Optimal Nonzero Set Point Regulation Via Fixed-Order Dynamic Compensation

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Abstract—**Standard LQG control theory is generalized to a regulation problem involving specified nonzero set points for the state and control variables and nonzero-mean disturbances. For generality, the results are obtained for the problem of fixed-order (i.e., not necessarily full-order) dynamic compensation. When the state, control, and disturbance offsets are set to zero and the compensator order is set equal to the plant dimension, the standard LQG result is recovered. These results provide the dynamic counterpart for the nonzero set point regulation results obtained in [1] via static controllers.**

I. Introduction

As discussed in [1], the standard quadratic performance criterion expresses the desire to maintain the state and control variables in the neighborhood of the origin. If regulation is desired about nonzero state and control offsets, then, in special cases, the set points can be translated to the origin and standard theory can be applied (see, e.g., [2, pp. 270–276]). In general, however, (see [1]) such a translation may either be suboptimal or impossible. The latter situation may occur, for example, if the number of state components with specified nonzero set points is greater than the number of controls, while the former is the case when the control offset is particularly costly.

Motivated by the work of Leizarowitz and Artstein [3], [4] on the more general problems of periodic and nonperiodic tracking, the nonzero set point problem was addressed in [1] for the case of static output-feedback controllers. The goal of the present note is to derive analogous results for the case of dynamic compensation considered by Leizarowitz in [5]. As in [1], the solution we obtain has the satisfying feature that the closed-loop dynamic-feedback-compensation gains are independent of the open-loop control components which arise from the state and control set points. Thus, if the state set point is changed during operation, then only the open-loop control components require updating. Consequently, there is no need to recalculate the closed-loop gains by solving Riccati equations in real time. The overall theory thus permits the treatment of step commands within standard LQG theory.

For generality the development herein incorporates several special features which provide additional flexibility in applications. These include: 1) constant disturbance vectors in addition to zero-mean additive plant and measurement noise (i.e., nonzero-mean disturbances); 2) correlated plant and measurement noise; 3) state/control performance cross-weighting; 4) arbitrary set points for selected linear combinations of the state and control variables (see L_1 and L_2 in the problem statement in Section III); and 5) fixed-order (i.e., full- or reduced-order) compensation. Because of the last feature, the results obtained in the present note also generalize the results of [6]. For clarity, we specialize the main result to the usual full-order LQG case.

II. Notation and Definitions

$\mathbb{R}, \mathbb{R}^{r\times s}, \mathbb{R}^r, \mathbb{E}$	Real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$, expectation.
$I_n, (\ )^T, (\ )^{\#}$	$n \times n$ identity, transpose, group generalized inverse.
asymptotically stable matrix	Matrix with eigenvalues in open left-half plane.
n, m, l, q, r, n_c	Positive integers.
$\tilde{n}$	$n + n_c$.
$x, u, y, x_c, \tilde{x}$	$n, m, l, n_c, \tilde{n}$-dimensional vectors.
A, B, C, D	$n \times n, n \times m, l \times n, l \times m$ matrices.
A_c, B_c, C_c	$n_c \times n_c, n_c \times l, m \times n_c$ matrices.
L_1, L_2	$q \times n, r \times m$ matrices.
δ_1, δ_2	q, r-dimensional set point vectors.
γ_1, γ_2	n, l-dimensional constant disturbance vectors.
α, α_c	m, n_c-dimensional control vectors.
$\tilde{\delta}, \tilde{\gamma}, \tilde{\alpha}$	$\begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix}, \begin{bmatrix}\gamma_1\\ B_c\gamma_2\end{bmatrix}, \begin{bmatrix}\alpha\\ \alpha_c\end{bmatrix}$.
$w_1(t), w_2(t)$	n, l-dimensional zero-mean white noise processes.
V_1, V_2	Intensities of w_1, w_2; $V_1 \geq 0, V_2 > 0$.
V_{12}	$n \times l$ cross intensity of w_1, w_2.
$\tilde{w}(t), \tilde{V}$	$\begin{bmatrix}w_1(t)\\ B_c w_2(t)\end{bmatrix}, \begin{bmatrix}V_1 & V_{12}B_c^T\\ B_c V_{12}^T & B_c V_2 B_c^T\end{bmatrix}$.
R_1, R_2	$q \times q$ and $r \times r$ state and control weightings; $R_1 \geq 0, R_2 \geq 0, L_2^T R_2 L_2 > 0$.
R_{12}	$q \times r$ cross weighting; $L_1^T R_1 L_1 - L_1^T R_{12} L_2 (L_2^T R_2 L_2)^{-1} L_2^T R_{12}^T L_1 \geq 0$.
$\tilde{R}$	$\begin{bmatrix}L_1^T R_1 L_1 & L_1^T R_{12} L_2 C_c\\ C_c^T L_2^T R_{12}^T L_1 & C_c^T L_2^T R_2 L_2 C_c\end{bmatrix}$.
$\tilde{A}, \tilde{B}$	$\begin{bmatrix}A & BC_c\\ B_c C & A_c + B_c D C_c\end{bmatrix}, \begin{bmatrix}B & 0\\ B_c D & I_{n_c}\end{bmatrix}$.
m, m_c	n, n_c-dimensional vectors.
$\tilde{m}$	$\begin{bmatrix}m\\ m_c\end{bmatrix}$.
$\tilde{R}_1$	$\begin{bmatrix}L_1^T R_1 L_1 - L_1^T R_{12} L_2 (L_2^T R_2 L_2)^{-1} L_2^T R_{12}^T L_1 & 0\\ 0 & 0\end{bmatrix}$.
$\tilde{R}_{12}, \tilde{R}_2, \tilde{R}_2^{\#}$	$\begin{bmatrix}L_2^T R_{12}^T L_1 & L_2^T R_2 L_2 C_c\\ 0 & 0\end{bmatrix}, \begin{bmatrix}L_2^T R_2 L_2 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}(L_2^T R_2 L_2)^{-1} & 0\\ 0 & 0\end{bmatrix}$.
$\tilde{N}, \tilde{S}$	$\begin{bmatrix}L_2^T R_{12}^T & L_2^T R_2\\ 0 & 0\end{bmatrix}, \begin{bmatrix}L_1^T R_1 & L_1^T R_{12}\\ C_c^T L_2^T R_{12}^T & C_c^T L_2^T R_2\end{bmatrix}$.

