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A Subspace Algorithm for Simultaneous Identification and Input Reconstruction

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I. Introduction

Systems with unknown inputs have received considerable attention [4–23, 25, 26, 28–30]. The unknown inputs may represent unknown external drivers, input uncertainty, or instrument faults. An active research area is state reconstruction with known model equations and unknown inputs. Approaches include full-order observers [5, 7, 10, 16, 17, 30], reduced-order observers [8, 9, 20, 22], geometric approach [4], and the trial-and-error approach [28]. A widely used approach is to model the unknown inputs as outputs of a known dynamic system and incorporate the input dynamics with the plant dynamics [1, 14]. However, this approach increases the dimension of the observer and is limited to specific types of inputs.

In [25, 26] input reconstruction is achieved inverting the known transfer function. More recently, methods for input reconstruction using optimal filters are developed in [5, 10, 11, 15, 29]. However, the methods of [5, 10, 11, 15, 25, 26, 29] for state reconstruction and input reconstruction require knowledge of the model equations.

A related problem is the concept of input and state observability, which is the ability to reconstruct the inputs and states using only output measurements. Necessary and sufficient conditions for input and state observability for continuous-time systems in terms of the invariant zeros of the system are presented in [5, 9, 13, 15, 20]. Input and state observability for discrete-time systems is considered in [15], while [10] considers a constructive algorithm to determine the observability of the unknown input and state.

Subspace identification algorithms are used to identify systems in state space form, and are naturally applicable to multi-input, multi-output (MIMO) systems [27]. The idea underlying subspace algorithms is that estimates of the state sequence in an unknown basis can be computed directly from input-output observations. Once the state estimates are available, state space matrices are estimated using least squares. These methods are computationally tractable and require no a priori information about the structure or order of the system.

In this paper, we examine conditions under which both the input and state can be estimated from the output measurements. We discuss necessary and sufficient conditions for a discrete-time system to be input and state observable and derive tests for input and state observability. Since no assumptions on the input are made, the unknown input can be either an unmodeled exogenous signal or an unknown function of the states.

We then develop a deterministic subspace identification algorithm for systems with arbitrary unknown inputs. When the conditions for input and state observability and persistency of excitations are satisfied, we show that the states, the state space matrices, and the unknown inputs can be estimated from the known inputs and the output measurements. No assumptions are imposed on the unknown inputs.

II. Input and State Observability: Strictly Proper Case

Consider the system

$$x_{k+1} = Ax_k + He_k, \tag{II.1}$$

$$y_k = Cx_k, \tag{II.2}$$

where $x_k \in \mathbb{R}^n$, $e_k \in \mathbb{R}^p$, $y_k \in \mathbb{R}^l$, $A \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{l\times n}$. Without loss of generality, we assume $l \le n$, $\operatorname{rank}(C) = l > 0$, and $\operatorname{rank}(H) = p > 0$. No assumptions on the unmeasured signal e_k are made. Hence, e_k can be either an exogenous input or a nonlinear, time-varying function of the present or past states.

Throughout this paper, r denotes a nonnegative integer. Furthermore, for convenience, every vector or matrix with zero rows or zero columns is an empty matrix. Define $\mathcal{Y}_r \in \mathbb{R}^{(r+1)l}$ and $\mathcal{E}_r \in \mathbb{R}^{(r+1)p}$ as

$$\mathcal{Y}_r \triangleq \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_r \end{bmatrix}, \quad \mathcal{E}_r \triangleq \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_r \end{bmatrix}. \tag{II.3}$$

Definition II.1. Let $r \ge 1$. Then the *input and state unobservable subspace* $\mathfrak{U}_r$ of (II.1), (II.2) is the subspace

$$\mathfrak{U}_r \triangleq \left\{ \begin{bmatrix} x_0 \\ \mathcal{E}_{r-1} \end{bmatrix} \in \mathbb{R}^{n+rp} \colon \mathcal{Y}_r = 0 \right\}. \tag{II.4}$$

We define $\Gamma_r \in \mathbb{R}^{(r+1)l\times n}$, $M_r \in \mathbb{R}^{(r+1)l\times rp}$, and $\Psi_r \in \mathbb{R}^{(r+1)l\times (n+rp)}$ by

$$\Gamma_r \triangleq \begin{bmatrix} C^{\mathrm{T}} & (CA)^{\mathrm{T}} & \cdots & (CA^r)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \tag{II.5}$$

$$M_r \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ CH & 0 & \cdots & 0 \\ CAH & CH & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{r-1}H & CA^{r-2}H & \cdots & CH \end{bmatrix}, \tag{II.6}$$

and

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$$\Psi_r \triangleq \left[\begin{array}{cc} \Gamma_r & M_r \end{array} \right]. \qquad \text{(II.7)}$$

