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Fixed-structure synthesis of induced-norm controllers

R. SCOTT ERWIN†¶, DENNIS S. BERNSTEIN‡ and DAVID A. WILSON§

This paper proposes a fixed-structure technique for synthesizing controllers that are optimal with respect to various operator norms. An optimal control problem is developed for each of these operator norms, and necessary conditions for sub-optimal performance are derived. Mixed-norm optimal control problems are also formulated. A continuation algorithm using quasi-Newton corrections is used to compute approximate solutions of the necessary conditions for a sequence of problems whose solutions approach an optimal controller. Optimal controllers with respect to each of the operator norms are synthesized for a 4th-order mass-spring-dashpot system.

1. Introduction

While the $\mathcal{H}_2$ system norm is a convenient and widely used criterion for optimal controller design, it is not an induced operator norm relating classes of input and output signals. Therefore, the $\mathcal{H}_2$ norm may not be an appropriate objective function for certain performance specifications and classes of disturbances. In Wilson (1989), expressions were developed for convolution operator norms induced by various combinations of input/output signal norms. An alternative characterization of system input–output properties is the Hankel operator, which provides a mapping from past inputs to future outputs (Glover 1984).

One practical application of induced norms to the design of control systems is the problem of actuator saturation. The induced norm from $\mathcal{L}_2$ to $\mathcal{L}_\infty$ provides a system gain between energy- and peak-excursion-type signal norms. Hence, if a bound is known for the energy of the disturbance applied to systems at rest (zero initial conditions), this induced norm can be used to bound state amplitude or actuator excursion (saturation).

Controller synthesis using induced norms for systems as performance measures has been considered in Beran (1996 a,b) utilizing a linear matrix inequality (LMI) approach. The approach taken is that of defining a feasibility problem, where a controller of fixed-order that yields a closed-loop system with an induced norm bounded by a chosen value is computed based on the solution to two coupled LMI problems. The numerical technique used in computing a solution employs an alternating projection/semidefinite programming approach. As noted in Beran and Grigoriadis (1996), this algorithm is not guaranteed to converge to a solution, although numerical studies have shown the algorithm to be effective on practical problems.

Unfortunately, the results of Beran (1996 a,b), while allowing reduced-order control system design, do not allow the designer complete freedom in choosing the control architecture. An alternative approach to LMI-based synthesis methods is provided by fixed-structure techniques. Such techniques have been considered in Erwin *et al.* (1998) as a generic approach to practical control system design. The approach is amenable to a large class of practical control architectures, such as the application and tuning SISO PID control loops to a multivariable plant, as well as reduced-order controllers for high-dimensional plants. The core of this approach is a decentralized static output feedback framework for problem definition. This framework can represent the varied class of control architectures described above within a unified framework. The resulting problems are then amenable to solution via a numerical solution technique such as the continuation algorithm used in Erwin *et al.* (1998).

In the present paper we use fixed-structure techniques for synthesizing controllers that are optimal with respect to three induced norms applied to the closed-loop system. After reviewing signal norms and defining notation in §2, we introduce differentiable approximations for three operator norms in §3. Section 4 reviews the decentralized static output feedback framework, which is then used in §5 to define a fixed-structure optimal control problem for each of the norms as well as mixed-norm optimal control problems. Section 6 presents a continuation algorithm using quasi-Newton updates for computing approximate solutions of the optimal control problems. Finally, §7 presents synthesis results for each of the operator norms using a 4th-order model of a lightly damped mass-spring-dashpot system.

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† Air Force Research Laboratory, AFRL/VSSV, Bldg. 472, Kirtland AFB NM 87117-5776, USA.
‡ Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2140, USA.
§ Department of Electronic and Jlectrical Engineering, The University of Leeds, Leeds LS2 9JT, UK.
¶ Author for correspondence. e-mail: richard.erwin@kirtland.af.mil

2. System norms

In this section we review the definitions of several norms for linear, time-invariant systems. Throughout this section let

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \quad (1)$$

be an $l \times m$ asymptotically stable rotational matrix transfer function and let $g: (-\infty, \infty) \to \mathbb{R}^{l\times m}$ be the corresponding impulse response matrix given by

$$g(t) = \begin{cases} 0, & t < 0 \\ C e^{At} B & t \geq 0 \end{cases} \quad (2)$$

We define the *r-norm* of $x \in \mathbb{R}^n$ as

$$\|x\|_r \triangleq \begin{cases} \left[\sum_{i=1}^{n} |x_i|^r\right]^{1/r}, & 1 \leq r < \infty \\ \max_{1\leq i \leq n} |x_i|, & r = \infty \end{cases} \quad (3)$$

and the (p, r)-*norm* of a measurable function $f: (a, b) \to \mathbb{R}^n$ by

$$\|f\|_{p,r} \triangleq \begin{cases} \left[\int_a^b \|f(t)\|_r^p \, dt\right]^{1/p}, & 1 \leq p < \infty \\ \operatorname{ess\,sup}_{a<t<b} \|f\|_r, & p = \infty \end{cases} \quad (4)$$

We also define the space

$$\mathcal{L}^{p,r}(a,b) \triangleq \{f: (a,b) \to \mathbb{R}^m : f \text{ is measurable and } \|f\|_{p,r} < \infty\} \quad (5)$$

Next define the convolution operator $\mathbf{G}: \mathcal{L}^{p,r}(-\infty, \infty) \to \mathcal{L}^{q,s}(-\infty, \infty)$ by

$$z(t) = (\mathbf{G}w)(t) \triangleq \int_{-\infty}^{\infty} g(t-\tau) w(\tau) \, d\tau, \quad t \in (-\infty, \infty) \quad (6)$$

and the Hankel operator $\Gamma_G: \mathcal{L}^{p,r}(0, \infty) \to \mathcal{L}^{q,s}(0, \infty)$ by

$$z(t) = (\Gamma_G w)(t) \triangleq \int_0^{\infty} g(t+\tau) w(-\tau) \, dt, \qquad t \in (0, \infty)$$

where $z(t) \in \mathbb{R}^m$ and $w(t) \in \mathbb{R}^l$. Induced norms for $\mathbf{G}$ and Γ_G can then be defined as

$$\|G(s)\|_{(q,s),(p,r)} \triangleq \sup_{\|w\|_{p,r}=1} \|\mathbf{G}w\|_{q,s} \quad (7)$$

for the convolution operator and

$$\|G(s)\|_{\mathrm{H}} \triangleq \sup_{\|w\|_{2,2}=1} \|\Gamma_G w\|_{2,2} \quad (8)$$

for the Hankel operator.

It was shown in Wilson (1989) that

$$\|G(s)\|_{(\infty,2),(2,2)} = \lambda_{\max}^{1/2}\left[\int_{-\infty}^{\infty} g(t) g^{\mathrm{T}}(t) \, dt\right] = \lambda_{\max}^{1/2}[CQC^{\mathrm{T}}] \quad (9)$$

$$\|G(s)\|_{(\infty,\infty),(2,2)} = \mathrm{d}_{\max}^{1/2}\left[\int_{-\infty}^{\infty} g(t) g^{\mathrm{T}}(t) \, dt\right] = \mathrm{d}_{\max}^{1/2}[CQC^{\mathrm{T}}] \quad (10)$$

where $\lambda_{\max}$ denotes the largest eigenvalue, $\mathrm{d}_{\max}$ denotes the maximum diagonal entry, and Q satisfies the matrix Lyapunov equation

$$0 = AQ + QA^{\mathrm{T}} + BB^{\mathrm{T}} \quad (11)$$

Furthermore, in Glover (1984), it was shown that

$$\|G(s)\|_{\mathrm{H}} = \lambda_{\max}(PQ) \quad (12)$$

where Q satisfies (11) and P satisfies the matrix Lyapunov equation

$$0 = A^{\mathrm{T}} P + PA + C^{\mathrm{T}} C \quad (13)$$

Finally, recall that the $\mathcal{H}_2$ norm of $G(s)$ is defined as

$$\|G(s)\|_2^2 \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{tr}[G(\jmath\omega) G^*(\jmath\omega)] \, d\omega = \operatorname{tr} CQC^{\mathrm{T}} \quad (14)$$

where Q satisfies (11). Note that for multi-input, multi-ouput systems, the $\mathcal{H}_2$ norm of a system is not defined as an induced norm.