For arbitrary $n \times n$ Q, P define:

$$Q_a \triangleq QC^T + V_{12}, \qquad P_a \triangleq B^T P + L_2^T R_{12}^T L_1,$$

$$A_Q \triangleq A - Q_a V_2^{-1} C^T, \qquad A_P \triangleq A - B(L_2^T R_2 L_2)^{-1} P_a.$$

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III. Dynamic Compensation for Nonzero Set Point Regulation

A. Nonzero Set Point Problem

Given the nth-order stabilizable and detectable plant

$$\dot{x}(t)=Ax(t)+Bu(t)+w_1(t)+\gamma_1, \qquad t\in[0,\infty), \tag{3.1}$$

$$y(t)=Cx(t)+Du(t)+w_2(t)+\gamma_2 \tag{3.2}$$

design a fixed-order dynamic compensator

$$\dot{x}_c(t)=A_cx_c(t)+B_cy(t)+\alpha_c, \tag{3.3}$$

$$u(t)=C_cx_c(t)+\alpha \tag{3.4}$$

which minimizes the steady-state performance criterion

$$\begin{aligned}J(A_c,B_c,C_c,\alpha,\alpha_c)\triangleq \lim_{t\to\infty}\mathbb{E}[(L_1x(t)-\delta_1)^TR_1(L_1x(t)-\delta_1)\\ +2(L_1x(t)-\delta_1)^TR_{12}(L_2u(t)-\delta_2)\\ +(L_2u(t)-\delta_2)^TR_2(L_2u(t)-\delta_2)].\end{aligned} \tag{3.5}$$

Remark 3.1: The cost functional (3.5) is identical to the LQG criterion (usually stated in terms of an averaged integral) with the exception of the shifted set points δ_1 and δ_2 and matrices L_1 and L_2 for selecting linear combinations of components of x and u.

The closed-loop system (3.1)–(3.4) can be written as

$$\dot{\tilde{x}}(t)=\tilde{A}\tilde{x}(t)+\tilde{B}\tilde{\alpha}+\tilde{w}(t)+\tilde{\gamma}, \qquad t\in[0,\infty) \tag{3.6}$$

where $\tilde{x}(t)\triangleq[x^T(t),\ x_c^T(t)]^T$ and the closed-loop disturbance $\tilde{w}(t)$ has nonnegative-definite intensity $\tilde{V}$. To analyze (3.6) define the covariance matrix

$$\tilde{Q}(t)\triangleq\mathbb{E}[(\tilde{x}(t)-\tilde{m}(t))(\tilde{x}(t)-\tilde{m}(t))^T]=\mathbb{E}[\tilde{x}(t)\tilde{x}^T(t)]-\tilde{m}(t)\tilde{m}^T(t)$$

where $\tilde{m}(t)\triangleq\mathbb{E}[\tilde{x}(t)]$. As shown in [1], $\tilde{Q}(t)$ and $\tilde{m}(t)$ satisfy

$$\dot{\tilde{Q}}(t)=\tilde{A}\tilde{Q}(t)+\tilde{Q}(t)\tilde{A}^T+\tilde{V}, \tag{3.7}$$

$$\dot{\tilde{m}}(t)=\tilde{A}\tilde{m}(t)+\tilde{B}\tilde{\alpha}+\tilde{\gamma}. \tag{3.8}$$

To guarantee that J is finite and independent of initial conditions, we restrict our attention to the set of admissible stabilizing compensators

$$\mathbb{S}\triangleq\{(A_c,B_c,C_c):\ \tilde{A}\text{ is asymptotically stable}\}.$$