Note that M_0 is an empty matrix and thus $\Psi_0 = \Gamma_0 = C$. Next, from (II.1), (II.2), we can write

$$\mathcal{Y}_r = \Gamma_r x_0 + M_r \mathcal{E}_{r-1} = \Psi_r \left[\begin{array}{c} x_0 \\ \mathcal{E}_{r-1} \end{array} \right], \qquad \text{(II.8)}$$

so that

$$\mathfrak{U}_r = \mathcal{N}(\Psi_r), \qquad \text{(II.9)}$$

where $\mathcal{N}$ denotes null space. Next, define the positive integer

$$r_0 \triangleq \begin{cases} \max\{\lceil \frac{n-l}{l-p} \rceil, 1\}, & p < l, \\ 1, & p = l, \end{cases} \qquad \text{(II.10)}$$

where $\lceil a \rceil$ denotes the smallest integer greater than or equal to a. Note that r_0 is not defined in the case $p > l$.

Proposition II.1. Assume that $n \geq 2$ and $p \leq l$. Then $r_0 \leq n - 1$.

Proposition II.2. Let $r \geq 1$. If $\mathfrak{U}_r = \{0\}$, then the following statements hold:

1) $p \leq l$.
2) If $p = l$, then $p = l = n$.
3) (A, C) is observable, that is, $\operatorname{rank}(\Gamma_{n-1}) = n$.
4) $r \geq r_0$.
5) $\operatorname{rank}(CH) = p$.
6) $\operatorname{rank}(\Psi_r) = \operatorname{rank}(\Psi_{r-1}) + p$ for all $r \geq r_0$.

Proposition II.3. Assume that either $p < l$ or $p = l = n$. Then $n + rp \leq (r + 1)l$ for all $r \geq r_0$.

Proposition II.3 implies that if $p < l$ or $p = l = n$, then, for all $r \geq r_0$, the number of columns of Ψ_r is less than or equal to the number of rows of Ψ_r.

Definition II.2. The system (II.1), (II.2) is *input and state observable* if $\mathfrak{U}_r = \{0\}$ for all $r \geq r_0$.

Definition II.2 implies that if (II.1), (II.2) is input and state observable, then, for all $r \geq r_0$, the initial condition x_0 and input sequence $\{e_i\}_{i=0}^{r-1}$ are uniquely determined from the measured output sequence $\{y_i\}_{i=0}^{r}$.

Theorem II.1. The following statements are equivalent:

1) (II.1), (II.2) is input and state observable.
2) For all $r \geq r_0$, $\mathcal{Y}_r = 0$ if and only if $\left[\begin{array}{c} x_0 \\ \mathcal{E}_{r-1} \end{array} \right] = 0$.
3) For all $r \geq r_0$, $\operatorname{rank}(\Psi_r) = n + rp$.
4) There exists $r \geq r_0$ such that $\operatorname{rank}(\Psi_r) = n + rp$.
5) $\operatorname{rank}(\Psi_{n-1}) = n + (n - 1)p$.

Proof. From Definition II.1 and Definition II.2 it follows that $1) \Rightarrow 2)$. Using (II.8), $2) \Rightarrow 3)$. The result $3) \Rightarrow 4)$ is immediate. To prove $4) \Rightarrow 5)$ let $n = 1$. Then $\Psi_0 = C$ and rank(C) = 1. Now, suppose $n \geq 2$. Since $\operatorname{rank}(\Psi_r) = n + rp$ it follows that $\operatorname{rank}(CH) = p$. Hence, for all $\hat{r} \geq r_0$, $\operatorname{rank}(\Psi_{\hat{r}}) = \operatorname{rank}(\Psi_{\hat{r}-1}) + p$. Hence, since $n - 1 \geq r_0$, we have $\operatorname{rank}(\Psi_{n-1}) = n + (n - 1)p$. Finally to show $5) \Rightarrow 1)$, we consider two cases. First, suppose $n = 1$. In this case C and H are nonzero scalars, and hence it follows that $\operatorname{rank}(\Psi_r) = n + rp$ for all $r \geq r_0$ and hence $\mathfrak{U}_r = \{0\}$ for all $r \geq r_0$. Next, suppose $n \geq 2$. In this case $\operatorname{rank}(\Psi_{n-1}) = n + (n - 1)p$ implies that $\operatorname{rank}(CH) = p$ and hence $\operatorname{rank}(\Psi_r) = \operatorname{rank}(\Psi_{r-1}) + p$ for all $r \geq r_0$. Next, since $n - 1 \geq r_0$, it follows that, for all $r \geq r_0$, $\operatorname{rank}(\Psi_r) = \operatorname{rank}(\Psi_{n-1}) + (r - n + 1)p$. Thus $\operatorname{rank}(\Psi_r) = n + rp$ for all $r \geq r_0$ and hence $\mathfrak{U}_r = \{0\}$ for all $r \geq r_0$. □

Theorem II.1 shows that (II.1), (II.2) is input and state observable if and only if Ψ_r has full column rank for all $r \geq r_0$. In this case the unique solution of (II.8) is

$$\left[\begin{array}{c} x_0 \\ \mathcal{E}_{r-1} \end{array} \right] = \Psi_r^\dagger \mathcal{Y}_r, \qquad \text{(II.11)}$$

where † represents the Moore-Penrose generalized inverse $\Psi_r^\dagger = (\Psi_r^{\mathrm{T}} \Psi_r)^{-1} \Psi_r^{\mathrm{T}}$.