Remark 1: It follows from the Cauchy interlacing theorem (Stewart and Sun 1990, p. 198) that

$$\mathrm{d}_{\max}(CQC^{\mathrm{T}}) \leq \lambda_{\max}(CQC^{\mathrm{T}}) \leq \operatorname{tr} CQC^{\mathrm{T}} \quad (15)$$

Hence (Rotea 1993), for multi-input, multi-output systems

$$\|G(s)\|_{(\infty,\infty),(2,2)} \leq \|G(s)\|_{(\infty,2),(2,2)} \leq \|G(s)\|_2 \quad (16)$$

while, for single-input, single-ouput systems

$$\|G(s)\|_{(\infty,\infty),(2,2)} = \|G(s)\|_{(\infty,2),(2,2)} = \|G(s)\|_2 \quad (17)$$

3. Approximation of the convolution and Hankel operator norms

Throughout this section let $G(s)$ be an asymptotically stable rational matrix transfer function with realization (1) and impulse response matrix (2). To develop a differentiable approximation to $\|G(s)\|_{(\infty,2),(2,2)}$, note that

$$\operatorname{tr} CQC^{\mathrm{T}} = \sum_{i=1}^{n} \lambda_i \quad (18)$$

where Q satisfies (11) and λ_i is the ith largest eigenvalue of CQC^{T}. Since λ_i^k is an eigenvalue of $(CQC^{\mathrm{T}})^k$, where $k > 0 \in \mathbb{R}$, it follows that

$$\mathrm{tr}\,[(CQC^{\mathrm{T}})^k] = \sum_{i=1}^{n} \lambda_i^k \tag{19}$$

Thus

$$\lim_{k\to\infty} (\mathrm{tr}\,[(CQC^{\mathrm{T}})^k])^{1/k} = \lambda_{\max}(CQC^{\mathrm{T}}) \tag{20}$$

In addition, the sequence is monotonically decreasing. Defining

$$\mathcal{J}_{1,k}(G) \triangleq (\mathrm{tr}\,[(CQC^{\mathrm{T}})^k])^{1,k} \tag{21}$$

where Q is the solution to (11), it follows that

$$\lim_{k\to\infty} \mathcal{J}_{1,k}(G) = \|G(s)\|^2_{(\infty,2),(2,2)} \tag{22}$$

Next, to develop a differentiable approximation to $\|G(s)\|_{(\infty,\infty),(2,2)}$, note that

$$\mathrm{d}_i[(I \circ CQC^{\mathrm{T}})^k] = (\mathrm{d}_i[I \circ CQC^{\mathrm{T}}])^k \tag{23}$$

where $\circ$ denotes Hadamard (entry-by-entry) multiplication, d_i stands for the ith diagonal entry, and $k > 0 \in \mathbb{R}$. Hence it follows that

$$\lim_{k\to\infty} (\mathrm{tr}\,[(I \circ CQC^{\mathrm{T}})^k])^{1/k} = \mathrm{d}_{\max}(CQC^{\mathrm{T}}) \tag{24}$$

Defining

$$\mathcal{J}_{2,k}(G) \triangleq (\mathrm{tr}\,[(I \circ CQC^{\mathrm{T}})^k])^{1/k} \tag{25}$$

where Q satisfies (11), it follows that

$$\lim_{k\to\infty} \mathcal{J}_{2,k}(G) = \|G(s)\|^2_{(\infty,\infty),(2,2)} \tag{26}$$

Finally, it can be seen that

$$\lim_{k\to\infty} (\mathrm{tr}\,[PQ)^k])^{1/k} = \lambda_{\max}(PQ) \tag{27}$$

where Q satisfies (11), P satisfies (13) and $k > 0 \in \mathbb{R}$. Defining

$$\mathcal{J}_{3,k}(G) \triangleq (\mathrm{tr}\,[(PQ)^k])^{1/k} \tag{28}$$

where Q satisfies (11) and P is the solution of (13), it follows that

$$\lim_{k\to\infty} \mathcal{J}_{3,k}(G) = \|G(s)\|^2_{\mathrm{H}} \tag{29}$$

4. The decentralized static output feedback framework

This section reviews the decentralized static output feedback framework for fixed-structure controller synthesis. As demonstrated in Erwin *et al.* (1998), this framework captures a large class of fixed-structure synthesis problems in a single format, thus permitting the use of a single computational algorithm. In particular, a given fixed-structure controller synthesis problem is recast as a decentralized static output feedback synthesis problem that yields the same closed-loop system.

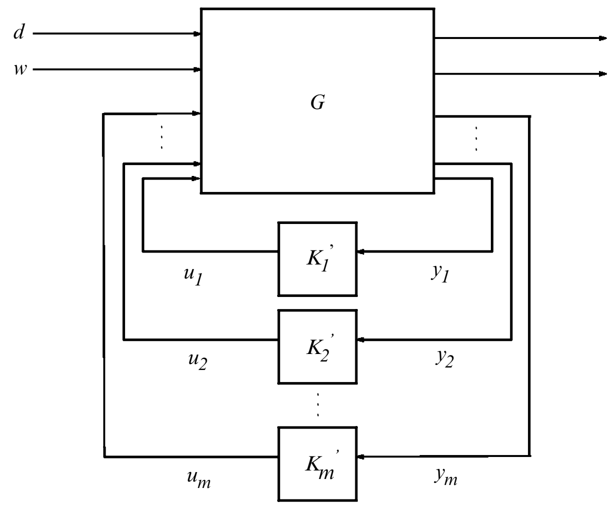

Figure 1. Decentralized static output feedback framework.

For the $(m+2)$-vector-input, $(m+2)$-vector-output decentralized system shown in figure 1, define

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \tag{30}$$

Here w and d represent exogenous inputs to the system, and e and z represent performance signals. Let $G(s)$ have the realization

$$G(s) \sim \left[\begin{array}{c|c:c:c} \mathcal{A} & \mathcal{B}_u & \mathcal{B}_d & \mathcal{B}_w \\ \hline \mathcal{C}_y & \mathcal{D}_{yu} & \mathcal{D}_{yd} & \mathcal{D}_{yw} \\ \hdashline \mathcal{C}_e & \mathcal{D}_{eu} & \mathcal{D}_{ed} & \mathcal{D}_{ew} \\ \hdashline \mathcal{C}_z & \mathcal{D}_{zu} & \mathcal{D}_{zd} & \mathcal{D}_{zw} \end{array}\right] \tag{31}$$

which represents the linear, time-invariant dynamic system

$$\dot{x} = \mathcal{A}x + \mathcal{B}_u u + \mathcal{B}_d d + \mathcal{B}_w w \tag{32}$$

$$y = \mathcal{C}_y x + \mathcal{D}_{yu} u + \mathcal{D}_{yd} d + \mathcal{D}_{yw} w \tag{33}$$

$$e = \mathcal{C}_e x + \mathcal{D}_{eu} u + \mathcal{D}_{ed} d + \mathcal{D}_{ew} w \tag{34}$$

$$z = \mathcal{C}_z x + \mathcal{D}_{zu} u + \mathcal{D}_{zd} d + \mathcal{D}_{zw} w \tag{35}$$

To represent decentralized static output feedback control with possibly repeated gains, we consider

$$u_i = \mathcal{K}_i' y_i, \qquad i = 1, \ldots, m \tag{36}$$

where the matrices $\mathcal{K}_i'$ are not necessarily distinct. Reordering the variables in (36) if necessary, we can rewrite (36) as

$$u = \mathcal{K}y \tag{37}$$

where $\mathcal{K}$ is an element of the set

$$\mathcal{U} \triangleq \{\mathcal{K} : \mathcal{K} = \text{block-diag}\,(I_{\phi_1} \otimes \mathcal{K}_1, \ldots, I_{\phi_v} \otimes \mathcal{K}_v)\} \tag{38}$$

v is the number of *distinct* gains $\mathcal{K}_i \in \mathbb{R}^{r_i \times c_i}$, and ϕ_i is the number of repetitions of gain $\mathcal{K}_i$. Note that $\mathcal{K}_1, \ldots, \mathcal{K}_v$ are not necessarily square matrices and that $\sum_{i=1}^{v} \phi_i = m$.

For convenience, define

$$L_{\mathcal{K}} \triangleq I - \mathcal{D}_{yu}\mathcal{K} \tag{39}$$

Assuming that $L_{\mathcal{K}}$ is non-singular, the closed-loop dynamics are given by

$$\dot{x} = \tilde{A}x + \tilde{B}_d d + \tilde{B}_w w \tag{40}$$

$$e = \tilde{C}_e + \tilde{D}_{ed} d + \tilde{D}_{ew} w \tag{41}$$

$$z = \tilde{C}_z x + \tilde{D}_{zd} d + \tilde{D}_{zw} w \tag{42}$$

where

$$\tilde{A} \triangleq \mathcal{A} + \mathcal{B}_u \mathcal{K} L_{\mathcal{K}}^{-1} \mathcal{C}_y, \qquad \tilde{B}_d \triangleq \mathcal{B}_d + \mathcal{B}_u \mathcal{K} L_{\mathcal{K}}^{-1} \mathcal{D}_{yd}$$

$$\tilde{B}_w \triangleq \mathcal{B}_w + \mathcal{B}_u \mathcal{K} L_{\mathcal{K}}^{-1} \mathcal{D}_{yw}, \qquad \tilde{C}_z \triangleq \mathcal{C}_z + \mathcal{D}_{zu} \mathcal{K} L_{\mathcal{K}}^{-1} \mathcal{C}_y$$

$$\tilde{D}_{zd} \triangleq \mathcal{D}_{zd} + \mathcal{D}_{zu} \mathcal{K} L_{\mathcal{K}}^{-1} \mathcal{D}_{yd}, \qquad \tilde{D}_{zw} \triangleq \mathcal{D}_{zw} + \mathcal{D}_{zu} \mathcal{K} L_{\mathcal{K}}^{-1} \mathcal{D}_{yw}$$

$$\tilde{C}_e \triangleq \mathcal{C}_e + \mathcal{D}_{eu} \mathcal{K} L_{\mathcal{K}}^{-1} \mathcal{C}_y, \qquad \tilde{D}_{ed} \triangleq \mathcal{D}_{ed} + \mathcal{D}_{eu} \mathcal{K} L_{\mathcal{K}}^{-1} \mathcal{D}_{yd}$$

$$\tilde{D}_{ew} \triangleq \mathcal{D}_{ew} + \mathcal{D}_{eu} \mathcal{K} L_{\mathcal{K}}^{-1} \mathcal{D}_{yw}$$

The closed-loop transfer function $\tilde{G}_{zw}(s)$ therefore has the realization

$$\tilde{G}_{zw}(s) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B}_w \\ \hline \tilde{C}_z & \tilde{D}_{zw} \end{array}\right] \tag{43}$$

while the closed-loop transfer function $\tilde{G}_{ed}(s)$ has the realization

$$\tilde{G}_{ed}(s) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B}_d \\ \hline \tilde{C}_e & \tilde{D}_{ed} \end{array}\right] \tag{44}$$

This paper will be concerned only with synthesis of centralized, strictly proper dynamic compensators. The equivalent decentralized static output feedback system for this fixed-structure control problem is given in Appendix A.

