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On the gap between H_2 and entropy performance measures in H_∞ control design *

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Abstract: Recent papers have considered the problem of minimizing an entropy functional subject to an H_∞ performance constraint. Since the entropy is an upper bound for the H_2 cost, there remains a gap between entropy minimization and H_2 minimization. In this paper we consider a generalized cost functional involving both H_2 and entropy aspects. This approach thus provides a means for optimizing H_2 performance within H_∞ control design.

Keywords: H_2 design, minimum entropy, mixed-norm H_2/H_∞ design

1. Introduction

It was recently shown in [1] that suboptimal H_∞ controllers can be characterized by means of modified Riccati equations. These equations were obtained by minimizing an H_2 performance bound subject to a constraint on the H_∞ performance. Subsequently it was shown that, in the equalized H_2/H_∞ weight case, the H_2 performance bound coincides with an entropy functional [4,5]. Although less familiar than the H_2 objective, the entropy functional is mathematically tractable within the context of H_∞ control theory.

In many practical applications, however, it may be desirable to minimize the H_2 cost directly. That is, although the entropy functional bounds the H_2 cost (in the equalized weight case), there may exist a 'gap' between these performance measures. Thus, the control law that minimizes the entropy need not also minimize the H_2 performance.

The goal of the present paper is to extend the approach of [1] to include both H_2 and entropy performance measures within the context of constrained H_∞ design. This multiobjective problem is treated by forming a convex combination of both performance measures. This approach is reminiscent of scalarization techniques for Pareto optimization [4].

For simplicity the present paper is confined to static full-state feedback control. Full- and reduced-order dynamic compensation as in [1] will be considered in a future paper.

Notation. Note: All matrices have real entries.

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$	real numbers, $r\times s$ real matrices, $\mathbb{R}^{r\times 1}$.
I_r, $(\ )^{\mathrm{T}}$, tr	$r\times r$ identity matrix, transpose, trace
n, m, d, q, q_∞	positive integers.
x, u	n, m-dimensional vectors.
A, B, K	$n\times n$, $n\times m$, $m\times n$ matrices.
$w(\cdot)$	L_2 disturbance signal in $\mathbb{R}^d$.

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D, V	$n\times d$, $n\times n$ matrices, $V=DD^{\mathrm{T}}$
E_1, E_2	$q\times n$, $q\times m$ matrices; $E_1^{\mathrm{T}}E_2=0$
R_1, R_2	$E_1^{\mathrm{T}}E_1$, $E_2^{\mathrm{T}}E_2$.
$\tilde{R}$	$R_1+K^{\mathrm{T}}R_2K$
$E_{1\infty}$, $E_{2\infty}$	$q_\infty\times n$, $q_\infty\times m$ matrices, $E_{1\infty}^{\mathrm{T}}E_{2\infty}=0$
$R_{1\infty}$, $R_{2\infty}$	$E_{1\infty}^{\mathrm{T}}E_{1\infty}$, $E_{2\infty}^{\mathrm{T}}E_{2\infty}$
$\tilde{R}_\infty$	$R_{1\infty}+K^{\mathrm{T}}R_{2\infty}K$.
α, β, γ	real numbers, positive number

2. Problem statement

Combined H_2/H_∞/Entropy Control Problem Consider the nth-order dynamic system

$$\dot{x}(t)=Ax(t)+Bu(t)+Dw(t),\quad t\in[0,\infty), \tag{2.1}$$

with feedback law

$$u(t)=Kx(t), \tag{2.2}$$

and H_2 and H_∞ performance variables

$$z_2(t)=E_1x(t)+E_2u(t), \tag{2.3}$$

$$z_\infty(t)=E_{1\infty}x(t)+E_{2\infty}u(t) \tag{2.4}$$

Then determine $K\in\mathbb{R}^{m\times n}$ satisfying the following design criteria.