Note that if no unknown inputs are present, that is, $p = 0$, then $\Psi_r = \Gamma_r$, and statement 5 of Theorem II.1 becomes the standard rank test for observability.

III. Input and State Observability: Exactly Proper Case

Next, we consider the system

$$x_{k+1} = Ax_k + He_k, \qquad \text{(III.1)}$$
$$y_k = Cx_k + Ge_k, \qquad \text{(III.2)}$$

where $G \in \mathbb{R}^{l \times p}$, while A, H, C, x_k, e_k, and y_k are defined as in (II.1), (II.2). Without loss of generality, we assume $l \leq n$, $\operatorname{rank}(C) = l > 0$, and $\operatorname{rank} \left[\begin{array}{c} H \\ G \end{array} \right] = p > 0$. Due to Ge_k, the output y_k is directly affected by e_k as well as by the past values of e_k. Therefore, we have

$$\mathcal{Y}_r = \bar{\Psi}_r \left[\begin{array}{c} x_0 \\ \mathcal{E}_r \end{array} \right], \qquad \text{(III.3)}$$

where $\mathcal{E}_r$ is defined by (II.3), $\bar{\Psi}_r \triangleq \left[\begin{array}{cc} \Gamma_r & \bar{M}_r \end{array} \right] \in \mathbb{R}^{(r+1)l \times [n+(r+1)p]}$, and

$$\bar{M}_r = \left[\begin{array}{cccc} G & 0 & \cdots & 0 \\ CH & G & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{r-1}H & CA^{r-2}H & \cdots & G \end{array} \right]. \qquad \text{(III.4)}$$

Furthermore, we have the following definition.

Definition III.1. Let $r \geq 0$. Then the *input and state unobservable subspace* $\bar{\mathfrak{U}}_r$ of (III.1), (III.2) is the subspace

$$\bar{\mathfrak{U}}_r \triangleq \left\{ \left[\begin{array}{c} x_0 \\ \mathcal{E}_r \end{array} \right] \in \mathbb{R}^{n+(r+1)p} \colon \mathcal{Y}_r = 0 \right\}. \qquad \text{(III.5)}$$

The input and state unobservable subspace is given by $\bar{\mathfrak{U}}_r = \mathcal{N}(\bar{\Psi}_r)$. Next, if $p < l$ then define

$$\bar{r}_0 \triangleq \lceil \tfrac{n}{l-p} \rceil - 1. \qquad \text{(III.6)}$$

Since $n > l - p$ it follows that $\bar{r}_0 \geq 1$.

Theorem III.1. *The following statements are equivalent:*

1) *(III.1), (III.2) is input and state observable.*
2) *For all $r \geq \bar{r}_0$, $\mathcal{Y}_r = 0$ if and only if* $\left[\begin{array}{c} x_0 \\ \mathcal{E}_r \end{array} \right] = 0$.

3) *rank*$(\bar{\Psi}_r) = n + (r+1)p$ *for all* $r \geq \bar{r}_0$.
4) *There exists* $r \geq \bar{r}_0$ *such that rank*$(\bar{\Psi}_r) = n+(r+1)p$.
5) *rank*$(\bar{\Psi}_{n-1}) = n(p+1)$.

Finally, if (III.1), (III.2) is input and state observable, then Theorem III.1 implies that $\bar{\Psi}_r$ has full column rank for all $r \geq \bar{r}_0$. In this case the unique solution of (III.3) is

$$\begin{bmatrix} x_0 \\ \mathcal{E}_r \end{bmatrix} = \bar{\Psi}_r^\dagger \mathcal{Y}_r. \tag{III.7}$$

IV. Connections with Multivariable Zeros

In this section, we reinterpret the input and state observability conditions for the strictly proper case in terms of multivariable transmission zeros.

For $\lambda \in \mathbb{C}$, define $v(\lambda) \in \mathbb{C}^{n-1}$ by

$$v(\lambda) = \begin{bmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{n-2} \end{bmatrix}^{\mathrm{T}} \tag{IV.1}$$

and $V(\lambda) \in \mathbb{C}^{[n+(n-1)p]\times(n+p)}$ by

$$V(\lambda) \triangleq \begin{bmatrix} -I_n & 0 \\ 0 & v(\lambda)\otimes I_p \end{bmatrix},$$

where $\otimes$ is the Kronecker product.

