleq \begin{bmatrix} 0_{1\times q_i-2} & 1 & 0_{1\times n-q_i-1} \end{bmatrix}$, and q_i depends on the cell from which measurements are obtained. We assume that $S_k = 0$ and $R_k = 0.1I_p$.

Let $n = 52$ so that (6.2) has order 50 and let $u_1(k) = T_1(k) = 300 + 5\sin(0.1k)$ and $u_2(k) = T_2(k) = 300 - 5\sin(0.01k)$. We choose $\Delta_x = 0.5$, $\Delta_t = 1$ and $\alpha = 0.1$ so that $\delta = 0.4$. Furthermore, we assume that the initial temperature at the ith cell is a random variable with mean 300 K and variance 5. Next, we let $\mathcal{I}_w = \{20, 30\}$, and assume that measurements from cells with indices $i \in \mathcal{I}_y = \{10, 11, 12, 24, 25, 26, 38, 39, 40\}$ are available so that $p = 9$. First, we consider an estimator of the form presented in [12], where

$$\hat{X}_{k+1} = A\hat{X}_k + Bu_k + \mathit{\Gamma} K_k(y_k - \hat{y}_k), \tag{6.5}$$

and, y_k and $\hat{y}_k$ are given by

$$y_k = [y_{1,k} \ \cdots \ y_{p,k}]^{\mathrm{T}},\ \hat{y}_k = [\hat{y}_{1,k} \ \cdots \ \hat{y}_{p,k}]^{\mathrm{T}}. \tag{6.6}$$

and $\mathit{\Gamma}$ is chosen so that only the cells with indices $i \in \mathcal{I}_y$ are in the range of $\mathit{\Gamma}$. Furthermore, we choose $L_k = I$ so that all the state estimates are weighted. The estimator gain and the covariance update equation are given by (3.14) and (3.15), respectively. The mean square error (MSE) in the state estimates and sum of the squares of error in estimates is shown in Figure 1 and Figure 2, respectively. Note that the classical Kalman filter updates all of the states directly and hence, the performance of the classical Kalman filter is better than (6.5) with $\mathit{\Gamma} \neq I$. The structure of the closed-loop dynamics of the classical Kalman filter and the spatially constrained Kalman filter given by (6.5) is shown in Figure 3. Note that the closed-loop dynamics of both these filters is not tridiagonal.

Next, we consider two estimators that yield tridiagonal closed-loop dynamics. First, we consider an estimator of the form

$$\hat{X}_{k+1} = A\hat{X}_k + Bu_k + \tilde{K}_k(y_k - \hat{y}_k), \tag{6.7}$$

where $\tilde{K}_k$ is obtained by using the classical Kalman filter gain expression and zeroing out certain elements, that is,

$$\tilde{K}_k = H \circ \left(AP_kC^{\mathrm{T}}(CP_kC^{\mathrm{T}} + R)^{-1}\right), \tag{6.8}$$

where $H \in \mathbb{R}^{50\times 9}$ is defined by

$$H_{i,j} = \begin{cases} 1, & \text{if } i \in \{\mathrm{L}(q_j), \ldots, \mathrm{R}(q_j)\}, \\ 0, & \text{else,} \end{cases} \tag{6.9}$$

where $L(\cdot)$ and $R(\cdot)$ are defined in (4.4). The covariance update is given by

$$P_{k+1} = (A - \tilde{K}_kC)P_k(A - \tilde{K}_kC)^{\mathrm{T}} + Q + \tilde{K}_kR\tilde{K}_k^{\mathrm{T}}. \tag{6.10}$$

Hence, we use the classical Kalman filter gain expression to obtain a filter gain and then zero out the elements in the gain matrix so that measurement $y_{i,k}$ of the state $X_{q_i,k}$ directly affects the state estimates at only the cells with indices $\mathrm{L}(q_i), \ldots, \mathrm{R}(q_i)$. Although, this procedure guarantees that the closed loop dynamics $A - \tilde{K}_kC$ is tridiagonal, the resulting estimator is suboptimal. The errors in state estimates obtained from this estimator are shown in Figures 1 and 2.