Hence, for $(A_c,B_c,C_c)\in\mathbb{S}$, $\tilde{Q}\triangleq\lim_{t\to\infty}\tilde{Q}(t)$ and $\tilde{m}\triangleq\lim_{t\to\infty}\tilde{m}(t)$ exist and satisfy

$$0=\tilde{A}\tilde{Q}+\tilde{Q}\tilde{A}^T+\tilde{V}, \tag{3.9}$$

$$0=\tilde{A}\tilde{m}+\tilde{B}\tilde{\alpha}+\tilde{\gamma}. \tag{3.10}$$

Since the value of J is independent of the internal realization of the transfer function corresponding to (3.3) and (3.4), without loss of generality we further restrict our attention to the set

$$\mathbb{S}'\triangleq\{(A_c,B_c,C_c)\in\mathbb{S}:\ (A_c,B_c)\text{ is controllable and }(A_c,C_c)\text{ is observable}\}.$$

Now $J(A_c,B_c,C_c,\alpha,\alpha_c)$ is given by

$$\begin{aligned}J(A_c,B_c,C_c,\alpha,\alpha_c)=\operatorname{tr}[\tilde{Q}+\tilde{m}\tilde{m}^T]\tilde{R}-2m^TL_1^TR_1\delta_1+\delta_1^TR_1\delta_1\\ +2m^TL_1^TR_{12}L_2\alpha-2m^TL_1^TR_{12}\delta_2\\ -2\delta_1^TR_{12}L_2C_cm_c-2\delta_1^TR_{12}L_2\alpha\\ +2\delta_1^TR_{12}\delta_2+2m_c^TC_c^TL_2^TR_2L_2\alpha-2m_c^TC_c^TL_2^TR_2\delta_2\\ -2\alpha^TL_2^TR_2\delta_2+\alpha^TL_2^TR_2L_2\alpha+\delta_2^TR_2\delta_2.\end{aligned} \tag{3.11}$$

To obtain closed-form expressions for the feedback gains we further restrict consideration to the set

$$\mathbb{S}''\triangleq\{(A_c,B_c,C_c)\in\mathbb{S}':\ \Omega>0\},$$

where

$$\Omega\triangleq\tilde{B}^T\tilde{A}^{-T}\tilde{R}_1\tilde{A}^{-1}\tilde{B}+(\tilde{R}_{12}\tilde{A}^{-1}\tilde{B}-\tilde{R}_{12})^T\tilde{R}_2^{\#}(\tilde{R}_{12}\tilde{A}^{-1}\tilde{B}-\tilde{R}_{12}).$$

The following factorization lemma is needed for the statement of the main result.

Lemma 3.1: Suppose $n\times n$ $\hat{Q}$, $\hat{P}$ are nonnegative definite and rank $\hat{Q}\hat{P}=n_c$. Then there exist $n_c\times n$ G, Γ and $n_c\times n_c$ invertible M such that

$$\hat{Q}\hat{P}=G^TM\Gamma, \tag{3.12}$$

$$\Gamma G^T=I_{n_c}. \tag{3.13}$$

Furthermore, G, M, and Γ are unique except for a change of basis in $\mathbb{R}^{n_c}$.

Proof: See [6]. □

As shown in [6], $\hat{Q}\hat{P}$ has a group generalized inverse $(\hat{Q}\hat{P})^{\#}=G^TM^{-1}\Gamma$, and the matrix

$$\tau\triangleq\hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#}=G^T\Gamma \tag{3.14}$$

is an oblique projection. A triple (G, M, Γ) satisfying (3.12) and (3.13) with $G,\Gamma\in\mathbb{R}^{n_c\times n}$, $M\in\mathbb{R}^{n_c\times n_c}$, and $n_c=\operatorname{rank}\hat{Q}\hat{P}$ will be called a *projective factorization* of $\hat{Q}\hat{P}$. Furthermore, define the complementary projection $\tau_\perp\triangleq I_n-\tau$. Optimizing (3.11) subject to (3.9) and (3.10) yields the following result illustrated in Fig. 1.

Theorem 3.1: Suppose $(A_c,B_c,C_c,\alpha,\alpha_c)$ solves the nonzero set point problem with $(A_c,B_c,C_c)\in\mathbb{S}''$. Then there exist $n\times n$ nonnegative-definite matrices Q, P, $\hat{Q}$, $\hat{P}$ such that, for some projective factorization (G, M, Γ) of $\hat{Q}\hat{P}$, A_c, B_c, C_c, α, and α_c are given by

$$A_c=\Gamma[A-B(L_2^TR_2L_2)^{-1}P_a-Q_aV_2^{-1}C+Q_aV_2^{-1}D(L_2^TR_2L_2)^{-1}P_a]G^T, \tag{3.15}$$

$$B_c=\Gamma Q_aV_2^{-1}, \tag{3.16}$$

$$C_c=-(L_2^TR_2L_2)^{-1}P_aG^T, \tag{3.17}$$

$$\begin{bmatrix}\alpha\\ \alpha_c\end{bmatrix}=\Omega^{-1}[(\tilde{R}_{12}-\tilde{B}^T\tilde{A}^{-T}\tilde{R})\tilde{A}^{-1}\tilde{\gamma}+(\tilde{N}-\tilde{B}^T\tilde{A}^{-T}\tilde{S})\tilde{\delta}] \tag{3.18}$$