Lemma IV.1. Assume that (A, C) is observable, $\mathrm{rank}(\Psi_{n-1}V(\lambda)) = n+p$ for all $\lambda \in \mathbb{C}$, and either $p < l$ or $p = l = n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{C}$ be distinct, then

$$\mathrm{rank}\left(\Psi_{n-1}\left[V(\lambda_1) \quad \cdots \quad V(\lambda_{n-1})\right]\right) = n + (n-1)p.$$

Outline of Proof. First, using Fact 2.10.24 in [2], we have

$$\mathrm{rank}\left(\Psi_{n-1}\begin{bmatrix} V(\lambda_1) & V(\lambda_2) \end{bmatrix}\right) \quad = \quad n+2p.$$

Next, let $2 < k < n-1$ be an integer and assume that

$$\mathrm{rank}\left(\Psi_{n-1}\begin{bmatrix} V(\lambda_1) & V(\lambda_2) & \cdots & V(\lambda_k) \end{bmatrix}\right) = n+kp.$$

Next, noting that it follows from Fact 5.13.3, p. 211 in [2] that $\mathrm{rank}\begin{bmatrix} V(\lambda_1) & \cdots & V(\lambda_{n-1}) \end{bmatrix} = n+(n-1)p$, and since $p < l$ or $p = l = n$, it follows that

$$\begin{aligned}
\mathrm{rank}&\left(\Psi_{n-1}\left[V(\lambda_1) \quad \cdots \quad V(\lambda_{k+1})\right]\right) \\
&= \mathrm{rank}\left(\Psi_{n-1}\left[V(\lambda_1) \quad \cdots \quad V(\lambda_k)\right]\right) \\
&\qquad + \mathrm{rank}\left(\Psi_{n-1}V(\lambda_{k+1})\right) \\
&- \dim\ \left(\mathcal{R}\left(\Psi_{n-1}\left[V(\lambda_1) \quad \cdots \quad V(\lambda_k)\right]\right)\right. \\
&\qquad \left.\cap\left(\mathcal{R}\left(\Psi_{n-1}V(\lambda_{k+1})\right)\right)\right). \\
&= n+(k+1)p.
\end{aligned}$$

Setting $k = n-2$ yields the result. □

Next, define the l by p rational transfer function matrix $L(s)$ by

$$L(s) \triangleq C(sI-A)^{-1}H. \tag{IV.2}$$

Furthermore, we assume that (A, H, C) is minimal. Then $\lambda \in \mathbb{C}$ is an *invariant zero* of the realization (A, H, C) if [31]

$$\mathrm{rank}\begin{bmatrix} \lambda I - A & H \\ C & 0 \end{bmatrix} < \mathrm{normalrank}\begin{bmatrix} sI - A & H \\ C & 0 \end{bmatrix}.$$

Since (A, H, C) is minimal, the transmission zeros of L are the invariant zeros of (A, H, C).

Lemma IV.2. The following statements are equivalent:

i) normalrank $L = p$ and L has no transmission zeros.

ii) For all $\lambda \in \mathbb{C}$, $\mathrm{rank}\begin{bmatrix} \lambda I - A & H \\ C & 0 \end{bmatrix} = n+p$.

Note that *ii*) in Lemma IV.2 implies that (II.1)-(II.2) has no invariant zeros. The following result provides equivalent conditions for Theorem II.1 in terms of multivariable zeros.

Theorem IV.1. The following statements are equivalent:

i) Either $p < l$ or $p = l = n$, and (A, H, C) has no invariant zeros.

ii) rank $(\Psi_{n-1}) = n+(n-1)p$.

Proof. To prove *i*) implies *ii*), it follows from *i*) that, for all $\lambda \in \mathbb{C}$, $\mathrm{rank}\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$, and thus (A, C) is observable. Hence

$$\mathrm{rank}\begin{bmatrix} 0 & -I_l \\ \Gamma_{n-1} & 0 \end{bmatrix} = n + l.$$

Furthermore, noting that

$$\begin{aligned}
\Upsilon &\triangleq \begin{bmatrix} 0 & -I_l \\ \Gamma_{n-1} & 0 \end{bmatrix}\begin{bmatrix} \lambda I - A & H \\ C & 0 \end{bmatrix} \\
&= \begin{bmatrix} -C & 0 \\ \lambda C - CA & CH \\ \vdots & \vdots \\ \lambda CA^{n-2} - CA^{n-1} & CA^{n-2}H \end{bmatrix},
\end{aligned}$$

it follows from Sylvester's inequality (Proposition 2.5.8 in [2]) that, for all $\lambda \in \mathbb{C}$,

$$\begin{aligned}
n+p &\geq \mathrm{rank}\Upsilon \\
&\geq \mathrm{rank}\begin{bmatrix} 0 & -I_l \\ \Gamma_{n-1} & 0 \end{bmatrix} + \mathrm{rank}\begin{bmatrix} \lambda I - A & H \\ C & 0 \end{bmatrix} - (n+l) \\
&= \quad n+p.
\end{aligned}$$