5. Optimal control problems

In this section we consider optimal control problems corresponding to each of the induced norms discussed in §2. The $\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,2}$-optimal control problem is

$$\min_{\mathcal{K}\in\mathcal{U}} \|\tilde{G}_{zw}(s)\|_{(\infty,2),(2,2)} \tag{45}$$

the $\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,2}$-optimal control problem is

$$\min_{\mathcal{K}\in\mathcal{U}} \|\tilde{G}_{zw}(s)\|||_{(\infty,\infty),(2,2)} \tag{46}$$

and the Hankel-norm optimal control problem is

$$\min_{\mathcal{K}\in\mathcal{U}} \|\tilde{G}_{zw}(s)\|_{\mathrm{H}} \tag{47}$$

Given the realization (43) for $\tilde{G}_{zw}(s)$, assuming that $\tilde{A}$ is asymptotically stable, and using the approximations (21), (25) and (28), the optimal control problems (45), (46) and (47) can be rewritten as

$$\min_{\mathcal{K}\in\mathcal{U}} \lim_{k\to\infty} (\mathrm{tr}\,[(\tilde{C}_z \tilde{Q}_{zw} \tilde{C}_z^{\mathrm{T}})^k])^{1/k} \tag{48}$$

$$\min_{\mathcal{K}\in\mathcal{U}} \lim_{k\to\infty} (\mathrm{tr}[(I \circ \tilde{C}_z \tilde{Q}_{zw} \tilde{C}_z^{\mathrm{T}})^k])^{1/k} \tag{49}$$

$$\min_{\mathcal{K}\in\mathcal{U}} \lim_{k\to\infty} (\mathrm{tr}[(\tilde{P}_{zw} \tilde{Q}_{zw})^k])^{1/k} \tag{50}$$

respectively, where $k > 0 \in \mathbb{R}$, and the matrices $\tilde{Q}_{zw}$ and $\tilde{P}_{zw}$ satisfy

$$0 = \tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w \tilde{B}_w^{\mathrm{T}} \tag{51}$$

$$0 = \tilde{A}^{\mathrm{T}}\tilde{P}_{zw} + \tilde{P}_{zw}\tilde{A} + \tilde{C}_z^{\mathrm{T}} \tilde{C}_z \tag{52}$$

respectively. Analogously, the fixed-structure $\mathcal{H}_2$-optimal problem is defined as

$$\min_{\mathcal{K}\in\mathcal{U}} \|\tilde{G}_{zw}(s)\|_2 \tag{53}$$

Given the realization (43) for $\tilde{G}_{zw}(s)$, where $\tilde{A}$ is asymptotically stable and the feedthrough term $\tilde{D}_{zw} = 0$, it follows from (14) that

$$\|\tilde{G}_{zw}(s)\|||_2^2 = \mathrm{tr}\,\tilde{C}_z \tilde{Q}_{zw} \tilde{C}_z^{\mathrm{T}} \tag{54}$$

where $\tilde{Q}_{zw}$ satisfies (51).

Next we consider several mixed-norm optimal control problems where the objective function is a convex combination of one norm ($\mathcal{H}_2$ or induced) applied to the system between input w and output z and another norm applied to the system between d and e. This mixed norm problem allows the synthesis of controllers satisfying multiple objectives, such as providing $\mathcal{H}_2$ performance while ensuring that actuator saturation constraints are not exceeded. Specifically, we consider mixed-norm optimal control problems involving the $\mathcal{H}_2$ norm and one of the operator norms described in §2. Specifically, we consider the mixed $\mathcal{H}_2/\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,2}$ control problem

$$\min_{\mathcal{K}\in\mathcal{U}} \alpha\|\tilde{G}_{zw}(s)\|_2 + (1-\alpha)\|\tilde{G}_{ed}(s)\|_{(\infty,2),(2,2)} \tag{55}$$

the mixed $\mathcal{H}_2/\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,\infty}$ control problem

$$\min_{\mathcal{K}\in\mathcal{U}} \alpha\|\tilde{G}_{zw}(s)|_2 + (1-\alpha)\|\tilde{G}_{ed}(s)\|_{(\infty,\infty),(2,2)} \tag{56}$$

and, finally, the mixed $\mathcal{H}_2$/Hankel control problem

$$\min_{\mathcal{K}\in\mathcal{U}} \alpha\|\tilde{G}_{zw}(s)\|_2 + (1-\alpha)\|\tilde{G}_{ed}(s)\|_{(\infty,\infty),(2,2)} \tag{56}$$

and, finally, the mixed $\mathcal{H}_2$/Hankel control problem

$$\min_{\mathcal{K}\in\mathcal{U}} \alpha\|\tilde{G}_{zw}(s)\|_2 + (1-\alpha)\|\tilde{G}_{ed}(s)\|_{\mathrm{H}} \tag{57}$$

In (57), the parameter $\alpha \in [0,1]$ plays the role of a weight in determining the relative importance of the $\mathcal{H}_2$ and Hankel portion of the control problem.

Using (54) and (21), (25) or (28), the mixed-norm optimal control problems (55), (56) or (57) can be rewritten as

$$\min_{\mathcal{K}\in\mathcal{U}} \lim_{k\to\infty} \alpha \operatorname{tr} \tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}} + (1-\alpha)(\operatorname{tr}[\tilde{C}_e\tilde{Q}_{ed}\tilde{C}_e^{\mathrm{T}})^k])^{1/k} \tag{58}$$

$$\min_{\mathcal{K}\in\mathcal{U}} \lim_{k\to\infty} \alpha \operatorname{tr} \tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}} + (1-\alpha)(\operatorname{tr}[(I\circ\tilde{C}_e\tilde{Q}_{ed}\tilde{C}_e^{\mathrm{T}})^k])^{1/k} \tag{59}$$

$$\min_{\mathcal{K}\in\mathcal{U}} \lim_{k\to\infty} \alpha \operatorname{tr} \tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}} + (1-\alpha)(\operatorname{tr}[(\tilde{P}_{ed}\tilde{Q}_{ed})^k])^{1/k} \tag{60}$$

where $k > 0 \in \mathbb{R}$, $\tilde{Q}_{zw}$ solves (51), and $\tilde{Q}_{ed}$ and $\tilde{P}_{ed}$ satisfy

$$0 = \tilde{Q}_{ed}\tilde{A} + \tilde{Q}_{ed}\tilde{A}^{\mathrm{T}} + \tilde{B}_d\tilde{B}_d^{\mathrm{T}} \tag{61}$$

$$0 = \tilde{A}^{\mathrm{T}}\tilde{P}_{ed} + \tilde{P}_{ed}\tilde{A} + \tilde{C}_e^{\mathrm{T}}\tilde{C}_e \tag{62}$$

Lagrangian functions and the resulting necessary conditions for optimality for each of the optimal control problems discussed in this section are given in Appendix B.

6. Algorithm description

To solve the non-linear optimization problems represented by (48)–(50) and (58)–(60), a continuation algorithm employing a general-purpose BFGS quasi-Newton correction (Dennis and Schnable 1983) is used. The line-search portions of the quasi-Newton correction were modified to include a constraint-checking subroutine that decreases the length of the search direction vector to ensure that the closed-loop system remains stable. Numerical experience indicates that this subroutine is usually invoked during only the first few iterations of a synthesis procedure.

The use of a continuation strategy is suggested by the relationship of the approximate cost functions (21), (25) and (28) with the standard $\mathcal{H}_2$-optimal control problem. For example, noting that the approximate cost functions (21) and (25) reduce to the closed-loop $\mathcal{H}_2$ norm for $k = 1$ naturally suggests the exponent k be used as a continuation parameter for convolution-norm optimal control problems. The quasi-Newton corrector is initialized with the $\mathcal{H}_2$- optimal controller for the approximate cost function for $k = 2$. The resulting controller is then used to initialize the next problem, where $k = 3$, etc. The resulting sequence of optimization problems, defined by an increasing sequence of values of k, can be solved until no further decrease in $\|\tilde{G}_{zw}(s)\|_{(\infty,2),(2,2)}$ or $\|\tilde{G}_{zw}(s)\|_{(\infty,\infty),(2,2)}$ is obtained by increasing k.