(i) the closed-loop system (2.1), (2.2) is asymptotically stable, i.e., $\tilde{A}\triangleq A+BK$ is Hurwitz,

(ii) for given $\gamma>0$, the $q_\infty\times d$ transfer function

$$G_\infty(s)\triangleq(E_{1\infty}+E_{2\infty}K)(sI_n-\tilde{A})^{-1}D \tag{2.5}$$

from disturbances $w(\cdot)$ to H_∞ performance variables z_∞ satisfies the H_∞-norm constraint

$$\|G_\infty\|_\infty<\gamma, \tag{2.6}$$

(iii) for $\mu\in[0,1]$ the cost functional

$$J(K)\triangleq\mu\|G_2\|_2^2+(1-\mu)I(G_\infty,\gamma) \tag{2.7}$$

is minimized, where

$$G_2(s)\triangleq(E_1+E_2K)(sI_n-\tilde{A})^{-1}D \tag{2.8}$$

is the $q\times d$ transfer function from disturbances w to H_2 performance variables z_2, and

$$I(G_\infty,\gamma)\triangleq-\lim_{s_0\to\infty}\left[\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty}\ln\left|\det\left(I_n-\gamma^{-2}G_\infty(\mathrm{j}\omega)G_\infty^*(\mathrm{j}\omega)\right)\right|\left[\frac{s_0^2}{s_0^2+\omega^2}\right]\mathrm{d}\omega\right] \tag{2.9}$$

is the entropy functional for the H_∞ performance variables z_∞.

Note that the problem statement involves both H_2 and H_∞ performance variables where for generality z_2 is not necessarily equal to z_∞. For convenience we omit H_2 and H_∞ cross weighting terms by assuming $E_1^{\mathrm{T}}E_2=0$ and $E_{1\infty}^{\mathrm{T}}E_{2\infty}=0$

As discussed in [5,6], the entropy functional (2.9) can be viewed as a measure of the distance from $\|G_\infty\|_\infty$ to γ. Like the H_2 norm, but unlike the H_∞ norm, however, the entropy $I(G_\infty,\gamma)$ accounts for $G_\infty(\mathrm{j}\omega)$ at all frequencies. Furthermore, it can be shown [2] that the entropy functional at infinity is equivalent to the exponential-of-quadratic cost of the Risk-Sensitive LQG Control Problem [8]

Remark 2.1. Note that (2.7) involves a convex combination of two scalar costs. By varying $\mu \in [0, 1]$, (2 7) can be viewed as the scalar representation of a multiobjective cost (see, e.g , [4]) By setting $\mu = 0$ we obtain an entropy/H_∞ control problem as in [1] However, it is important to stress that if $\mu = 1$ then the entropy functional is excluded from the cost functional (2.7) so that the optimization procedure is unable to enforce (2 9). In this case the bound (2.6) plays no role and the standard H_2 LQR problem is obtained. The practical value of this formulation is the case $\mu \approx 1$ in which the role of the entropy functional (2.9) is deemphasized and the optimization problem corresponds to minimizing the *actual* H_2 cost while *enforcing* the H_∞ constraint (2.6)

3. Reformulation of the control problem

In this section we reformulate the combined H_2/H_∞/Entropy Control Problem to facilitate the development of optimality conditions. First, we present a sufficient condition that enforces the disturbance attenuation constraint (2.6). For arbitrary $K \in \mathbb{R}^{m\times n}$ define the notation

$$\tilde{R} \triangleq R_1 + K^{\mathrm{T}}R_2K, \qquad \tilde{R}_\infty \triangleq R_{1\infty} + K^{\mathrm{T}}R_{2\infty}K, \qquad V \triangleq DD^{\mathrm{T}}.$$

Lemma 3.1. *Let $K \in \mathbb{R}^{m\times n}$ be given and assume there exists a nonnegative-definite matrix $\mathcal{Q} \in \mathbb{R}^{n\times n}$ satisfying*

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q} + V. \tag{3.1}$$

Then

$$(\tilde{A},\ D) \text{ is stabilizable} \tag{3.2}$$

if and only if

$$\tilde{A} \text{ is Hurwitz}. \tag{3.3}$$

In this case, the following statements hold

(i) *the transfer function G_∞ satisfies*

$$\| G_\infty \|_\infty \le \gamma; \tag{3.4}$$

(ii) *if $\| G_\infty \|_\infty < \gamma$ then*

$$I(G_\infty, \gamma) \le \operatorname{tr} \mathcal{Q}\tilde{R}_\infty; \tag{3.5}$$

(iii) *the transfer function G_2 is given by*

$$\| G_2 \|_2^2 = \operatorname{tr} Q\tilde{R}, \tag{3.6}$$

where the $n \times n$ matrix Q satisfies

$$0 = \tilde{A}Q + Q\tilde{A}^{\mathrm{T}} + V, \tag{3.7}$$

(iv) *the solution Q to (3.7) satisfies the bound*

$$Q \le \mathcal{Q} \tag{3.8}$$

and hence

$$\| G_2 \|_2^2 \le \operatorname{tr} \mathcal{Q}\tilde{R}; \tag{3.9}$$

(v) *all real symmetric solutions to (3.1) are nonnegative definite;*

(vi) *there exists a (unique) minimal solution to (3.1) in the class of real symmetric solutions;*