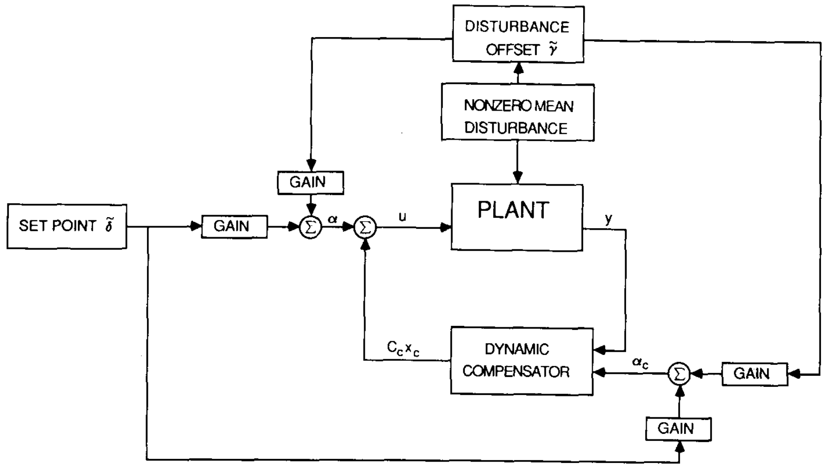

Fig. 1.

and such that Q, P, $\hat{Q}$, and $\hat{P}$ satisfy

$$0=AQ+QA^T+V_1-Q_aV_2^{-1}Q_a^T+\tau_\perp Q_aV_2^{-1}Q_a^T\tau_\perp^T, \tag{3.19}$$

$$0=A^TP+PA+L_1^TR_1L_1-P_a^T(L_2^TR_2L_2)^{-1}P_a + \tau_\perp^TP_a^T(L_2^TR_2L_2)^{-1}P_a\tau_\perp, \tag{3.20}$$

$$0=A_P\hat{Q}+\hat{Q}A_P^T+Q_aV_2^{-1}Q_a^T-\tau_\perp Q_aV_2^{-1}Q_a^T\tau_\perp^T, \tag{3.21}$$

$$0=A_Q^T\hat{P}+\hat{P}A_Q+P_a^T(L_2^TR_2L_2)^{-1}P_a-\tau_\perp^TP_a^T(L_2^TR_2L_2)^{-1}P_a\tau_\perp, \tag{3.22}$$

$$\operatorname{rank}\hat{Q}=\operatorname{rank}\hat{P}=\operatorname{rank}\hat{Q}\hat{P}=n_c. \tag{3.23}$$

Proof: See Section IV. □

Remark 3.2: The results of [6] are a special case of Theorem 3.1. To see this set $\delta_1=\gamma_1=0$, $\delta_2=0$, $\gamma_2=0$, $L_1=I_n$, and $L_2=I_m$, which yields the results of [6] with the added features of correlated plant/measurement noise (V_{12}), cross weighting (R_{12}), and a direct transmission term (D) in the plant dynamics.

As discussed in [6], in the full-order (LQG) case $n_c=n$ the Lyapunov equations (3.21) and (3.22) for $\hat{Q}$ and $\hat{P}$ are superfluous. In this case $G=\Gamma^{-1}$ and thus $G=\Gamma=\tau=I_n$ without loss of generality. To develop further connections with standard LQG theory, assume

$$L_1=I_n,\ L_2=I_m,\ R_{12}=0,\ V_{12}=0 \tag{3.24}$$

and define

$$\mathfrak{R}\triangleq\begin{bmatrix}R_1 & 0\\ 0 & C_c^TR_2C_c\end{bmatrix},\ \mathfrak{R}_1\triangleq\begin{bmatrix}R_1 & 0\\ 0 & 0\end{bmatrix},$$

$$\mathfrak{R}_{12}\triangleq\begin{bmatrix}0 & R_2C_c\\ 0 & 0\end{bmatrix},\ \mathfrak{R}_2\triangleq\begin{bmatrix}R_2 & 0\\ 0 & 0\end{bmatrix},$$

$$\mathfrak{N}\triangleq\begin{bmatrix}0 & R_2\\ 0 & 0\end{bmatrix},\ \mathcal{S}\triangleq\begin{bmatrix}R_1 & 0\\ 0 & C_c^TR_2\end{bmatrix}.$$