Alternatively, a continuation can be performed utilizing the weighting parameter α in any mixed-norm optimal control (45)–(47). To see this, we note that the mixed norm problem reduces to the standard $\mathcal{H}_2$-optimal control problem if the system matrices are chosen such that $d = w$ and $e = z$, with $\alpha = 1$. The continuation parameter α is then adjusted towards zero, and the quasi-Newton correction is applied to the $\mathcal{H}_2$-optimal controller. A second smaller value of α is then chosen, and the correction applied again, etc. As $\alpha \to 0$, it can be seen from (58)–(60) that the resulting controller approaches the optimal induced norm controller. This continuation approach is particularly useful for approaching the synthesis of purely Hankel-norm optimal controllers (47), since the $\mathcal{H}_2$-optimal controller is not a solution of the approximate Hankel-norm cost function (28) for any value of the exponent k.

7. Numerical results

The example considered involves a lightly damped single degree-of-freedom mass-spring-dashpot system, as shown in figure 2. We let the velocity of the mass be the measured signal, which is corrupted by an exogenous noise signal v. For performance, we wish to bring the cart to rest without utilizing excessive control effort. The equations describing this system are then given by

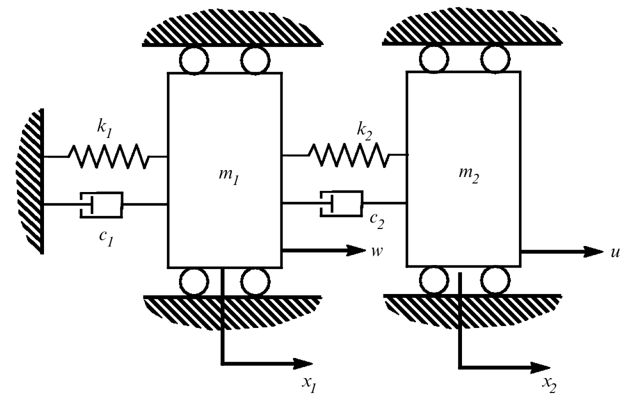

Figure 2. Mechanical system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2)/m_1 & k_2/m_1 & -(c_1+c_2)/m_1 & c_2/m_1 \\ k_2/m_2 & -k_2/m_2 & c_2/m_2 & -c_2/m_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \tag{63}$$

$$y = [1 \quad 0 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + [0 \quad 1] \begin{bmatrix} w \\ v \end{bmatrix} \tag{64}$$

$$z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} u \tag{65}$$

The numerical values used for controller synthesis are: $k_1 = k_2 = 4$, $m_1 = m_2 = 2$ and $c_1 = c_2 = 0.01$.

7.1. *$\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,2}$-optimal synthesis*

Utilizing the exponent k as the continuation parameter (as discussed in §6), we begin with the $\mathcal{H}_2$-optimal controller, yielding the optimal solution for (21) with $k = 1$. The continuation is then described by a sequence of quasi-Newton corrections corresponding to increasing values of k using the approximate cost function $\mathcal{J}_{1,k}$ given by (21). As indicated by (9), one would expect the maximum eignevalue of $\tilde{C}_z^{\mathrm{T}} \tilde{Q} \tilde{C}_z$ to decrease during this continuation. Figure 3 displays the minimum and maximum eigenvalues of $\tilde{C}_z^{\mathrm{T}} \tilde{Q} \tilde{C}_z$ versus the continuation parameter k for the resulting closed-loop systems, showing that this is indeed the case.

Figure 4 presents a parametric tradeoff curve showing the values of the $\|\tilde{G}_{zw}(s)\|_2^2$ versus $\tilde{G}_{zw}(s)\|_{(\infty,2),(2,2)}$ at each value of the continuation parameter k, demonstrating the loss of $\mathcal{H}_2$ performance as the $\mathcal{H}_2$-optimal controller evolves during the continuation process towards the $\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty}$,2-optimal controller.

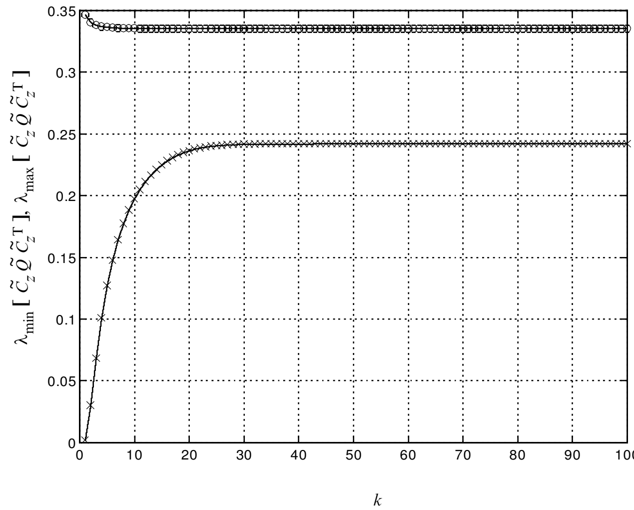

Figure 3. Minimum and maximum values of the eigenvalues of $\tilde{C}_z^{\mathrm{T}} \tilde{Q} \tilde{C}_z$ versus k for $\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,\in}$-optimal synthesis.

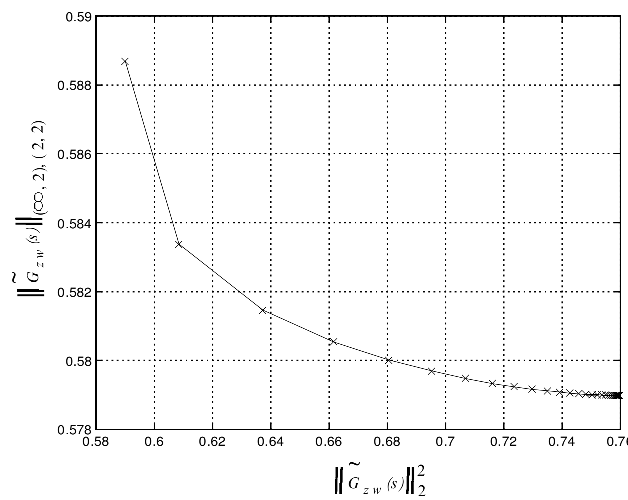

Figure 4. $\|\tilde{G}_{zw}(s)\|_2^2$ versus $\|\tilde{G}_{zw}(s)\|_{(\infty,2),(2,2)}$ for $\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,2}$-optimal synthesis.

7.2. *$\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,\infty}$-optimal synthesis*

As in §7.1, beginning with the $\mathcal{H}_2$-optimal controller $(k = 1)$, a continuation using increasing values of k was formulated as described in §6, this time with the quasi-Newton corrections aimed at the approximate cost function $\mathcal{J}_{2,k}$ given by (25). Again, as indicated by (10), we expect the maximum diagonal element of $\tilde{C}_z^{\mathrm{T}} \tilde{Q} \tilde{C}_z$ to decrease during the continuation on k, and figure 5 demonstrates that this is indeed the case. Figure 6 shows the gain in $\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,\infty}$ performance at the expense of $\mathcal{H}_2$ performance as the controller evolves from the original $\mathcal{H}_2$-optimal during the continuation on the exponent k.

7.3. *Hankel-norm optimal synthesis*

For Hankel-operator norm synthesis, a mixed-norm optimization problem was used to obtain the optimal Hankel-norm controller as discussed in §6. Defining the system matrices such that $d = w$ and $e = z$, a continuation employing the weighting parameter α was employed as described in §6, with the $\mathcal{H}_2$-optimal controller providing the initial solution for $\alpha = 1$. The continuation is then described by a sequence of quasi-Newton corrections for a decreasing sequence of values

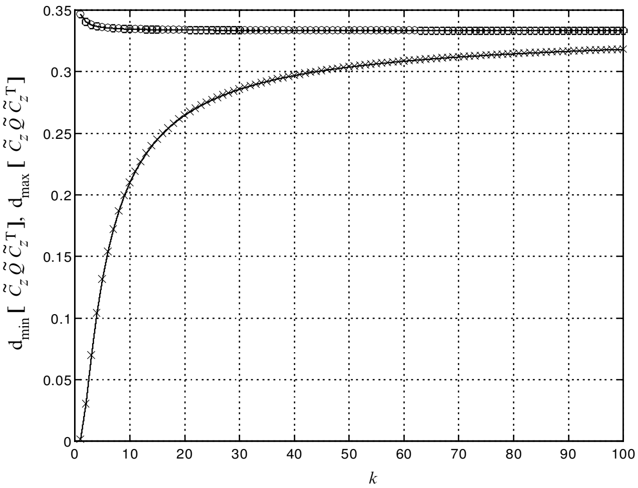

Figure 5. Minimum and maximum values of the diagonal elements of $\tilde{G}_z^{\mathrm{T}}\tilde{Q}\tilde{C}_z$ versus k for $\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,\infty}$-optimal synthesis.