(vii) $\mathcal{Q}$ *is the minimal solution to* (3.1) *if and only if*

$$\operatorname{Re} \lambda_i(\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty) \leq 0; \tag{3.10}$$

(viii) $\| G_\infty \|_\infty < \gamma$ *if and only if* $\tilde{A} + \gamma^{-2}\mathcal{Q}\tilde{R}_\infty$ *is Hurwitz, where* $\mathcal{Q}$ *is the minimal solution to* (3.1);

(ix) *if* $\mathcal{Q}$ *is the minimal solution to* (3.1) *and* $\| G_\infty \|_\infty < \gamma$, *then*

$$I(G_\infty, \gamma) = \operatorname{tr} \mathcal{Q}\tilde{R}_\infty. \tag{3.11}$$

Proof The proof of (3.2)–(3.4) and (3.6)–(3.9) is similar to the proof of Lemma 2.1 given in [1] Assuming $\tilde{A}$ is Hurwitz, (v) follows by writing $\mathcal{Q} = \int_0^\infty e^{\tilde{A}t}[\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q} + V]\, e^{\tilde{A}^{\mathrm{T}}t}\, dt$. Result (vi) is given by Theorem 2.1 of [3], while (vii) follows from Theorem 2.1 of [3] and Theorem 2 of [7]. Statement (viii) follows from [6]. Finally, (ix) is given in [5,6], while (3.5) follows from (3.11). □

Remark 3.1. Consider the equalized weight case $z_2 = z_\infty$ so that $G_2 = G_\infty$. In this case it follows from (3.9) and (3.11) that

$$\| G_2 \|_2^2 \leq I(G_2, \gamma), \tag{3.12}$$

i.e., the entropy is an upper bound for the H_2 cost (see also [5,6]) If the H_∞ disturbance attenuation constraint is sufficiently relaxed, i.e., $\gamma \to \infty$, then it can be shown [5,6] that the entropy functional (2.9) coincides with the H_2 cost, i.e.,

$$I(G_2, \infty) = \| G_2 \|_2^2 = \operatorname{tr} Q\tilde{R} \tag{3.13}$$

Remark 3.2. The treatment of the entropy functional appears to be difficult when $\| G_\infty \|_\infty = \gamma$. This case was not considered in [6].

Lemma 3.1 shows that the H_∞ disturbance attenuation constraint is enforced when a nonnegative-definite solution to (3.1) is known to exist and $\tilde{A}$ is Hurwitz. Furthermore, all such solutions provide upper bounds for the H_2 performance $\| G_2 \|_2^2$ Also, if $\mathcal{Q}$ is the minimal solution to (3.1), then the entropy functional (2.9) is given by (3.11) Then, the combined H_2/H_∞/Entropy Control Problem can be recast as the following Auxiliary Optimization Problem We shall say $K \in \mathbb{R}^{m \times n}$ is *admissible* if $\tilde{A}$ is Hurwitz and $\| G_\infty \|_\infty < \gamma$.

Auxiliary Optimization Problem. For $\mu \in [0, 1]$, determine admissible $K \in \mathbb{R}^{m \times n}$ that minimizes

$$J(K) = \mu \operatorname{tr} Q\tilde{R} + (1 - \mu) \operatorname{tr} \mathcal{Q}\tilde{R}_\infty, \tag{3.14}$$

where Q, $\mathcal{Q} \geq 0$ satisfy (3.7) and (3.1)

4. Sufficient conditions for optimality

In this section we state sufficient conditions for characterizing full-state feedback controllers guaranteeing closed-loop stability and constrained H_∞ disturbance attenuation. For convenience in stating the main result we assume

$$R_2 = \alpha^2 \hat{R}_2, \qquad R_{2\infty} = \beta^2 \hat{R}_2, \tag{4.1}$$

where α, β are real numbers and $\hat{R} \in \mathbb{R}^{m \times m}$ is positive definite The general case in which (4.1) does not hold is discussed later in Remark 4.1. Also define

$$\Sigma \triangleq B\hat{R}_2^{-1}B^{\mathrm{T}} \tag{4.2}$$

Theorem 4.1. *Suppose there exist $n \times n$ nonnegative-definite matrices Q, P, $\mathcal{Q}$, $\mathcal{P}$ satisfying*