Hence $\mathrm{rank}\Upsilon = n+p$. Next, for all $\lambda \in \mathbb{C}$, we have

$$n+p = \mathrm{rank}\begin{bmatrix} I_n & 0 & \cdots & 0 \\ \lambda I_n & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{n-1}I_n & \lambda^{n-2}I_n & \cdots & I_n \end{bmatrix}\Upsilon. \tag{IV.3}$$

Next, using (**??**), (IV.3) becomes

$$\mathrm{rank}\left(\Psi_{n-1}V(\lambda)\right) = n+p.$$

Finally, let $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{C}$ be distinct. Then, it follows from Lemma IV.1 and Lemma 2.5.2 [2] that

$$\begin{aligned}
n+(n-1)p &= \mathrm{rank}\left(\Psi_{n-1}\begin{bmatrix} V(\lambda_1) & \cdots & V(\lambda_{n-1}) \end{bmatrix}\right) \\
&\leq \mathrm{rank}(\Psi_{n-1}).
\end{aligned} \tag{IV.4}$$

However, since $\mathrm{rank}(\Psi_{n-1}) \leq n+(n-1)p$, it follows that $\mathrm{rank}(\Psi_{n-1}) = n+(n-1)p$.

Next, to prove *ii*) implies *i*), suppose there exists $\lambda \in \mathbb{C}$ such that $\mathrm{rank}\begin{bmatrix} \lambda I - A & H \\ C & 0 \end{bmatrix} < n+p$. Then there exist

$\tilde{x}_0 \in \mathbb{C}^n$ and $\tilde{e} \in \mathbb{C}^p$ such that $\begin{bmatrix} \tilde{x}_0 \\ \tilde{e} \end{bmatrix}$ is nonzero and

$$(\lambda I - A)\tilde{x}_0 + H\tilde{e} = 0 \tag{IV.5}$$

and

$$C\tilde{x}_0 = 0. \tag{IV.6}$$

Premultiplying (IV.5) by C and using (IV.6) yields

$$-CA\tilde{x}_0 + CH\tilde{e} = 0. \tag{IV.7}$$

Next, premultiplying (IV.5) by CA yields

$$\lambda CA\tilde{x}_0 - CA^2\tilde{x}_0 + CAH\tilde{e} = 0. \tag{IV.8}$$

Using (IV.7) in (IV.8) yields

$$-CA^2\tilde{x}_0 + CAH\tilde{e} + \lambda CH\tilde{e} = 0. \tag{IV.9}$$

Similarly, premultiplying (IV.5) by $CA^2, CA^3, \ldots, CA^{n-2}$, and writing the resulting equations in matrix form yields

$$\Psi_{n-1} \begin{bmatrix} \tilde{x}_0 \\ \tilde{\mathcal{E}}_{n-2} \end{bmatrix} = 0, \tag{IV.10}$$

where $\tilde{\mathcal{E}}_{n-2} \triangleq \begin{bmatrix} \tilde{e}^{\mathrm{T}} & \lambda\tilde{e}^{\mathrm{T}} & \cdots & \lambda^{n-2}\tilde{e}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{C}^{(n-1)p}$. Since $\begin{bmatrix} \tilde{x}_0 \\ \tilde{e} \end{bmatrix} \neq 0$, it follows that $\begin{bmatrix} \tilde{x}_0 \\ \tilde{\mathcal{E}}_{n-2} \end{bmatrix} \neq 0$. However, since $\operatorname{rank}(\Psi_{n-1}) = n + (n-1)p$, it follows that $\begin{bmatrix} \tilde{x}_0 \\ \tilde{\mathcal{E}}_{n-2} \end{bmatrix} = 0$, which contradicts $\begin{bmatrix} \tilde{x}_0 \\ \tilde{\mathcal{E}}_{n-2} \end{bmatrix} \neq 0$. Hence $\operatorname{rank} \begin{bmatrix} \lambda I - A & H \\ C & 0 \end{bmatrix} = n + p$ for all $\lambda \in \mathbb{C}$. Furthermore, using Proposition II.2, it follows that either $p < l$ or $p = l = n$. □

Note that $i)$ in the above result is same as the sufficient condition for input observability presented in [20].

V. State Estimation with Unknown Inputs and Unknown Dynamics

Consider the system

$$x_{k+1} = Ax_k + Bu_k + He_k, \tag{V.1}$$

$$y_k = Cx_k + Du_k + Ge_k, \tag{V.2}$$

where x_k, y_k, e_k, A, C, H, G are as in Section 2, $u_k \in \mathbb{R}^m$, $B \in \mathbb{R}^{n\times m}$, and $D \in \mathbb{R}^{l\times m}$. Furthermore, u_k is a known input, whereas e_k is an unknown signal. The system (V.1), (V.2) is *input and state observable* if it is input and state observable with $u_k \equiv 0$. We consider the problem of estimating the state sequence using measurements of inputs u_k and outputs y_k, while A, B, C, D, H, G, and e_k are unknown. The problem of estimating A, B, C, D, H, G, and e_k is considered in the next section. We assume (A, B) is controllable, $p \le l$ is known, but the order n of the system is unknown. In this section we assume that $G \neq 0$ so that (V.1), (V.2) corresponds to the exactly proper case (III.1), (III.2). The case $G = 0$ is discussed later.