Finally, consider the estimator given by (2.4), where for all $i = 1, \ldots, 9$, $\mathit{\Gamma}_i$ is chosen according to (4.6) so that only the state estimates $\hat{X}_{q_i-1,k}, \ldots, \hat{X}_{q_i+1,k}$ are in the range of $\mathit{\Gamma}_i$. The filter gain and the covariance update equation are given by (3.13) and (3.5), respectively. The MSE in the state estimates, and the sum of the squares of error in temperature estimates are shown Figure 1 and Figure 2, respectively. It can be seen from Figure 3 that the closed-loop dynamics is tridiagonal. Note that the performance of

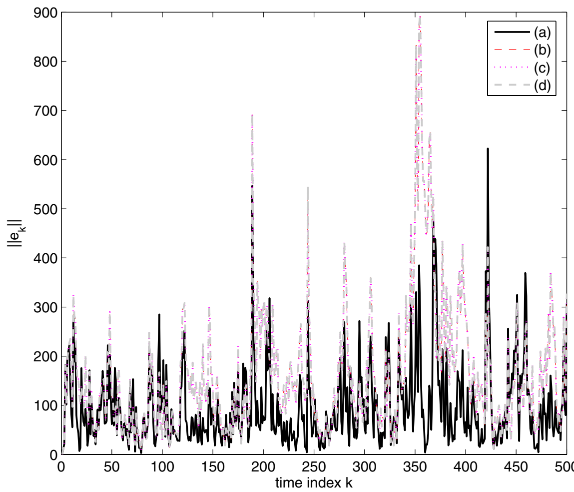

Fig. 1. MSE in state estimates obtained using the one-step filter in (6.5) with $\Gamma \neq I$ is denoted as (b). In this case the estimates of the temperature of only cells $i \in \mathcal{I}_y$ are updated. Since only specific state estimates are updated, the performance of the filter with $\Gamma \neq I$ is not as good as the classical Kalman filter which is denoted as (a). The MSE in state estimates obtained using the suboptimal estimator in (6.7) and the optimal estimator given by Proposition IV.1, are denoted by (c) and (d), respectively.

the optimal estimator in (6.5), the suboptimal estimator in (6.7), and the optimal estimator with selective scalar output injection are similar. However, as shown in Figure 2, the performance of the estimator in (6.5) is slightly better since it uses the measurement $y_{i,k}$ to directly update the state estimates at all of the cells with indices $i \in \mathcal{I}_{\rm y}$ and their neighboring cells. On the other hand, the optimal estimator with spatially constrained scalar output injection uses the measurement $y_{i,k}$ to update the estimate at only cells with indices $q_i - 1, \ldots, q_i + 1$. The performance of both these estimators is slightly better than the estimator in (6.7) that zeros out elements of the filter gain obtained using the classical Kalman filter gain expression.

VII. Conclusion

This paper presents an extension of the Kalman filter that uses specific subset of the measurement to directly update only a specific subset of the state estimates rather than the entire state estimate. The ability to perform selective output injection allows us to structure the estimator gain so that the close-loop dynamics is banded. The one-step and the two-step forms of this filter are generally not equivalent. In future work, we plant to consider the problem of determining the exact conditions that guarantee the existence of a steady state covariance for linear time-invariant dynamics.

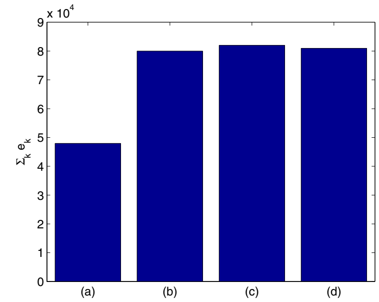

Fig. 2. Sum of the squares of error in temperature estimates obtained using (a)-the classical Kalman filter and (b)-the one-step filter in (6.5) with $\Gamma_k \neq I$. Although the suboptimal estimator in (6.7) guarantees closed-loop banded diagonal dynamics, the performance of the optimal estimator given by Proposition III.1 with Γ_i chosen according to section IV (denoted by (d)) is slightly better than the estimator in (6.7) (denoted by (c)).

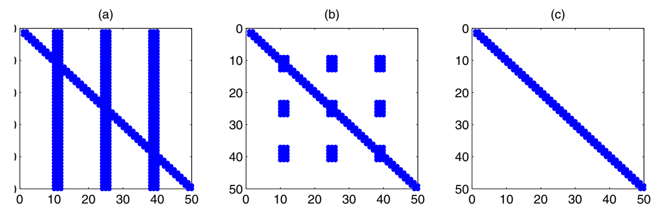

Fig. 3. Structure of the closed-loop dynamics of the classical Kalman filter is shown in (a). The colored dots indicate the location of nonzero elements of $\tilde{A}_k$. The structure of the closed-loop dynamics of the estimator in (6.5) is shown in (b). Note that the tridiagonal structure of A is not preserved in both these cases. Alternately, the tridiagonal structure is preserved by the optimal estimator in Proposition III.1 with $\Gamma_{i,k}$ chosen according to section IV (shown in (c)).

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