84-1035

THE OPTIMAL PROJECTION EQUATIONS FOR FIXED-ORDER DYNAMIC COMPENSATION OF DISTRIBUTED-PARAMETER SYSTEMS*

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Abstract

One of the major difficulties in designing implementable active controllers for distributed parameter systems such as flexible space structures is that such systems are inherently infinite dimensional while controller dimension is severely constrained by on-line computing capability. Suboptimal approaches to this problem usually either seek a distributed parameter control law or design a low-order dynamic controller for an approximate high-order finite-element model. This paper presents a more direct approach by deriving explicit optimality conditions for finite-dimensional steady-state fixed-order dynamic compensation of infinite-dimensional systems. In contrast to the pair of operator Riccati equations for the "full-order" LQG case, the optimal fixed-order dynamic compensator is characterized by *four* operator equations (two modified Riccati equations and two modified Lyapunov equations) coupled by a projection whose rank is precisely equal to the order of the compensator and which determines the optimal compensator gains. The coupling represents a graphic portrayal of the demise of the classical separation principle for the reduced-order controller case. The results obtained apply to a semigroup formulation in Hilbert space and thus are applicable to control problems involving a broad range of specific partial and hereditary differential equations.

1. Introduction

Numerous techniques have been proposed for the problem of designing an optimal finite-dimensional fixed-order dynamic compensator for an infinite-dimensional system. Generally speaking, most of these methods can be divided into two main categories. The first category, largely associated with the engineering literature, consists of methods that first replace the infinite-dimensional system with a discretized and truncated model and then seek a relatively low-order controller based upon the approximate model. A survey of design techniques proposed for this latter step can be found in Ref. 2; see also Ref. 3. Methods of the second category, associated with the mathematical literature, initially seek a control law for the infinite-dimensional system of a correspondingly infinite-dimensional nature.[1,9,10] Practical implementation in this case requires subsequent approximation by a finite-dimensional controller.[11]

A more direct approach[18,23] is to both retain the infinite-dimensional model and fix the order of the finite-dimensional compensator. Although this idea is conceptually the most appealing, progress in this direction has undoubtedly been impeded by the lack of optimality conditions such as are available for the infinite-dimensional controller case, i.e., the operator Riccati equations. The purpose of this paper is to make significant progress in filling this gap by presenting new, explicit conditions for characterizing the optimal finite-dimensional fixed-order dynamic compensator for an infinite-dimensional system. In contrast to the pair of operator Riccati equations for the LQG case, the optimal steady-state fixed-order dynamic compensator is characterized by four coupled operator equations (two modified Riccati equations and two modified Lyapunov equations). This coupling, by means of a projection (idempotent) operator whose rank is precisely equal to the order of the compensator, represents a graphic portrayal of the demise of the classical separation principle for the finite-dimensional reduced-order controller case. The optimal gains and compensator dynamics matrix are determined by the solutions of the modified Riccati and Lyapunov equations and by a factorization of the product of the solutions of the pair of modified Lyapunov equations. Considerable insight into the compensator structure is obtained since the projection operator determines control and observation subspaces. Because of the use of a projection in the form of a state-truncation operation in related model-reduction schemes,[21] these equations have been termed the "optimal projection equations".[17] In this regard it is briefly pointed out in this paper that the mathematical steps involved in characterizing the projection are analogous to the model-reduction method of Ref. 21. An in-depth investigation into this topic is reserved for Ref. 17.

It should be stressed that an important problem which is beyond the scope of the present paper is stabilizability, i.e., the existence of a dynamic compensator of a given order such that the closed-loop system is stable. Our approach is to assume that the set of stabilizing compensators is nonempty and then characterize the optimal compensator should it exist. We note that stabilizing compensators do exist for the class of problems considered in Refs. 4, 8 and 27.

It is important to point out that the results of this paper can be immediately adapted to finite-dimensional systems. One need only specialize the Hilbert space characterizing the dynamical system to a finite-dimensional Euclidean space. Then all "dense domain" considerations can be ignored, adjoints can be interpreted as transposes and other obvious simplifications can be invoked. The only mathematical aspect requiring attention is the treatment of white noise which, for convenient handling of the infinite-dimensional case, is interpreted according to Ref. 1. For the finite-dimensional case, however, the standard classical notions suffice and the results go through with virtually no modifications. The finite-dimensional case has been discussed in Refs. 14-16. Proofs of the results in the present paper can be found in Refs. 5 and 6.