In this case $\mathcal{S}''$ becomes

$$\hat{\mathcal{S}}''\triangleq\{(A_c,B_c,C_c)\in\mathcal{S}':\hat{\Omega}>0\}$$

where

$$\hat{\Omega}\triangleq\tilde{B}^T\tilde{A}^{-T}\mathfrak{R}_1\tilde{A}^{-1}\tilde{B}+(\mathfrak{R}_{12}\tilde{A}^{-1}\tilde{B}-\mathfrak{R}_{12})^T\mathfrak{R}_2^{\#}(\mathfrak{R}_{12}\tilde{A}^{-1}\tilde{B}-\mathfrak{R}_{12}).$$

Corollary 3.1: Let $n_c=n$, assume (3.24) is satisfied, and suppose $(A_c, B_c, C_c, \alpha, \alpha_c)$ solves the full-order nonzero set point problem with $(A_c, B_c, C_c)\in\hat{\mathcal{S}}''$. Then there exist $n\times n$ nonnegative-definite matrices Q, P such that A_c, B_c, C_c, α, and α_c are given by

$$A_c=A-BR_2^{-1}B^TP-QC^TV_2^{-1}C+QC^TV_2^{-1}DR_2^{-1}B^TP,$$

$$B_c=QC^TV_2^{-1},$$

$$C_c=-R_2^{-1}B^TP,$$

$$\begin{bmatrix}\alpha\\ \alpha_c\end{bmatrix}=\hat{\Omega}^{-1}[(\mathfrak{R}_{12}-\tilde{B}^T\tilde{A}^{-T}\mathfrak{R})\tilde{A}^{-1}\tilde{\gamma}+(\mathfrak{N}-\tilde{B}^T\tilde{A}^{-T}\mathcal{S})\tilde{\delta}]$$

and such that Q, P satisfy

$$0=AQ+QA^T+V_1-QC^TV_2^{-1}CQ,$$

$$0=A^TP+PA+R_1-PBR_2^{-1}B^TP.$$

Remark 3.3: Note that by setting $\delta_1=\gamma_1=0$, $\delta_2=0$, $\gamma_2=0$, and $D=0$, Corollary 3.1 yields the standard LQG result.

Remark 3.4: It is easy to see that in the full-order case $n_c=n$ a solution to the nonzero set point problem exists as long as $\hat{\Omega}$ is positive definite. In the reduced-order case, however, the situation is more complex. For details, see [8].

IV. Proof of Theorem 3.1

To optimize (3.11) over the open set $\mathcal{S}''$ subject to the constraints (3.9) and (3.10), form the Lagrangian

$$\mathcal{L}(A_c, B_c, C_c, \alpha, \alpha_c)\triangleq\operatorname{tr}\{\lambda_0J(A_c,B_c,C_c,\alpha,\alpha_c)+(\tilde{A}\tilde{Q}+\tilde{Q}\tilde{A}^T+\tilde{V})\tilde{P}+\tilde{\lambda}^T(\tilde{A}\tilde{m}+\tilde{B}\tilde{\alpha}+\tilde{\gamma})\}$$

where the Lagrange multipliers $\lambda_0\geq 0$, $\tilde{\lambda}\in\mathbb{R}^{\tilde{n}}$, and $\tilde{P}\in\mathbb{R}^{\tilde{n}\times\tilde{n}}$ are not all zero. Setting $\partial\mathcal{L}/\partial\tilde{Q}=0$ and using the fact that $\tilde{A}$ is asymptotically stable, it follows that $\lambda_0=1$ without loss of generality.

Now partition $\tilde{n}\times\tilde{n}$ $\tilde{Q}$, $\tilde{P}$ into $n\times n$, $n\times n_c$, $n_c\times n_c$ subblocks and $\tilde{\lambda}\in\mathbb{R}^{\tilde{n}}$ into $\mathbb{R}^n$ and $\mathbb{R}^{n_c}$ components as

$$\tilde{Q}=\begin{bmatrix}Q_1 & Q_{12}\\ Q_{12}^T & Q_2\end{bmatrix},\ \tilde{P}=\begin{bmatrix}P_1 & P_{12}\\ P_{12}^T & P_2\end{bmatrix},\ \tilde{\lambda}=\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}.$$

Thus, the stationarity conditions are given by

$$\frac{\partial\mathcal{L}}{\partial\tilde{P}}=\tilde{A}\tilde{Q}+\tilde{Q}\tilde{A}^T+\tilde{V}=0, \tag{4.1}$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{Q}}=\tilde{A}^T\tilde{P}+\tilde{P}\tilde{A}+\tilde{R}=0, \tag{4.2}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{m}} = \tilde{R}\tilde{m} + \frac{1}{2}\tilde{A}^T\tilde{\lambda} + \tilde{R}_{12}^T\tilde{\alpha} - \tilde{S}\tilde{\delta} = 0, \tag{4.3}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{\alpha}} = \begin{bmatrix} [L_2^TR_2L_2 \;\; 0]\tilde{\alpha} + [L_2^TR_{12}^TL_1 \;\; L_2^TR_2L_2C_c]\tilde{m} - [L_2^TR_{12} \;\; L_2^TR_2]\tilde{\delta} + \frac{1}{2}B^T\lambda_1 + \frac{1}{2}D^TB_c^T\lambda_2 \\ \frac{1}{2}\lambda_2 \end{bmatrix} = 0, \tag{4.4}$$