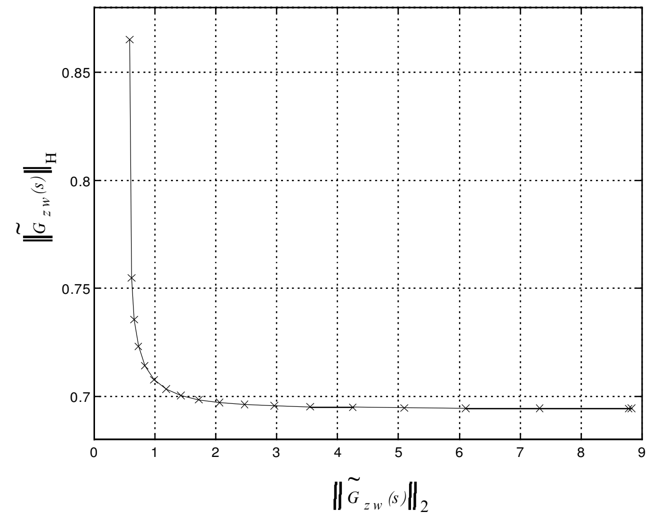

Figure 7. $\|\tilde{G}_{zw}(s)\|_{\mathrm{H}}$ versus $\|\tilde{G}_{zw}(s)\|_2^2$ for Hankel-norm optimal syntheis.

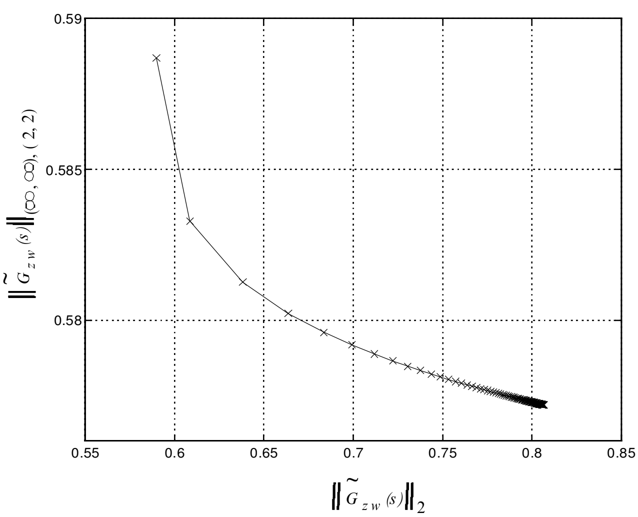

Figure 6. $\|\tilde{G}_{zw}(s)\|_{(\infty,\infty),(2,2)}$ versus $\|\tilde{G}_{zw}(s)\|_2^2$ for $\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,\infty}$-optimal synthesis.

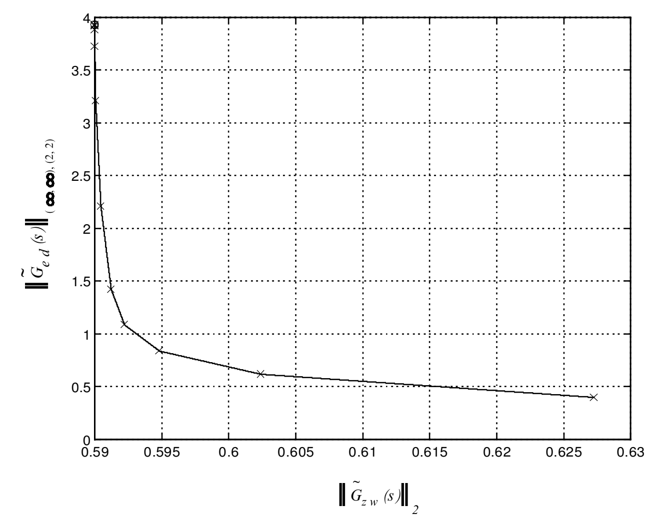

Figure 8. $\|\tilde{G}_{ed}(s)\|_{(\infty,\infty),(2,2)}$ versus $\tilde{G}_{zw}(s)\|_2$ for mixed-norm optimal control.

of the continuation parameter α using the mixed-norm cost function (60). Figure 7 presents the parametric tradeoff curve showing the values of $\|\tilde{G}_{zw}(s)\|_2^2$ versus $\|\tilde{G}_{zw}(s)\|_{\mathrm{H}}$ for each value of the continuation parameter α. The exponent $k = 10$ was held constant for all values of α.

7.4. *Mixed-norm optimal control*

As a final example, we now consider synthesis of controllers that are optimal with respect to a mixed $\mathcal{H}_2/\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,\infty}$ performance measure. Assume the actuator for the mass-spring-dashpot example described above saturates at $|u| = 1$, and furthermore suppose the measurement noise signal v satisfies $\|v\|_{2,2} \le 1$. Defining the system matrices such that $e = u$ and $d = v$, the problem then is to determine a controller which minimizes $\|\tilde{G}_{zw}(s)\|_2$ subject to $\|\tilde{G}_{ed}(s)\|\|_{(\infty,\infty),(2,2)} < 1$. Note that the $\mathcal{H}_2$-optimal controller in this case yields $\|\tilde{G}_{zw}(s)\|_2 = 0.5900$, while $\|\tilde{G}_{ed}(s)\|_{(\infty,\infty),(2,2)} = 3.9287$, and thus is not a feasible solution to the problem. We also note that the open-loop value of $\|G_{zw}(s)\|_2 = 2.7386$.

Mixed-norm controllers were synthesized for 20 values of the weighting variable $\alpha \in (0,1)$. Figure 8 shows the resulting tradeoff curve between $\|\tilde{G}_{zw}(s)\|_2$ and $\|\tilde{G}_{ed}(s)\|_{(\infty,\infty),(2,2)}$ as α moves from $1 - 1^{-12}$ to 1×10^{-12}. At $\alpha = 0.94544$, $\|\tilde{G}_{ed}(s)\|_{(\infty,\infty),(2,2)} = 0.8367$, and therefore the design specification is met. This design yields $\|\tilde{G}_{zw}(s)\|_2 = 0.5948$. A triangle pulse disturbance was input to the system as the measurement noise signal

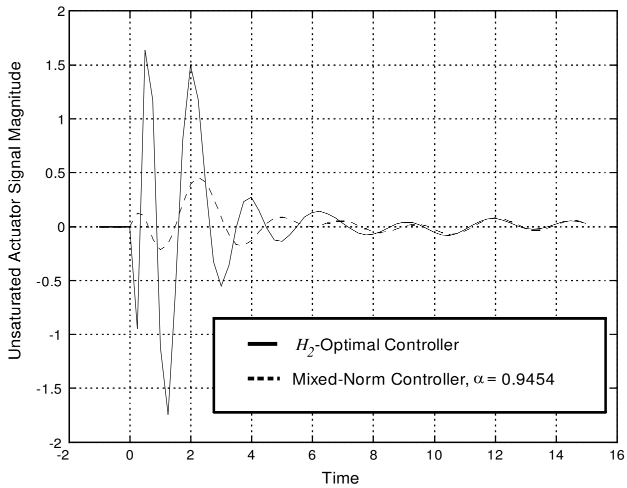

Figure 9. Unsaturated actuator signals for $\mathcal{H}_2$-optimal controller and mixed $\mathcal{H}_2/\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,\infty}$ optimal controller ($\alpha = 0.945\,44$).

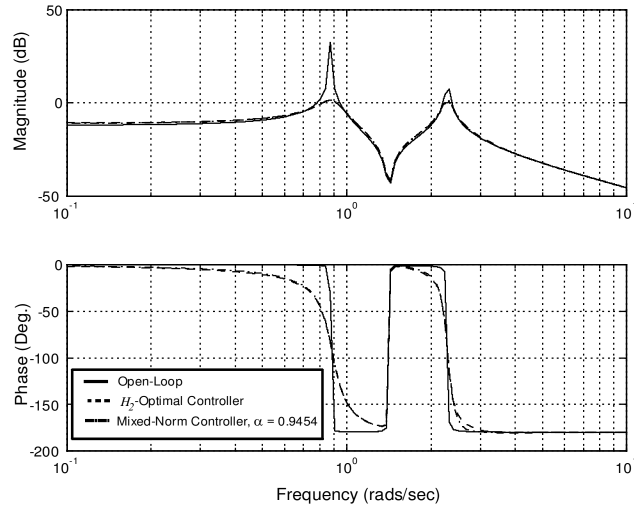

Figure 10. Closed-loop frequency response from w to $\dot{x}$ for $\mathcal{H}_2$-optimal controller and mixed $\mathcal{H}_2/\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,\infty}$ optimal controller ($\alpha = 0.945\,44$).

v. The maximum amplitude of the pulse is $\sqrt{6}$ and its duration is 0.25, and therefore $\|v\|_{2,2} = 1$.