* This work was sponsored by the Dept. of the Air Force. The U.S. Government assumes no responsibility for the information presented.

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2. Preliminaries and Problem Statement

Let H and H' denote real separable Hilbert spaces with inner product $<\cdot,\cdot>$ and let $B(H,H')$, $B_1(H,H')$ and $B_2(H,H')$ denote, respectively, the spaces of bounded, trace class and Hilbert-Schmidt operators from H into H'.[1,13,19,25] If $H = H'$ then write $B(H) \triangleq B(H,H')$, etc. The adjoint of $L \in B(H,H')$ is L^*, $L^{-*} \triangleq (L^*)^{-1}$ and $\rho(L)$ denotes the rank of L. $L \in B(H)$ is nonnegative definite if $L = L^*$ and $<Lx,x> \geq 0$, $x \in H$. With respect to fixed orthonormal bases in Euclidean spaces we identify $\mathbb{R}^{m\times n} = B(\mathbb{R}^n, \mathbb{R}^m)$. The transposes of $x \in \mathbb{R}^n$ and $M \in \mathbb{R}^{m\times n}$ are denoted by x^T and M^T and $M^{-T} \triangleq (M^T)^{-1}$. I_n is the n×n identity matrix and I_H is the identity operator on H.

We consider the following steady-state fixed-order dynamic-compensation problem. Given the control system

$$\dot{x}(t) = Ax(t) + Bu(t) + H_1 w(t)$$

$$y(t) = Cx(t) + H_2 w(t),$$

design a fixed-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t),$$

$$u(t) = C_c x_c(t)$$

to minimize the performance criterion

$$J(A_c,B_c,C_c) \triangleq \lim_{t_f\to\infty} \mathbb{E}[<R_1 x(t),x(t)> + u(t)^T R_2 u(t)].$$

The following data are assumed. The state x is an element of a real separable Hilbert space H and the state differential equation is interpreted in the weak sense (see, e.g., Ref. 1, pp. 229, 317). The closed, densely defined operator A: $D(A) \subset H \to H$ generates a strongly continuous semigroup e^{At}, $t \geq 0$. The control $u \in \mathbb{R}^m$, $B \in B(\mathbb{R}^m,H)$ and the operator $R_1 \in B_1(H)$ and the matrix $R_2 \in \mathbb{R}^{m\times m}$ are nonnegative definite and positive definite, respectively. $w(\cdot)$ is a "standard white noise process" in $L_2((0,\infty),H')$ (see Ref. 1, p. 314), where H' is a real separable Hilbert space, $H_1 \in B_2(H',H)$, $H_2 \in B(H,\mathbb{R}^\ell)$ and "$\mathbb{E}$" denotes expectation. We assume that $H_1 H_2^* = 0$, i.e., the disturbance and measurement noises are independent, and that $V_2 \triangleq H_2 H_2^*$ is positive definite, i.e., all measurements are noisy. Note that $V_1 \triangleq H_1 H_1^*$ is trace class. The observation $y \in \mathbb{R}^\ell$ and $C \in B(H,\mathbb{R}^\ell)$. The dimension of the compensator state x_c is of fixed order n_c and the optimization is performed over the matrices A_c, B_c and C_c. Under these and the following assumptions, J is independent of x(0) and q(0).

In order to guarantee the existence of $J(A_c,B_c,C_c)$ we confine (A_c,B_c,C_c) to the set of stabilizing compensators

$$A \triangleq \{(A_c,B_c,C_c): e^{\tilde{A}t} \text{ is exponentially stable}\},$$

where

$$\tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}$$

is a closed, densely defined operator on $D(\tilde{A}) \triangleq D(A) \times \mathbb{R}^{n_c} \subset \tilde{H}$ and $\tilde{H} \triangleq H \oplus \mathbb{R}^{n_c}$ is a real separable Hilbert space with inner product $<\tilde{x}_1,\tilde{x}_2> = <x_1,x_2> + x_{c1}^T x_{c2}$, $\tilde{x}_i \triangleq (x_i,x_{ci})$. Since the value of J is independent of the internal realization of the compensator, we can further restrict (A_c,B_c,C_c) to

$$A_+ \triangleq \{(A_c,B_c,C_c) \in A: (A_c,B_c) \text{ is controllable and } (C_c,A_c) \text{ is observable}\}.$$