$$0=(A-\Sigma M)Q+Q(A-\Sigma M)^{\mathrm{T}}+V, \tag{4.3}$$

$$0=(A-\Sigma M)^{\mathrm{T}}P+P(A-\Sigma M)+\mu R_1+\mu\alpha^2 M^{\mathrm{T}}\Sigma M, \tag{4.4}$$

$$0=(A-\Sigma M)\mathcal{Q}+\mathcal{Q}(A-\Sigma M)^{\mathrm{T}}+\gamma^{-2}\mathcal{Q}R_{1\infty}\mathcal{Q}+\gamma^{-2}\beta^2\mathcal{Q}M^{\mathrm{T}}\Sigma M\mathcal{Q}+V, \tag{4.5}$$

$$0=\left(A-\Sigma M+\gamma^{-2}\mathcal{Q}\left[R_{1\infty}+\beta^2M^{\mathrm{T}}\Sigma M\right]\right)^{\mathrm{T}}\mathcal{P}+\mathcal{P}\left(A-\Sigma M+\gamma^{-2}\mathcal{Q}\left[R_{1\infty}+\beta^2M^{\mathrm{T}}\Sigma M\right]\right)$$
$$+(1-\mu)R_{1\infty}+(1-\mu)\beta^2M^{\mathrm{T}}\Sigma M, \tag{4.6}$$

and

$$\mu\alpha^2Q+(1-\mu)\beta^2\mathcal{Q}+\gamma^{-2}\beta^2\mathcal{Q}\mathcal{P}\mathcal{Q}>0, \tag{4.7}$$

where

$$M \triangleq (PQ+\mathcal{P}\mathcal{Q})\left(\mu\alpha^2Q+(1-\mu)\beta^2\mathcal{Q}+\gamma^{-2}\beta^2\mathcal{Q}\mathcal{P}\mathcal{Q}\right)^{-1}, \tag{4.8}$$

and let K be given by

$$K=-\hat{R}_2^{-1}B^{\mathrm{T}}M. \tag{4.9}$$

Then $(\tilde{A}, D)$ is stabilizable if and only if $\tilde{A}$ is Hurwitz In this case,

$$\|G_2\|_2^2=\operatorname{tr} Q\left(R_1+\alpha^2M^{\mathrm{T}}\Sigma M\right), \tag{4.10}$$

$$\|G_\infty\|_\infty \le \gamma, \tag{4.11}$$

and, if $\|G_\infty\|_\infty<\gamma$, then

$$I(G_\infty,\gamma)\le \operatorname{tr}\mathcal{Q}\left(R_{1\infty}+\beta^2M^{\mathrm{T}}\Sigma M\right). \tag{4.12}$$

If, in addition, $A-\Sigma M+\gamma^{-2}\mathcal{Q}(R_{1\infty}+\beta^2M^{\mathrm{T}}\Sigma M)$ is Hurwitz, then

$$I(G_\infty,\gamma)=\operatorname{tr}\mathcal{Q}\left(R_{1\infty}+\beta^2M^{\mathrm{T}}\Sigma M\right). \tag{4.13}$$

Proof. First we obtain necessary conditions for the Auxiliary Optimization Problem and then show, by construction, that these conditions serve as sufficient conditions for closed-loop stability and prespecified disturbance attenuation. Thus, to optimize (3.14) subject to (3.1) and (3.7), form the Lagrangian

$$\mathcal{L}(K,Q,\mathcal{Q},P,\mathcal{P}) \triangleq \operatorname{tr}\left[\lambda\left[\mu Q\tilde{R}+(1-\mu)\mathcal{Q}\tilde{R}_\infty\right]\right.$$
$$\left.+\left(\tilde{A}Q+Q\tilde{A}^{\mathrm{T}}+V\right)P+\left(\tilde{A}\mathcal{Q}+\mathcal{Q}\tilde{A}^{\mathrm{T}}+\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\mathcal{Q}+V\right)\mathcal{P}\right], \tag{4.14}$$

where the Lagrange multipliers $\lambda \ge 0$ and P, $\mathcal{P} \in \mathbb{R}^{n\times n}$ are not all zero. By viewing K, Q, and $\mathcal{Q}$ as independent variables, we obtain

$$\frac{\partial\mathcal{L}}{\partial Q}=\tilde{A}^{\mathrm{T}}P+P\tilde{A}+\lambda\mu\tilde{R}, \tag{4.15}$$

$$\frac{\partial\mathcal{L}}{\partial\mathcal{Q}}=\left(\tilde{A}+\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\right)^{\mathrm{T}}\mathcal{P}+\mathcal{P}\left(\tilde{A}+\gamma^{-2}\mathcal{Q}\tilde{R}_\infty\right)+\lambda(1-\mu)\tilde{R}_\infty \tag{4.16}$$