Figure 9 shows the resulting unsaturated actuator signals for both the $\mathcal{H}_2$-optimal controller and the mixed $\mathcal{H}_2/\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,\infty}$ optimal controller for $\alpha = 0.945\,44$. We see that the $\mathcal{H}_2$-optimal controller violates the actuator saturation constraint of $u \leq 1$ for this particular disturbance, while the mixed-norm optimal controller keeps the control signal within the saturation limit. To examine the loss of $\mathcal{H}_2$ performance incurred by the mixed-norm controller, figure 10 presents the closed-loop frequency response plots from the exogenous plant disturbance w to the performance variable x_1. As shown in the plot, very little performance in terms of frequency response was sacrificed in order to satisfy the actuator constraint.

8. Discussion

In this paper we have presented differentiable approximations for two convolution operator norms and the Hankel operator norm for linear time-invariant dynamic systems. Using these approximations we defined optimal control problems for each of the system norms, and also introduced several mixed-norm optimal control problems. The decentralized static output feedback controller synthesis framework for fixed-structure synthesis was used to formulate Lagrangian functions for the optimal control problems. Necessary conditions for optimality were derived, and a continuation algorithm employing quasi-Newton updates was used to compute approximate solutions to these equations for a 2nd-order mass-spring-dashpot example. The results of a mixed $\mathcal{H}_2/\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,\infty}$ optimal control problem for this example demonstrated the applicability of these operator norms to the problem of actuator saturation.

Numerical experience with LMI-based approaches to the $\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,2}$-optimal synthesis problem (Beran 1996 a,b) yielded optimal closed-loop systems with the minimum and maximum eigenvalues of $\tilde{C}_z^{\mathrm{T}}\tilde{Q}\tilde{C}_z$ of equal values. Although the authors know of no theoretical results that proves this is always the case, figure 3 shows the values approaching a fixed bias with respect to each other, potentially indicating that a local minimum has been reached and is being tracked during the continuation. This could be an indication that the integer values of k chosen for the continuation represented too coarse a grid, for example. In addition, the continuation algorithm approach utilized here does not provide any *a priori* bounds on how many continuation steps must be used to obtain the true optimal controller using this approach, although in practice the algorithm will iterate until a performance level is obtained.

In view of this, this paper's introduction of the differentiable approximations to the various induced system norms allows many numerical algorithms that rely on differentiable objective functions to be applied to the problem solution while keeping the decentralized static output feedback problem formulation intact. Probability-one homotopy algorithms (Ge *et al.* 1994, Collins *et al.* 1997), for example, have demonstrated excellent numerical robustness when employed to solve such non-convex optimal-control problems. The decentralized static output feedback problem formulation provides a convenient interface between specific fixed-structure problems and more generic numerical solution algorithms.

An interesting future line of inquiry would be a comparison of the computational burden, accuracy, and numerical stability of these various algorithms (including the alternating LMI approach) on benchmark problems. Some reports on the required computational burden for the continuation algorithm used in this work can be found in Erwin *et al.* (1998). Recent research into the use of randomized algorithms (Vidyasagar 1997, 1998) for attacking non-convex optimization problems have produced some promising initial results, and could also be amenable to the solution of the induced-norm problem.

Acknowledgments

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Appendix A. Decentralized static output feedback realization of centralized, strictly proper dynamic compensation problem

Consider a plant of the form

$$\dot{x}_p = Ax_p + B\hat{u} + F_1 w \tag{66}$$

$$\hat{y} = Cx_p + D\hat{u} + F_2 w \tag{67}$$

$$z = E_1 x_p + E_2 \hat{u} + E_0 w \tag{68}$$

in feedback with an n_cth-order strictly proper compensator having the realization

$$\dot{x}_c = A_c x_c + B_c \hat{y} \tag{69}$$

$$\hat{u} = C_c x_c \tag{70}$$

A realization for the closed-loop system consisting of (66)–(68), (69), and (70) is given by

$$\tilde{G}(s) \sim \left[\begin{array}{cc|c} A & BC_c & F_1 \\ B_cC & A_c + B_cDC_c & B_cF_2 \\ 0 & 0 & 0 \\ \hdashline E_1 & E_2C_c & E_0 \end{array}\right] \tag{71}$$

This system can be written as decentralized static output feedback with $m = v = 3$, $\phi_1 = \phi_2 = \phi_3 = 1$, $G(s)$ given by

$$G(s) \sim \left[\begin{array}{cc|ccc:c} A & 0 & 0 & 0 & B & F_1 \\ 0 & 0 & I & I & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & D & F_2 \\ 0 & I & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline E_1 & 0 & 0 & 0 & E_2 & E_0 \end{array}\right] \tag{72}$$

and $\mathcal{K}$ denoting the block-diagonal matrix

$$\mathcal{K} = \text{block-diag}(A_c, B_c, C_c) \tag{73}$$

This yields

$$L_{\mathcal{K}} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & -DC_c \\ 0 & 0 & I \end{bmatrix} \tag{74}$$

which is non-singular.

Appendix B

Note that a matrix $\mathcal{K} \in \mathcal{U}$ as given by (38) can be written as

$$\mathcal{K} = \sum_{i=1}^{v} \sum_{j=1}^{\phi_i} Q_{\mathrm{L}ij} \mathcal{K}_i Q_{\mathrm{R}ij} \tag{75}$$

where $Q_{\mathrm{L}ij}$ and $Q_{\mathrm{R}ij}$ are defind as

$$Q_{\mathrm{L}ij} \triangleq \begin{bmatrix} 0_{r_1\phi_1 \times r_i} \\ 0_{r_2\phi_2 \times r_i} \\ \vdots \\ 0_{r_{i-1}\phi_{i=1} \times r_i} \\ 0_{r_i(j-1) \times r_i} \\ I_{r_i} \\ 0_{r_i(\phi_i - j) \times r_i} \\ 0_{r_{i=1}\phi_{i+1} \times r_i} \\ \vdots \\ 0_{r_v\phi_v \times r_i} \end{bmatrix} \qquad Q_{\mathrm{R}ij} \triangleq \begin{bmatrix} 0_{c_1\phi_1 \times c_i} \\ 0_{c_2\phi_2 \times c_i} \\ \vdots \\ 0_{c_{i-1}\phi_{i-1} \times c_i} \\ 0_{c_i(j-1) \times c_i} \\ I_{c_i} \\ 0_{c_i(\phi_i - j) \times c_i} \\ 0_{c_{i+1}\phi_{i+1} \times c_i} \\ \vdots \\ 0_{c_v\phi_v \times c_i} \end{bmatrix}^{\mathrm{T}} \tag{76}$$

B.1. $\mathcal{H}_2$ *norm* $\|G(s)\|_2$

The partial derivatives of

$$\mathcal{L}(\tilde{P}_{zw},\tilde{Q}_{zw},\mathcal{K}_i) = \operatorname{tr}\tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}} + \operatorname{tr}\tilde{P}_{zw}[\tilde{A}\tilde{Q}_{zw} + Q_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w\tilde{B}_w^{\mathrm{T}}] \quad (77)$$

are

$$\frac{\partial\mathcal{L}}{\partial\tilde{P}_{zw}} = \tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w\tilde{B}_w^{\mathrm{T}} \quad (78)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{Q}_{zw}} = \tilde{A}^{\mathrm{T}}\tilde{P}_{zw} + \tilde{P}_{zw}\tilde{A} + \tilde{C}_z^{\mathrm{T}}\tilde{C}_z \quad (79)$$

$$\frac{\partial\mathcal{L}}{\partial\mathcal{K}_i} = 2\sum_{j=1}^{\phi_i} Q_{\mathrm{L}ij}^{\mathrm{T}}(I + \mathcal{D}_{yu}^{\mathrm{T}}L_{\mathcal{K}}^{-\mathrm{T}}\mathcal{K}^{\mathrm{T}}) \times [\mathcal{D}_{zu}^{\mathrm{T}}\tilde{C}_z\tilde{Q}_{zw}\mathcal{C}_y + \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{zw}\tilde{B}_w\mathcal{D}_{yw}^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{zw}\tilde{Q}_{zw}\mathcal{C}_y^{\mathrm{T}}]L_{\mathcal{K}}^{-\mathrm{T}}Q_{\mathrm{R}ij}^{\mathrm{T}} \quad (80)$$

B.2. *Convolution operator norm* $\|G(s)\|_{(\infty,2),(2,2)}$

The partial derivatives of

$$\mathcal{L}(\tilde{P}_{zw},\tilde{Q}_{zw},\mathcal{K}_i) = (\operatorname{tr}[(\tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}})^k])^{1/k} + \operatorname{tr}\tilde{P}_{zw}[\tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w\tilde{B}_w^{\mathrm{T}}] \quad (81)$$

are

$$\frac{\partial\mathcal{L}}{\partial\tilde{P}_{zw}} = \tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w\tilde{B}_w^{\mathrm{T}} \quad (82)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{Q}_{zw}} = \tilde{A}^{\mathrm{T}}\tilde{P}_{zw} + \tilde{P}_{zw}\tilde{A} + (\operatorname{tr}[(\tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}})^k])^{1/k-1} \times \tilde{C}_z^{\mathrm{T}}(\tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}})^{k-1}\tilde{C}_z \quad (83)$$

$$\frac{\partial\mathcal{L}}{\partial\mathcal{K}_i} = 2\sum_{j=1}^{\phi_i} Q_{\mathrm{L}ij}^{\mathrm{T}}(I + \mathcal{D}_{yu}^{\mathrm{T}}L_{\mathcal{K}}^{-\mathrm{T}}\mathcal{K}^{\mathrm{T}})\{\mathcal{B}_u^{\mathrm{T}}\tilde{P}_{zw}\tilde{B}_w\mathcal{D}_{yw}^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{zw}\tilde{Q}_{zw}\mathcal{C}_y^{\mathrm{T}} + (\operatorname{tr}[(\tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}})^k])^{1/k-1} \times \mathcal{D}_{zu}^{\mathrm{T}}(\tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}})^{k-1}\tilde{C}_z\tilde{Q}_{zw}\mathcal{C}_y^{\mathrm{T}}\}L_{\mathcal{K}}^{-\mathrm{T}}Q_{\mathrm{R}ij}^{\mathrm{T}} \quad (84)$$