3. Characterization of the Optimal Projection and Symmetrized Equations

In order to state our main result we require a factorization lemma (Lemma 3.3) concerning the product of two finite-rank nonnegative-definite operators. Since the existence of such a factorization is crucial to Theorem 3.1, we first discuss simultaneous diagonalization of pairs of matrices and then generalize to the case of finite-rank operators. It should be noted that since H is a real Hilbert space we restrict our attention to matrices with real entries.

Let $U \in \mathbb{R}^{n\times n}$. We shall say U is positive (resp., nonnegative) diagonal if U is diagonal with positive (resp., nonnegative) diagonal elements. U is semisimple (Ref. 24, p. 13), or nondefective (Ref. 22, p. 375), if U has n linearly independent eigenvectors (i.e., U has a diagonal Jordan canonical form over the complex field). Call U real (resp., positive, nonnegative) semisimple if U is semisimple with real (resp., positive, nonnegative) eigenvalues. Note that U is real (resp., positive, nonnegative) semisimple if and only if there exists n×n invertible Φ such that $\Phi U \Phi^{-1}$ is diagonal (resp., positive diagonal, nonnegative diagonal).

The following terminology concerns simultaneous diagonalization.[24] Let n×n U, V be symmetric matrices. Then U and V are cogrediently diagonalizable if there exists n×n invertible Φ such that both $\Phi U \Phi^T$ and $\Phi V \Phi^T$ are diagonal. U and V are contragrediently diagonalizable if there exists n×n invertible Φ such that both $\Phi U \Phi^T$ and $\Phi^{-T} V \Phi^{-1}$ are diagonal. Since these two situations coincide when Φ is orthogonal, we shall say in this case that U and V are orthogonally diagonalizable.

The following lemma gives sufficient conditions under which symmetric U, V are cogrediently and contragrediently diagonalizable. Although this result goes beyond our needs, it serves the useful purpose of bringing together related results from the literature and hence places in perspective the results we actually require (see Ref. 22, p. 428 and Ref. 24, pp. 122-123).

Lemma 3.1. Suppose that U, $V \in \mathbb{R}^{n\times n}$ are symmetric. Then if either i) one of U and V is positive definite or ii) both U and V are nonnegative definite, then U and V are cogrediently and contragrediently diagonalizable.

Corollary 3.1. Suppose U, $V \in \mathbb{R}^{n\times n}$ are nonnegative definite. Then UV is nonnegative semisimple.

In generalizing the preceding results to the case in which U and V are finite-rank selfadjoint operators on H, we shall make use of the (infinite-) matrix representation of an operator with respect to an orthonormal basis. Note that all matrix representations given here will consist of real entries since the Hilbert spaces are real. Also, recall that every selfadjoint operator has a diagonal matrix representation with respect to some orthonormal basis.

Since orthogonal transformations correspond to a change in orthonormal basis, let us say, in analogy to the matrix case, that U, V $\in B(H)$ are orthogonally diagonalizable if there exists an orthonormal basis for H with respect to which both U and V have diagonal matrix representations (Ref. 12, p. 181). Also in analogy to the finite-dimensional case, call U $\in B(H)$ semisimple (resp., real semisimple, nonnegative semisimple) if there exists invertible $L \in B(H)$ such that LUL^{-1} is normal (resp., selfadjoint, nonnegative definite). This implies that LUL^{-1} has a complete set of orthonormal eigenvectors and, in the real-semisimple or nonnegative-semisimple cases, has real or nonnegative eigenvalues. Furthermore, we shall say that self-adjoint $\hat{Q}$, $\hat{P} \in B(H)$ are contragrediently diagonalizable if there exists invertible $L \in B(H)$ such that $L\hat{Q}L^*$ and $L^{-*}\hat{P}L^{-1}$ are orthogonally diagonalizable. Cogredient diagonalization is not needed and hence will not be discussed in the sequel. The next result is based upon Lemma 3.1 and upon Theorem 2.1, p. 240 of Ref. 12.