$$\frac{\partial \mathcal{L}}{\partial A_c} = P_{12}^TQ_{12} + P_2Q_2 + \frac{1}{2}\lambda_2 m_c^T = 0, \tag{4.5}$$

$$\frac{\partial \mathcal{L}}{\partial B_c} = P_{12}^TV_{12} + P_2B_cV_2 + (P_{12}^TQ_1 + P_2Q_{12}^T)C^T + (P_{12}^TQ_{12} + P_2Q_2)C_c^TD^T + \frac{1}{2}\lambda_2 m^TC^T + \frac{1}{2}\lambda_2\gamma_2^T + \frac{1}{2}\lambda_2\alpha^TD^T = 0, \tag{4.6}$$

$$\frac{\partial \mathcal{L}}{\partial C_c} = L_2^TR_{12}^TL_1Q_{12} + L_2^TR_2L_2C_cQ_2 + L_2^TR_{12}^TL_1mm_c^T + L_2^TR_2L_2C_cm_cm_c^T + B^T(P_1Q_{12} + P_{12}Q_2) + D^TB_c^T(P_{12}^TQ_{12} + P_2Q_2) - L_2^TR_{12}^T\delta_1m_c^T$$
$$+ L_2^TR_2L_2\alpha m_c^T - L_2^TR_2\delta_2m_c^T + \frac{1}{2}B^T\lambda_1m_c^T = 0. \tag{4.7}$$

Expanding (4.1) and (4.2) yields

$$0 = AQ_1 + Q_1A^T + V_1 + BC_cQ_{12}^T + Q_{12}C_c^TB^T, \tag{4.8}$$

$$0 = AQ_{12} + Q_{12}A_c^T + BC_cQ_2 + Q_1C^TB_c^T + V_{12}B_c^T + Q_{12}C_c^TD^TB_c^T, \tag{4.9}$$

$$0 = A_cQ_2 + Q_2A_c^T + B_cCQ_{12} + Q_{12}^TC^TB_c^T + B_cV_2B_c^T$$
$$+ B_cDC_cQ_2 + Q_2C_c^TD^TB_c^T, \tag{4.10}$$

$$0 = A^TP_1 + P_1A + L_1^TR_1L_1 + C^TB_c^TP_{12}^T + P_{12}B_cC, \tag{4.11}$$

$$0 = A^TP_{12} + P_{12}A_c + C^TB_c^TP_2 + P_1BC_c + L_1^TR_{12}L_2C_c + P_{12}B_cDC_c, \tag{4.12}$$

$$0 = A_c^TP_2 + P_2A_c + C_c^TB^TP_{12} + P_{12}^TBC_c + C_c^TL_2^TR_2L_2C_c$$
$$+ C_c^TD^TB_c^TP_2 + P_2B_cDC_c. \tag{4.13}$$

Next, note that (4.4) implies that $\lambda_2 = 0$, and thus (4.5) can be written as

$$-P_2^{-1}P_{12}^TQ_{12}Q_2^{-1} = I_{n_c}. \tag{4.14}$$

The existence of Q_2^{-1} and P_2^{-1} follows from the fact that (A_c, B_c, C_c) is minimal. See [6] for details. Now define the $n \times n$ matrices

$$Q \triangleq Q_1 - Q_{12}Q_2^{-1}Q_{12}^T, \; P \triangleq P_1 - P_{12}P_2^{-1}P_{12}^T,$$

$$\hat{Q} \triangleq Q_{12}Q_2^{-1}Q_{12}^T, \; \hat{P} \triangleq P_{12}P_2^{-1}P_{12}^T,$$

$$\tau \triangleq -Q_{12}Q_2^{-1}P_2^{-1}P_{12}^T$$

and the $n_c \times n$, $n_c \times n_c$, and $n_c \times n$ matrices

$$G \triangleq Q_2^{-1}Q_{12}^T, \quad M \triangleq Q_2P_2, \quad \Gamma \triangleq -P_2^{-1}P_{12}^T.$$

Note that $\tau = G^T\Gamma$. Clearly, Q, P, $\hat{Q}$, and $\hat{P}$ are symmetric and nonnegative definite.