B.3. *Convolution operator norm* $\|G(s)\|_{(\infty,\infty),(2,2)}$

The partial derivatives of

$$\mathcal{L}(\tilde{P}_{zw},\tilde{Q}_{zw},\mathcal{K}_i) = (\operatorname{tr}[(I\circ\tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}})^k])^{1/k} + \operatorname{tr}\tilde{P}_{zw}[\tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w\tilde{B}_w^{\mathrm{T}}] \quad (85)$$

are

$$\frac{\partial\mathcal{L}}{\partial\tilde{P}_{zw}} = \tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w\tilde{B}_w^{\mathrm{T}} \quad (86)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{Q}_{zw}} = \tilde{A}^{\mathrm{T}}\tilde{P}_{zw} + \tilde{P}_{zw}\tilde{A} + (\operatorname{tr}[(I\circ\tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}})^k])^{1/k-1} \times \tilde{C}_z^{\mathrm{T}}(I\circ\tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}})^{k-1}\tilde{C}_z \quad (87)$$

$$\frac{\partial\mathcal{L}}{\partial\mathcal{K}_i} = 2\sum_{j=1}^{\phi_i} Q_{\mathrm{L}ij}^{\mathrm{T}}(I + \mathcal{D}_{yu}^{\mathrm{T}}L_{\mathcal{K}}^{-\mathrm{T}}\mathcal{K}^{\mathrm{T}})\{\mathcal{B}_u^{\mathrm{T}}\tilde{P}_{zw}\tilde{B}_w\tilde{D}_{yw}^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{zw}\tilde{Q}_{zw}\mathcal{C}_y^{\mathrm{T}} + (\operatorname{tr}[(I\circ\tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}})^k])^{1/k-1} \times \mathcal{D}_{zu}^{\mathrm{T}}(I\circ\tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}})^{k-1}\tilde{C}_z\tilde{Q}_{zw}\mathcal{C}_y^{\mathrm{T}}\}L_{\mathcal{K}}^{-\mathrm{T}}Q_{\mathrm{R}ij}^{\mathrm{T}} \quad (88)$$

B.4. *Hankel operator norm* $\|G(s)\|_{\mathrm{H}}$

The partial derivatives of

$$\mathcal{L}(\bar{P},\bar{Q},\tilde{P}_{zw},\tilde{Q}_{zw},\mathcal{K}_i) = (\operatorname{tr}[(\tilde{P}_{zw}\tilde{Q}_{zw})^k])^{1/k} + \operatorname{tr}\bar{P}[\tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w\tilde{B}_w^{\mathrm{T}}] + \operatorname{tr}\bar{Q}[\tilde{A}^{\mathrm{T}}\tilde{P}_{zw} + \tilde{P}_{zw}\tilde{A} + \tilde{C}_z^{\mathrm{T}}\tilde{C}_z] \quad (89)$$

are

$$\frac{\partial\mathcal{L}}{\partial\bar{P}} = \tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w\tilde{B}_w^{\mathrm{T}} \quad (90)$$

$$\frac{\partial\mathcal{L}}{\partial\bar{Q}} = \tilde{A}^{\mathrm{T}}\tilde{P}_{zw} + \tilde{P}_{zw}\tilde{A} + \tilde{C}_z^{\mathrm{T}}\tilde{C}_z \quad (91)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{P}_{zw}} = \tilde{A}\bar{Q} + \bar{Q}\tilde{A}^{\mathrm{T}}(\operatorname{tr}[(\tilde{P}_{zw}\tilde{Q}_{zw})^k])^{1/k-1} \times (\tilde{Q}_{zw}\tilde{P}_{zw})^{k-1}\tilde{Q}_{zw} \quad (92)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{Q}_{zw}} = \tilde{A}^{\mathrm{T}}\bar{P} + \bar{P}\tilde{A} + (\operatorname{tr}[(\tilde{P}_{zw}\tilde{Q}_{zw})^k])^{1/k-1} \times \tilde{P}_{zw}(\tilde{Q}_{zw}\tilde{P}_{zw})^{k-1} \quad (93)$$

$$\frac{\partial\mathcal{L}}{\partial\mathcal{K}_i} = 2\sum_{j=1}^{\phi_i} Q_{\mathrm{L}ij}^{\mathrm{T}}(I + \mathcal{D}_{yu}^{\mathrm{T}}L_{\mathcal{K}}^{-\mathrm{T}}\mathcal{K}^{\mathrm{T}})[\mathcal{B}_u^{\mathrm{T}}\bar{P}\tilde{B}_w\mathcal{D}_{yw}^{\mathrm{T}} + \mathcal{D}_{zu}^{\mathrm{T}}\tilde{C}_z\bar{Q}\mathcal{C}_y^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}}\bar{P}\tilde{Q}_{zw}\mathcal{C}_y^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{zw}\bar{Q}\mathcal{C}_y^{\mathrm{T}}]L_{\mathcal{K}}^{-\mathrm{T}}Q_{\mathrm{R}ij}^{\mathrm{T}} \quad (94)$$

B.5. *Mixed* $\mathcal{H}_2/\mathcal{L}_{2,2}$*-to-*$\mathcal{L}_{\infty,2}$ *control*

The partial derivatives of

$$\mathcal{L}(\tilde{P}_{zw},\tilde{Q}_{zw},\tilde{P}_{ed},\tilde{Q}_{ed},\mathcal{K}_i) = \alpha\,\mathrm{tr}\,[\tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}}] + (1-\alpha)(\mathrm{tr}\,[(\tilde{C}_e\tilde{Q}_{ed}\tilde{C}_e^{\mathrm{T}})^k])^{1/k} + \mathrm{tr}\,\tilde{P}_{zw}[\tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w\tilde{B}_w^{\mathrm{T}}] + \mathrm{tr}\,\tilde{P}_{ed}[\tilde{A}\tilde{P}_{ed} + \tilde{P}_{ed}\tilde{A} + \tilde{B}_d\tilde{B}_d^{\mathrm{T}}] \quad (95)$$

are

$$\frac{\partial\mathcal{L}}{\partial\tilde{P}_{zw}} = \tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w\tilde{B}_w^{\mathrm{T}} \quad (96)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{Q}_{zw}} = \tilde{A}^{\mathrm{T}}\tilde{P}_{zw} + \tilde{P}_{zw}\tilde{A} + \alpha\tilde{C}_z^{\mathrm{T}}\tilde{C}_z \quad (97)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{P}_{ed}} = \tilde{A}\tilde{Q}_{ed} + \tilde{Q}_{ed}\tilde{A}^{\mathrm{T}} + \tilde{B}_d\tilde{B}_d^{\mathrm{T}} \quad (98)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{Q}_{ed}} = \tilde{A}^{\mathrm{T}}\tilde{P}_{ed} + \tilde{P}_{ed}\tilde{A} + (1-\alpha)(\mathrm{tr}\,[(\tilde{C}_e\tilde{Q}_{ed}\tilde{C}_e^{\mathrm{T}})^k])^{1/k-1} \times \tilde{C}_e^{\mathrm{T}}(\tilde{C}_e\tilde{Q}_{ed}\tilde{C}_e^{\mathrm{T}})^{k-1}\tilde{C}_e \quad (99)$$

$$\frac{\partial\mathcal{L}}{\partial\mathcal{K}_i} = 2\sum_{j=1}^{\phi_i} \mathcal{Q}_{\mathrm{L}ij}^{\mathrm{T}}(I + \mathcal{D}_{yu}^{\mathrm{T}}L_{\mathcal{K}}^{-\mathrm{T}}\mathcal{K}^{\mathrm{T}})\,\{\alpha\mathcal{D}_{zu}^{\mathrm{T}}\tilde{C}_z\tilde{Q}_{zw}\mathcal{C}_y^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{zw}\tilde{B}_w\mathcal{D}_{yw}^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{zw}\tilde{Q}_{zw}\mathcal{C}_y^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{ed}\tilde{B}_d\mathcal{D}_{yd}^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{ed}\tilde{Q}_{ed}\mathcal{C}_y^{\mathrm{T}} + (1-\alpha)(\mathrm{tr}\,[(\tilde{C}_e\tilde{Q}_{ed}\tilde{C}_e^{\mathrm{T}})^k])^{1/k-1} \times \mathcal{D}_{eu}^{\mathrm{T}}(\tilde{C}_e\tilde{Q}_{ed}\tilde{C}_e^{\mathrm{T}})^{k-1}\tilde{C}_e\tilde{Q}_{ed}\mathcal{C}_y^{\mathrm{T}}\}L_{\mathcal{K}}^{-\mathrm{T}}\mathcal{Q}_{\mathrm{R}ij}^{-\mathrm{T}} \quad (100)$$