Lemma 3.2. Suppose $\hat{Q}$, $\hat{P} \in B(H)$ have finite rank and are nonnegative definite. Then $\hat{Q}$ and $\hat{P}$ are contragrediently diagonalizable.

We now have the following generalization of Corollary 3.1.

Corollary 3.2. Suppose $\hat{Q}$, $\hat{P} \in B(H)$ have finite rank and are nonnegative definite. Then $\hat{Q}\hat{P}$ is nonnegative semisimple.

The next result is a straightforward consequence of Corollary 3.2.

Lemma 3.3. Suppose $\hat{Q}$, $\hat{P} \in B(H)$ have finite rank, are nonnegative definite and $\rho(\hat{Q}\hat{P}) = n_c$. Then there exist G, $\Gamma \in B(H, \mathbb{R}^{n_c})$ and $n_c \times n_c$ positive-semisimple M such that

$$\hat{Q}\hat{P} = G^*M\Gamma, \tag{3.1}$$

$$\Gamma G^* = I_{n_c}. \tag{3.2}$$

We shall refer to G, $\Gamma \in B(H, \mathbb{R}^{n_c})$ and $n_c \times n_c$ positive-semisimple M satisfying (3.1) and (3.2) as a (G,M,Γ)-factorization of $\hat{Q}\hat{P}$. Also, define the notation

$$\tau_\perp \triangleq I_H - \tau$$

and

$$\Sigma \triangleq BR_2^{-1}B^*, \quad \bar{\Sigma} \triangleq C^*V_2^{-1}C.$$

Main Theorem. Suppose $(A_c, B_c, C_c) \in A_+$ solves the steady-state fixed-order dynamic-compensation problem. Then there exist nonnegative-definite Q, P, $\hat{Q}$, $\hat{P} \in B_1(H)$ such that A_c, B_c and C_c are given by

$$A_c = \Gamma(A - Q\bar{\Sigma} - \Sigma P)G^*, \tag{3.3}$$

$$B_c = \Gamma QC^*V_2^{-1}, \tag{3.4}$$

$$C_c = -R_2^{-1}B^*PG^*, \tag{3.5}$$

for some (G,M,Γ)-factorization of $\hat{Q}\hat{P}$, and such that with $\tau \triangleq G^*\Gamma$ the following conditions are satisfied:

$$\rho(\hat{Q}) = \rho(\hat{P}) = \rho(\hat{Q}\hat{P}) = n_c, \tag{3.6}$$

$$Q: D(A^*) \to D(A), \quad P: D(A) \to D(A^*), \tag{3.7}$$

$$\hat{Q}: H \to D(A), \quad \hat{P}: H \to D(A^*), \tag{3.8}$$

$$0 = AQ + QA^* + V_1 - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^*, \tag{3.9}$$

$$0 = A^*P + PA + R_1 - P\Sigma P + \tau_\perp^* P\Sigma P\tau_\perp, \tag{3.10}$$

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^* + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^*, \tag{3.11}$$

$$0 = (A - Q\bar{\Sigma})^*\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P - \tau_\perp^* P\Sigma P\tau_\perp. \tag{3.12}$$

Remark 3.1. When H is finite dimensional and $n_c = \dim H$ (i.e., the full-order case), the (G,M,Γ)-factorization of $\hat{Q}\hat{P}$ is given by $G = \Gamma = I_H$ and $M = \hat{Q}\hat{P}$. Since $\tau = I_H$, and thus $\tau_\perp = 0$, (3.9) and (3.10) yield the familiar Riccati equations.

Remark 3.2. Replacing x_c by Sx_c, where S is invertible, yields the "equivalent" compensator $(SA_cS^{-1}, SB_c, C_cS^{-1})$. Since $J(A_c,B_c,C_c) = J(SA_cS^{-1}, SB_c, C_cS^{-1})$ one would expect the Main Theorem to apply also to $(SA_cS^{-1}, SB_c, C_cS^{-1})$. This is indeed the case since transformation of the compensator state basis corresponds to the alternative factorization