Let $N+1$ be the number of available measurements, and let i be an integer such that $n \le i$ and $2i - 1 < N$. Define $U_{0|2i-1} \in \mathbb{R}^{2mi\times(N-2i+2)}$, $U_{\mathrm{p}} \in \mathbb{R}^{mi\times(N-2i+2)}$, and $U_{\mathrm{f}} \in \mathbb{R}^{mi\times(N-2i+2)}$ by

$$U_{0|2i-1} \triangleq \left[\begin{array}{cccc} u_0 & u_1 & \cdots & u_{N-2i+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{i-1} & u_i & \cdots & u_{N-i} \\ \hline u_i & u_{i+1} & \cdots & u_{N-i+1} \\ \hdashline u_{i+1} & u_{i+2} & \cdots & u_{N-i+2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{2i-1} & u_{2i} & \cdots & u_N \end{array}\right] \tag{V.3}$$

$$= \left[\begin{array}{c} U_{0|i-1} \\ \hline U_{i|2i-1} \end{array}\right] = \left[\begin{array}{c} U_{\mathrm{p}} \\ \hline U_{\mathrm{f}} \end{array}\right]. \tag{V.4}$$

$$= \left[\begin{array}{c} U_{0|i} \\ \hdashline U_{i+1|2i-1} \end{array}\right] = \left[\begin{array}{c} U_{\mathrm{p}}^{+} \\ \hdashline U_{\mathrm{f}}^{-} \end{array}\right], \tag{V.5}$$

where $U_{\mathrm{p}}^{+} \in \mathbb{R}^{(i+1)m\times(N-2i+2)}$ and $U_{\mathrm{f}}^{-} \in \mathbb{R}^{(i-1)m\times(N-2i+2)}$. The subscript p denotes 'past' and the subscript f denotes 'future'. The output block-Hankel matrices $Y_{0|2i-1}$, Y_{p}, Y_{f}, Y_{p}^{+} and Y_{f}^{-} are defined as in (V.3) - (V.5) with u replaced by y. The unknown-input block-Hankel matrices $E_{0|2i-1}$, E_{p}, E_{f}, E_{p}^{+} and E_{f}^{-} are defined as in (V.3) - (V.5) with u replaced by e. Furthermore, define the past input-output data $W_{\mathrm{p}} \triangleq \begin{bmatrix} U_{\mathrm{p}} \\ Y_{\mathrm{p}} \end{bmatrix} \in \mathbb{R}^{i(m+l)\times(N-2i+2)}$ and the future input-output data $W_{\mathrm{f}} \triangleq \begin{bmatrix} U_{\mathrm{f}} \\ Y_{\mathrm{f}} \end{bmatrix} \in \mathbb{R}^{i(m+l)\times(N-2i+2)}$. Finally, define the block-Toeplitz matrix $\Omega_i \in \mathbb{R}^{(i+1)l\times(i+1)m}$ by

$$\Omega_i \triangleq \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{i-1}B & CA^{i-2}B & \cdots & D \end{bmatrix}, \tag{V.6}$$

and for $0 \le r \le 2i$ define the state sequences $X_r \in \mathbb{R}^{n\times(N-2i+2)}$ by

$$X_r \triangleq \begin{bmatrix} x_r & x_{r+1} & \cdots & x_{N-2i+r} & x_{N-2i+r+1} \end{bmatrix}. \tag{V.7}$$

Lemma V.1. *If (V.1), (V.2) is input and state observable, then the row space of X_i is contained in the intersection of the row space of W_{p} and the row space of W_{f}.*

Proof: From (V.1) and (V.2),

$$Y_{\mathrm{p}} = \Gamma_{i-1}X_0 + \bar{M}_{i-1}E_{\mathrm{p}} + \Omega_{i-1}U_{\mathrm{p}}, \tag{V.8}$$

$$Y_{\mathrm{f}} = \Gamma_{i-1}X_i + \bar{M}_{i-1}E_{\mathrm{f}} + \Omega_{i-1}U_{\mathrm{f}}. \tag{V.9}$$

Since the system is input and state observable, (V.9) can be written as

$$\begin{bmatrix} X_i \\ E_{\mathrm{f}} \end{bmatrix} = \begin{bmatrix} -\bar{\Psi}_{i-1}^{\dagger}\Omega_{i-1} & \bar{\Psi}_{i-1}^{\dagger} \end{bmatrix} W_{\mathrm{f}}. \tag{V.10}$$

The rest of the proof follows along the lines of the proof in [24]. □

To calculate the state sequence, we require the following definition.