B.6. *Mixed $\mathcal{H}_2/\mathcal{L}_{2,2}$-to-$\mathcal{L}_{\infty,\infty}$ control*

The partial derivatives of

$$\mathcal{L}(\tilde{P}_{zw},\tilde{Q}_{zw},\tilde{P}_{ed},\tilde{Q}_{ed},\mathcal{K}_i) = \alpha\,\mathrm{tr}\,[\tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}}] + (1-\alpha) \times (\mathrm{tr}\,[I\circ\tilde{C}_e\tilde{Q}_{ed}\tilde{C}_e^{\mathrm{T}})^k])^{1/k} + \mathrm{tr}\,\tilde{P}_{zw}[\tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w\tilde{B}_w^{\mathrm{T}}] + \mathrm{tr}\,\tilde{P}_{ed}[\tilde{A}\tilde{P}_{ed} + \tilde{P}_{ed}\tilde{A} + \tilde{B}_d\tilde{B}_d^{\mathrm{T}}] \quad (101)$$

are

$$\frac{\partial\mathcal{L}}{\partial\tilde{P}_{zw}} = \tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w\tilde{B}_w^{\mathrm{T}} \quad (102)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{Q}_{zw}} = \tilde{A}^{\mathrm{T}}\tilde{P}_{zw} + \tilde{P}_{zw}\tilde{A} + \alpha\tilde{C}_z^{\mathrm{T}}\tilde{C}_z \quad (103)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{P}_{ed}} = \tilde{A}\tilde{Q}_{ed} + \tilde{Q}_{ed}\tilde{A}^{\mathrm{T}} + \tilde{B}_d\tilde{B}_d^{\mathrm{T}} \quad (104)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{Q}_{ed}} = \tilde{A}^{\mathrm{T}}\tilde{P}_{ed} + \tilde{P}_{ed}\tilde{A} + (1-\alpha) \times (\mathrm{tr}\,[(I\circ\tilde{C}_e\tilde{Q}_{ed}\tilde{C}_e^{\mathrm{T}})^k])^{1/k-1} \times \tilde{C}_e^{\mathrm{T}}(I\circ\tilde{C}_e\tilde{Q}_{ed}\tilde{C}_e^{T})^{k-1}\tilde{C}_e \quad (105)$$

$$\frac{\partial\mathcal{L}}{\partial\mathcal{K}_i} = 2\sum_{j=1}^{\phi_i} \mathcal{Q}_{\mathrm{L}ij}^{\mathrm{T}}(I + \mathcal{D}_{yu}^{\mathrm{T}}L_{\mathcal{K}}^{-\mathrm{T}}\mathcal{K}^{\mathrm{T}})\,\{\alpha\mathcal{D}_{zu}^{\mathrm{T}}\tilde{C}_z\tilde{Q}_{zw}\mathcal{C}_y^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{zw}\tilde{B}_w\mathcal{D}_{yw}^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{zw}\tilde{Q}_{zw}\mathcal{C}_y^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{ed}\tilde{B}_d\mathcal{D}_{yd}^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}}\tilde{P}_{ed}\tilde{Q}_{ed}\mathcal{C}_y^{\mathrm{T}} + (1-\alpha)(\mathrm{tr}\,[(I\circ\tilde{C}_e\tilde{Q}_{ed}\tilde{C}_e^{\mathrm{T}})^k])^{1/k-1} \times \mathcal{D}_{eu}^{\mathrm{T}}(I\circ\tilde{C}_e\tilde{Q}_{ed}\tilde{C}_e^{\mathrm{T}})^{k-1}\tilde{C}_e\tilde{Q}_{ed}\mathcal{C}_y^{\mathrm{T}}\}L_{\mathcal{K}}^{-\mathrm{T}}\mathcal{Q}_{\mathrm{R}ij}^{\mathrm{T}} \quad (106)$$

B.7. *Mixed $\mathcal{H}_2$/Hankel control*

The partial derivatives of

$$\mathcal{L}(\tilde{P}_{zw},\tilde{Q}_{zw},\tilde{P}_{ed},\tilde{Q}_{ed},\bar{P}_{ed},\bar{Q}_{ed},\mathcal{K}_i) = \alpha\,\mathrm{tr}\,(\tilde{C}_z\tilde{Q}_{zw}\tilde{C}_z^{\mathrm{T}}) + (1-\alpha)(\mathrm{tr}\,[(\tilde{P}_{ed}\tilde{Q}_{ed})^k])^{1/k} + \mathrm{tr}\,\tilde{P}_{zw}(\tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w\tilde{B}_w^{\mathrm{T}}) + \mathrm{tr}\,\bar{P}_{ed}[\tilde{A}\tilde{Q}_{ed} + \tilde{Q}_{ed}\tilde{A}^{\mathrm{T}} + \tilde{B}_d\tilde{B}_d^{\mathrm{T}}] + \mathrm{tr}\,\bar{Q}_{ed}[\tilde{A}^{\mathrm{T}}\tilde{P}_{ed} + \tilde{P}_{ed}\tilde{A} + \tilde{C}_e^{\mathrm{T}}\tilde{C}_e] \quad (107)$$

are

$$\frac{\partial\mathcal{L}}{\partial\tilde{P}_{zw}} = \tilde{A}\tilde{Q}_{zw} + \tilde{Q}_{zw}\tilde{A}^{\mathrm{T}} + \tilde{B}_w\tilde{B}_w^{\mathrm{T}} \quad (108)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{Q}_{zw}} = \tilde{A}^{\mathrm{T}}\tilde{P}_{zw}\tilde{A} + \alpha\tilde{C}_z^{\mathrm{T}}\tilde{C}_z \quad (109)$$

$$\frac{\partial\mathcal{L}}{\partial\bar{Q}_{ed}} = \tilde{A}^{\mathrm{T}}\tilde{P}_{ed} + \tilde{P}_{ed}\tilde{A} + \tilde{C}_e^{\mathrm{T}}\tilde{C}_e \quad (110)$$

$$\frac{\partial\mathcal{L}}{\partial\bar{P}_{ed}} = \tilde{A}\tilde{Q}_{ed} + \tilde{Q}_{ed}\tilde{A}^{\mathrm{T}} + \tilde{B}_d\tilde{B}_d^{\mathrm{T}} \quad (111)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{P}_{ed}} = \tilde{A}\bar{Q}_{ed} + \bar{Q}_{ed}\tilde{A}^{\mathrm{T}} + (1-\alpha)(\mathrm{tr}\,[\tilde{P}_{ed}\tilde{Q}_{ed})^k])^{1/k-1} \times (\tilde{Q}_{ed}\tilde{P}_{ed})^{k-1}\tilde{Q}_{ed} \quad (112)$$

$$\frac{\partial\mathcal{L}}{\partial\tilde{Q}_{ed}} = \tilde{A}^{\mathrm{T}}\bar{P}_{ed} + \bar{P}_{ed}\tilde{A} + (1-\alpha)(\mathrm{tr}\,[(\tilde{P}_{ed}\tilde{Q}_{ed})^k])^{1/k-1} \times \tilde{P}_{ed}(\tilde{Q}_{ed}\tilde{P}_{ed})^{k-1} \quad (113)$$

$$\frac{\partial \mathcal{L}}{\partial \mathcal{K}_i} = 2\sum_{j=1}^{\phi_i} Q_{\mathrm{L}ij}^{\mathrm{T}}(I + \mathcal{D}_{yu}^{\mathrm{T}} L_{\mathcal{K}}^{-\mathrm{T}} \mathcal{K}^{\mathrm{T}})\{\alpha \mathcal{D}_{zu}^{\mathrm{T}} \tilde{C}_z \tilde{Q}_{zw} \mathcal{C}_y^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}} \tilde{P}_{zw} \tilde{B}_w \mathcal{D}_{yw}^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}} \tilde{P}_{zw} \tilde{Q}_{zw} \mathcal{C}_y^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}} \bar{P}_{ed} \tilde{B}_d \mathcal{D}_{yd}^{\mathrm{T}} + \mathcal{D}_{eu}^{\mathrm{T}} \tilde{C}_e \bar{Q}_{ed} \mathcal{C}_y^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}} \bar{P}_{ed} \tilde{Q}_{ed} \mathcal{C}_y^{\mathrm{T}} + \mathcal{B}_u^{\mathrm{T}} \tilde{P}_{ed} \bar{Q}_{ed} \mathcal{C}_y^{\mathrm{T}}\} L_{\mathcal{K}}^{-\mathrm{T}} Q_{\mathrm{R}ij}^{\mathrm{T}} \quad (114)$$

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