Next note that with the above definitions, (4.14) is equivalent to (3.13) and that (3.12) holds. Hence, $\tau = G^T\Gamma$ is idempotent, i.e., $\tau^2 = \tau$. Sylvester's inequality yields (3.23). Note also that

$$\hat{Q} = \tau\hat{Q}, \quad \hat{P} = \hat{P}\tau.$$

The components of $\tilde{Q}$ and $\tilde{P}$ can be written in terms of Q, P, $\hat{Q}$, $\hat{P}$, G, and Γ as

$$Q_1 = Q + \hat{Q}, \quad P_1 = P + \hat{P},$$

$$Q_{12} = \hat{Q}\Gamma^T, \quad P_{12} = -\hat{P}G^T,$$

$$Q_2 = \Gamma\hat{Q}\Gamma^T, \quad P_2 = G\hat{P}G^T.$$

The expressions (3.16) and (3.17) follow from (4.6) and (4.7) by using the n_c and n components of (4.4), respectively, and the above identities. Next, computing either Γ(4.9)–(4.10) or G(4.12) + (4.13) yields (3.15). Substituting this expression for A_c into (4.8)–(4.13) it follows that (4.10) = Γ(4.9) and (4.13) = G(4.12). Thus, (4.10) and (4.13) are superfluous and can be omitted. Next, using (4.8) + $G^T\Gamma$(4.9)G − (4.9)G − [(4.9)G]^T and $G^T\Gamma$(4.9)G − (4.9)G − [(4.9)G]^T yields (3.19) and (3.21). Using (4.11) + Γ^TG(4.12)Γ − (4.12)Γ − [(4.12)Γ]^T and Γ^TG(4.12)Γ − (4.12)Γ − [(4.12)Γ]^T yields (3.20) and (3.22).

To obtain (3.18) note that (4.4) can be rewritten as

$$\tilde{R}_2\tilde{\alpha} + \tilde{R}_{12}\tilde{m} - \tilde{N}\tilde{\delta} + \frac{1}{2}\tilde{B}^T\tilde{\lambda} = 0. \tag{4.15}$$

Next, note that (4.3) is equivalent to

$$\frac{1}{2}\tilde{\lambda} = -\tilde{A}^{-T}\tilde{R}\tilde{m} + \tilde{A}^{-T}\tilde{S}\tilde{\delta} - \tilde{A}^{-T}\tilde{R}_{12}^T\tilde{\alpha}. \tag{4.16}$$

Substituting (4.16) into (4.15) yields

$$(\tilde{R}_2 - \tilde{B}^T\tilde{A}^{-T}\tilde{R}_{12}^T)\tilde{\alpha} + (\tilde{R}_{12} - \tilde{B}^T\tilde{A}^{-T}\tilde{R})\tilde{m} + (\tilde{B}^T\tilde{A}^{-T}\tilde{S} - \tilde{N})\tilde{\delta} = 0. \tag{4.17}$$

Next, note that (3.10) is equivalent to

$$\tilde{m} = -\tilde{A}^{-1}\tilde{B}\tilde{\alpha} - \tilde{A}^{-1}\tilde{\gamma}. \tag{4.18}$$

Now, substituting (4.18) into (4.17) yields

$$(\tilde{R}_2 - \tilde{R}_{12}\tilde{A}^{-1}\tilde{B} - \tilde{B}^T\tilde{A}^{-T}\tilde{R}_{12}^T + \tilde{B}^T\tilde{A}^{-T}\tilde{R}\tilde{A}^{-1}\tilde{B})\tilde{\alpha} = (\tilde{N} - \tilde{B}^T\tilde{A}^{-T}\tilde{S})\tilde{\delta}$$
$$+ (\tilde{R}_{12} - \tilde{B}^T\tilde{A}^{-T}\tilde{R})\tilde{A}^{-1}\tilde{\gamma}. \tag{4.20}$$

Finally, note that the coefficient of $\tilde{\alpha}$ in (4.20) is equivalent to Ω and thus (4.20) yields (3.18). □

V. Concluding Remarks

The results of the present note can be combined with the results of [1] to obtain nonstrictly proper controllers leading to a generalization of [7]. Current research is focused on extending the results of the present note to larger classes of command and disturbance signals.

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A Study of Controllability and Time-Optimal Control of a Robot Model with Drive Train Compliances and Actuator Dynamics

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Abstract—**The problems of robot controllability and time-optimal control where drive train compliances and actuator dynamics are incorporated in the mathematical model is the subject of this note. This study demonstrates the conditions that ensure the existence of a time-optimal control, and establishes controllability of the augmented model (robot and actuator) in open- and closed-loop form. This note describes a procedure for the derivation of easily computable functional inequalities which represent upper bounds on the norm of the augmented system's time response.**

I. INTRODUCTION

To obtain the control strategy of mechanical manipulators, various control schemes are presented in the available literature. A few examples are resolved control [1], inverse problems technique [2], and resolved acceleration control [3]. In most cases, the control scheme involves the computation of the appropriate generalized forces by the equation

$$H(\theta)\theta^{(2)}+K(\theta,\dot\theta)+R(\theta)=q$$

where θ and q are the vectors of the generalized coordinates and forces, respectively, H is the moment of inertia matrix, K is a vector specifying centrifugal and Coriolis effects, and R is a vector specifying gravitational effects.

In much of the literature the actuators providing the drive torques are modeled as pure torque sources. However, this approach is in most cases a simplification of the realistic models of the system [4]–[8].