Definition V.1. The sequences $\{u_k\}_{k=1}^{N}$ and $\{e_k\}_{k=1}^{N}$ are *persistently exciting* for (V.1), (V.2) if

$$\operatorname{rank}\begin{bmatrix} X_0 \\ E_{0|2i-1} \\ U_{0|2i-1} \end{bmatrix} = n + 2pi + 2mi. \qquad \text{(V.11)}$$

Theorem V.1. *If the system (V.1), (V.2) is input and state observable and the sequences $\{u_k\}_{k=1}^N$ and $\{e_k\}_{k=1}^N$ are persistently exciting, then the intersection of the row spaces of $W_{\rm p}$ and $W_{\rm f}$ is equal to the row space of X_i.*

Proof. The proof is similar to the proofs in [24]. □

Let $\hat{X}_i$ denote an estimate of the state sequence X_i. Using Theorem V.1, we compute $\hat{X}_i$ as the intersection of the row spaces of $W_{\rm p}$ and $W_{\rm f}$. One way to compute this intersection is by orthogonally projecting the row space of $W_{\rm p}$ onto the row space of $W_{\rm f}$ [27]. Thus

$$\hat{X}_i \triangleq W_{\rm f} W_{\rm p}^{\rm T} (W_{\rm p} W_{\rm p}^{\rm T})^{\dagger} W_{\rm p}. \qquad \text{(V.12)}$$

Note that to calculate $\hat{X}_i$, we use measurements of u_k and y_k, however, knowledge of e_k is not required.

A numerically efficient way to compute $\hat{X}_i$ is to use the LQ decomposition of $\begin{bmatrix} W_{\rm p} \\ W_{\rm f} \end{bmatrix}$ [27] denoted by

$$\begin{bmatrix} W_{\rm p} \\ W_{\rm f} \end{bmatrix} = LQ^{\rm T} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} Q_1^{\rm T} \\ Q_2^{\rm T} \end{bmatrix}, \qquad \text{(V.13)}$$

where $L \in \mathbb{R}^{2i(m+l)\times 2i(m+l)}$ is lower triangular, L_{11}, L_{21}, $L_{22} \in \mathbb{R}^{i(m+l)\times i(m+l)}$, $Q \in \mathbb{R}^{(N-2i+2)\times 2i(m+l)}$ is orthogonal, and Q_1, $Q_2 \in \mathbb{R}^{(N-2i+2)\times i(m+l)}$. Then, the intersection of row spaces of W_p and W_f is computed as $L_{21}Q_1^{\rm T}$. An estimate $\hat{X}_i$ of the state sequence X_i can then be obtained by using a singular value decomposition to calculate a basis for the row space of $L_{21}Q_1^{\rm T}$,

Similarly, estimates $\hat{X}_{i+1}$ of the state sequence X_{i+1} are obtained by computing the intersection of the row spaces of $\begin{bmatrix} U_{\rm p}^+ \\ Y_{\rm p}^+ \end{bmatrix}$ and $\begin{bmatrix} U_{\rm f}^- \\ Y_{\rm f}^- \end{bmatrix}$.

Next, assume $G = 0$ in (V.1), (V.2). This case corresponds to the no-feedthrough case, and the following result considers state estimation with unknown inputs and unknown dynamics.

Theorem V.2. *Assume that (V.1) and (V.2) with $G = 0$ is input and state observable. If the input sequences $\{u_k\}_{k=1}^N$ and $\{e_k\}_{k=1}^N$ are persistently exciting, then the intersection of the row spaces of $\begin{bmatrix} U_{\rm p} \\ Y_{\rm p}^+ \end{bmatrix}$ and $\begin{bmatrix} U_{\rm f} \\ Y_{\rm f} \end{bmatrix}$ is the row space of X_i.*

VI. Simultaneous Model Estimation and Input Reconstruction

In this section we consider the problem of estimating the state space matrices A, B, C, D, H, G and e_k of (V.1), (V.2) using estimates $\hat{X}_i$ of the state sequence X_i and measurements of u_k and y_k. To do this we write

$$\begin{bmatrix} X_{i+1} \\ Y_{i|i} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_i \\ U_{i|i} \end{bmatrix} + \begin{bmatrix} H \\ G \end{bmatrix} E_{i|i}.$$

We use a two-step procedure to estimate A, B, C, D, H, and G. First we estimate the matrices A, B, C, and D by solving the least squares problem

$$\underset{A,B,C,D}{\operatorname{argmin}} \left|\left| R_{i|i} \right|\right|_2, \qquad \text{(VI.1)}$$

where the residuals $R_{i|i}$ are defined as

$$R_{i|i} \triangleq \begin{bmatrix} \hat{X}_{i+1} \\ Y_{i|i} \end{bmatrix} - \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} \hat{X}_i \\ U_{i|i} \end{bmatrix}. \qquad \text{(VI.2)}$$