$$\hat{Q}\hat{P} = (S^{-T}G)^T(SMS^{-1})(S\Gamma).$$

Next we give an alternative characterization of the optimal projection τ by demonstrating how it can be expressed in terms of the Drazin pseudoinverse of $\hat{Q}\hat{P}$. Since $\hat{Q}\hat{P}$ has finite rank, its Drazin inverse exists (see Theorem 6, p. 108 of Ref. 20). Since $(\hat{Q}\hat{P})^2 = G^*M^2\Gamma$, and hence $\rho(\hat{Q}\hat{P})^2 = \rho(\hat{Q}\hat{P})$, the "index"[7,20] of $\hat{Q}\hat{P}$ is 1. In this case the Drazin inverse is traditionally called the group inverse and is denoted by $(\hat{Q}\hat{P})^{\#}$ (see, e.g., Ref. 7, p. 124, or Ref. 26).

Proposition 3.1. Let $\hat{Q}$, $\hat{P}$ and τ be as in Theorem 3.1. Then

$$\tau = \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#}.$$

Proof. It is easy to verify that the conditions characterizing the Drazin inverse[20] for the case that $\hat{Q}\hat{P}$ has index 1 are satisfied by $G^*M^{-1}\Gamma$. Hence $(\hat{Q}\hat{P})^{\#} = G^*M^{-1}\Gamma$ and (3.2) yields the desired result. □

The next result is useful in making connections with Ref. 21.

Proposition 3.2. Suppose $\hat{Q}$, $\hat{P} \in B(H)$ and G, $\Gamma \in B(H, \mathbb{R}^{n_c})$ and $n_c \times n_c$ real-semisimple M satisfy (3.1) and (3.2). Then there exists invertible $L \in B(H)$ and an orthonormal basis for H with respect to which

$$\hat{Q}\hat{P} = L^{-1}\begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} L, \tag{3.13}$$

$$G^*\Gamma = L^{-1}\begin{bmatrix} I_{n_c} & 0 \\ 0 & 0 \end{bmatrix} L, \tag{3.14}$$

where $\Lambda \triangleq \operatorname{diag}(\lambda_1,\ldots,\lambda_{n_c})$ and $\lambda_1,\ldots,\lambda_{n_c}$ are the eigenvalues of M.

We can now point out some interesting similarities between the technique used to obtain τ from $\hat{Q}\hat{P}$ and certain methods appearing in the model-reduction literature. In Ref. 21, for example, the positive-definite controllability and observability gramians,

$$W_c \triangleq \int_0^\infty e^{At}BB^Te^{A^Tt}dt, \quad W_o \triangleq \int_0^\infty e^{A^Tt}C^TCe^{At}dt,$$

are contragrediently diagonalized, i.e., Φ is chosen so that $\Phi^{-1}W_c\Phi^{-T}$ and $\Phi^TW_o\Phi$ are both positive diagonal. If Φ is chosen so that these matrices are also equal, the resulting model is said to be "internally balanced". The magnitudes of the diagonal components are then used as a guide for determining a suitable reduced-order model. Specifically, the order of the reduced model is chosen to be the number of "large" eigenvalues in the product of the Gramians and the reduced model is obtained by applying the projection

$$\begin{bmatrix} I_{n_c} & 0 \\ 0 & 0 \end{bmatrix}$$

in the transformed ("balanced") basis. Note that Proposition 3.2 shows that our "optimal" projection τ is indeed of this form in the basis with respect to which $\hat{Q}\hat{P}$ is diagonal (and which may very well be different from the balanced coordinates). Hence one would suspect that $\hat{Q}$ and $\hat{P}$ are somehow analogous to W_c and W_o. Indeed, since the order of the reduced model is <u>chosen</u> such that (3.6) is, in a sense, approximately satisfied, it is not surprising that in the optimal model-reduction problem, W_c and W_o can be shown[17] to be approximations to $\hat{Q}$ and $\hat{P}$.

Since Theorem 3.1 applies to closed-loop dynamic compensation with quadratic optimization, further comparison with the model-reduction literature is not feasible. It is important to point out, however, that because of the demise of the separation principle as graphically portrayed by the presence of τ in all four equations (3.8)-(3.12), it should not be expected that either an LQG design for a reduced-order model or a reduced-order LQG controller would correspond to an <u>optimal</u> fixed-order dynamic compensator as characterized by Theorem 3.1.

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