The objective of this note is to study controllability and to investigate the conditions which ensure the existence of a control function that transfers the augmented model of the mechanical system, the actuator's dynamics, and the drive train's compliances, from a given initial position to a desired target in a minimum time. The model and the approach are useful for the design of a linear controller and can be used as a point of departure for a more general model of a robot arm.

II. THE MATHEMATICAL MODEL

The Lagrange formulation of a multilink mechanical system is given by

$$d(\partial L/\partial\dot\theta_i)/dt-\partial L/\partial\theta_i=q_i, \qquad i=1,2\cdots n \tag{1}$$

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where $L = T - V$. T and V are the kinetic and potential energies of the system, respectively.

Let p_i be the ith generalized momentum [9]. Using Legender's dual transformation

$$p_i=\partial L/\partial\dot\theta_i, \qquad i=1,2,\cdots n. \tag{2}$$

Since L is a quadratic function in $\dot\theta_1$, p is linear in $\dot\theta$ for any given θ, i.e.,

$$p_i=\sum_{j=1}^{n} a_{ij}(\theta)\dot\theta_j, \qquad i=1,2,\cdots n \tag{3}$$

with $a_{ij}(\theta) = \partial^2L/\partial\dot\theta_i\partial\dot\theta_j$.

The inertial matrix is $H = [\partial^2L/\partial\dot\theta_i\partial\dot\theta_j]_{n\times n}$ with det $(H) = h(\theta) > 0$, $\forall\theta$, where det $(\cdot)$ is the determinant of $(\cdot)$. Now, from (1), (2), and (3) we have

$$\dot p_i=\partial L/\partial\theta_i+q_i, \qquad i=1,2,\cdots n, \tag{4}$$

$$\dot\theta_i=\sum_j b_{ij}(\theta)p_j, \qquad i=1,2,\cdots,n. \tag{5}$$

Using (5) one obtains

$$\partial L/\partial\theta_s=\left[\sum_{j=1}^{n}\sum_{i=1}^{n} c_{sij}(\theta)p_ip_j\right]/(\det(H))^2. \tag{6}$$

Equations (4)–(6) constitute the state equations of the n-link mechanical system which can be written as

$$\dot z(t)=F(z(t))+Bq(t),\ z(t_0)=z_0 \tag{7}$$

where the vectors $z = [p^T\theta^T]^T$, $q = [q_1q_2\cdots q_n]^T$, and $F = [F_1F_2\cdots F_{2n}]^T$ are in Euclidean vector space with the usual norm $\|z\|^2 = \Sigma_{i=1}^{2n}(z_i)^2$. We also have

$$F_s=\left[\sum_{j=1}^{n}\sum_{i=1}^{n} c_{sij}p_ip_j\right]/[\det(H)]^2, \qquad s=1,2,\cdots,n$$

$$=\left[\sum_{i=1}^{n} d_{si}p_i\right]/\det(H), \qquad s=n+1,n+2,\cdots,2n$$

and $B = [{}^{I}_{0}]$, where I is the $n\times n$ identity matrix.

As an example, the exact equations for the two-link mechanical system which is confined to move in the vertical plane are given by

$$\dot p_1=[p_1p_2l_1l_{c2}m_2\sin(\theta_2-\theta_1)\det(H)-[0.5p_1^2I_2+0.5p_2^2(I_1+m_2l_1^2)$$
$$-p_1p_2E]2El_1l_{c2}m_2\sin(\theta_2-\theta_1)]/(\det(H))^2$$
$$-(m_1gl_{c1}+m_2gl_1)\sin\theta_1+q_1=F_1(p_1,p_2,\theta_1,\theta_2)+q_1$$

$$\dot p_2=-[p_1p_2l_1l_{c2}m_2\sin(\theta_2-\theta_1)\det(H)-[0.5p_1^2I_2+0.5p_2^2(I_1+m_2l_1^2)$$
$$-p_1p_2E]2El_1l_{c2}m_2\sin(\theta_2-\theta_1)]/(\det(H))^2-m_2gl_{c2}\sin\theta_2+q_2$$
$$=F_2(p_1,p_2,\theta_1,\theta_2)+q_2$$

$$\dot\theta_1=(p_1I_2-p_2E)/\det(H)=F_3(p_1,p_2,\theta_1,\theta_2)$$

$$\dot\theta_2=(p_2(I_1+m_2l_1^2)-p_1E)/\det(H)=F_4(p_1,p_2,\theta_1,\theta_2) \tag{8}$$

where $E = l_1l_{c2}m_2\cos(\theta_2-\theta_1)$, m_i and l_i are the mass and the length of the ith link, respectively, I_i is the moment of inertia of the ith link with respect to the ith joint, and l_{ci} is the distance from the ith joint to the center of gravity of the ith link.

The term det (H) is a trigonometric function of, and periodical in, θ_i. This function attains its minimum in the interval $0 \le \theta_i \le 2\pi$, $i = 1, 2, \cdots n$, and therefore

$$\det(H)\ge k>0, \qquad \forall z\in R^{2n}. \tag{9}$$

We turn now to the dynamics of the robot's drivers. The robot is