Although $\begin{bmatrix} X_{i+1} \\ Y_{i|i} \end{bmatrix}$ is a linear combination of $\begin{bmatrix} X_i \\ U_{i|i} \end{bmatrix}$ and $E_{i|i}$, the term due to $E_{i|i}$ is ignored in the least squares problem (VI.1). Thus $E_{i|i}$ is interpreted as noise, and hence unbiased estimates of the state space matrices are not guaranteed. However, if $\begin{bmatrix} x_k \\ u_k \end{bmatrix}$ and e_k are uncorrelated then unbiased estimates of A, B, C, and D are obtained using (VI.1). Next, we estimate $\begin{bmatrix} H \\ G \end{bmatrix}$ and $E_{i|i}$ by forming the singular value decomposition

$$\begin{aligned} R_{i|i} = U\Sigma V^{\rm T} \approx U\hat{\Sigma}V^{\rm T} &= (U\hat{\Sigma}^{1/2})(\hat{\Sigma}^{1/2}V^{\rm T}) \\ &= \begin{bmatrix} \hat{H} \\ \hat{G} \end{bmatrix} \hat{E}_{i|i}, \end{aligned} \qquad \text{(VI.3)}$$

where $\hat{\Sigma}$ contains the p dominant singular values from Σ, while $\begin{bmatrix} \hat{H} \\ \hat{G} \end{bmatrix} \triangleq U\hat{\Sigma}^{1/2}$ and $\hat{E}_{i|i} \triangleq \hat{\Sigma}^{1/2}V^{\rm T}$.

Furthermore, consider the case in which e_k is a nonlinear function of the states, that is, $e_k = h(x_k)$, where $h: \mathbb{R}^n \to \mathbb{R}^p$. We then assume that $h(x_k)$ can be expanded in terms of basis functions as $h(x_k) = \theta f_h(x_k)$, where $f_h \colon \mathbb{R}^n \to \mathbb{R}^s$ are basis functions, and $\theta \in \mathbb{R}^{p\times s}$ are unknown coefficients of the basis-function expansion. Next, we estimate θ by solving the least squares problem

$$\underset{\theta}{\operatorname{argmin}} \left|\left| \hat{E}_{i|i} - \theta f_h(\hat{X}_i) \right|\right|_2. \qquad \text{(VI.4)}$$

When noise terms are present in (V.1) and (V.2) the states are estimated by obliquely projecting the row space of $Y_{\rm f}$ along the row space of $U_{\rm f}$ into the row space of $W_{\rm p}$ similar to the procedure presented in [27]. The least squares problems for calculating the state space matrices remain the same as (VI.1), (VI.3) and (VI.4).

VII. Example

We consider a system comprised of $n = 6$ compartments or subsystems that exchange energy through mutual interactions [3]. Applying conservation of energy yields, for $i = 1, \ldots, n$,

$$x_{k+1,i} = x_{k,i} - \beta x_{k,i} + \alpha(x_{k,i+1} - x_{k,i}) - \alpha(x_{k,i} - x_{k,i-1}),$$

where $0 < \beta < 1$ is the loss coefficient and $0 < \alpha < 0$ is the flow coefficient. In addition, we assume that a known input enters compartment 1, while an unknown input enters compartment 2. The outputs are the energy states in compartments 2 and 3. It then follows that

$$x_{k+1} = Ax_k + Bu_k + He_k, \quad \text{(VII.1)}$$
$$y_k = Cx_k, \quad \text{(VII.2)}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}, H \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{2\times n}$ are defined as

$$A \triangleq \begin{bmatrix} 1-\beta-\alpha & \alpha & 0 & \cdots & 0 \\ \alpha & 1-\beta-\alpha & \alpha & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha & 1-\beta-\alpha \end{bmatrix},$$

$$B \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}, \quad H \triangleq \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix}^{\mathrm{T}},$$
$$C \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \end{bmatrix}.$$

It can be verified that (VII.1), (VII.2) is input and state observable.

To generate data for identification, we set $\alpha = 0.3$ and $\beta = 0.1$, and corrupt the system equations with process noise and measurement noise with standard deviation 0.1. The known input is a realization of a white noise process, while the unknown input is a realization of a white noise process with impulses at time 20s and 80s. In Figure 1 the actual unknown input and the reconstructed unknown input is shown.

VIII. CONCLUSIONS

In this paper, we introduced the concept of input and state observability, that is, conditions under which both the unknown input and state can be estimated from the output measurements. We discussed sufficient and necessary conditions for a discrete-time system to be input and state observable. Next, we developed a subspace identification algorithm that identified the state space matrices and reconstructed the unknown input using output measurements and known inputs.

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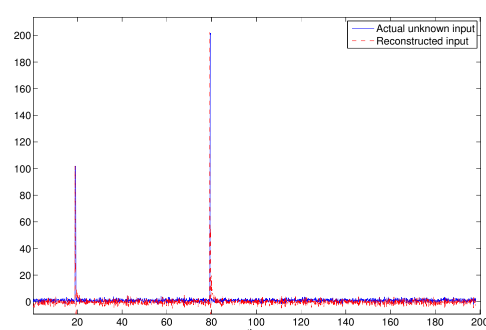

Fig. 1. The actual unknown input and the reconstructed unknown input.