If both $\tilde{A}$ and $\tilde{A}+\gamma^{-2}\mathcal{Q}\tilde{R}_\infty$ are Hurwitz, then $\lambda=0$ implies $P=0$ and $\mathcal{P}=0$. Hence, it can be assumed without loss of generality that $\lambda=1$. Furthermore, note that P and $\mathcal{P}$ are nonnegative-definite. Thus the stationary conditions with $\lambda = 1$ are given by

$$\frac{\partial \mathscr{L}}{\partial Q} = \tilde{A}^{\mathrm{T}}P + P\tilde{A} + \mu\tilde{R} = 0, \tag{4.17}$$

$$\frac{\partial \mathscr{L}}{\partial \mathscr{Q}} = \left(\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}_\infty\right)^{\mathrm{T}}\mathscr{P} + \mathscr{P}\left(\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}_\infty\right) + (1-\mu)\tilde{R}_\infty = 0, \tag{4.18}$$

$$\frac{\partial \mathscr{L}}{\partial K} = \mu R_2 KQ + (1-\mu)R_{2\infty}K\mathscr{Q} + \gamma^{-2}R_{2\infty}K\mathscr{Q}\mathscr{P}\mathscr{Q} + B^{\mathrm{T}}(PQ + \mathscr{P}\mathscr{Q}) = 0. \tag{4.19}$$

Assuming (4.1), (4.19) implies (4.9). Next, with K given by (4.9), (4.3)–(4.6) are equivalent to (3.7), (4.17), (3.1), and (4.18), respectively. It now follows from Lemma 3.1 that the stabilizability condition is equivalent to the stability of $\tilde{A}$. In this case the H_∞ disturbance attenuation constraint (4.11) holds, the entropy is bounded as in (4.12), and the H_2 cost is given by (4.10). If, finally, $\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}_\infty$ is Hurwitz, then the entropy is given by (4.13), which is a restatement of (3.11). □

Remark 4.1. Condition (4.1) was assumed for convenience only. When (4.1) does not hold, K is given by

$$K = -\mathrm{vec}^{-1}\left\{\Omega^{-1}\,\mathrm{vec}\left[B^{\mathrm{T}}(PQ + \mathscr{P}\mathscr{Q})\right]\right\}, \tag{4.20}$$

where 'vec' denotes the column stacking operator, and Ω is defined by

$$\Omega \triangleq \mu R_2 \otimes Q + (1-\mu)R_{2\infty} \otimes \mathscr{Q} + \gamma^{-2}R_{2\infty} \otimes \mathscr{Q}\mathscr{P}\mathscr{Q}, \tag{4.21}$$

where $\otimes$ denotes Kronecker product. Since $\Omega \geq 0$, (4.20) is valid if $\Omega > 0$, which is a generalization of (4.7). When (4.1) does not hold, however, (4.3)–(4.6) cannot be used and must be replaced by (3.7), (4.17), (3.1) and (4.18), respectively.

5. Specializations of Theorem 4.1

To draw connections with the existing literature, a series of specializations of Theorem 4.1 is now given. We begin by considering the case of an entropy functional only, i.e., $\mu = 0$. In this case, set $R_1 = 0$, $\alpha = 0$ (i.e., $R_2 = 0$) so that (4.3) is superfluous and (4.4) implies $P = 0$. Furthermore, (4.9) becomes

$$K = -R_{2\infty}^{-1}B^{\mathrm{T}}\mathscr{P}S \tag{5.1}$$

and $\mathscr{Q}$, $\mathscr{P}$ satisfy

$$0 = (A - \Sigma_\infty\mathscr{P}S)\mathscr{Q} + \mathscr{Q}(A - \Sigma_\infty\mathscr{P}S)^{\mathrm{T}} + \gamma^{-2}\mathscr{Q}R_{1\infty}\mathscr{Q} + \gamma^{-2}\beta^2\mathscr{Q}S^{\mathrm{T}}\mathscr{P}\Sigma_\infty\mathscr{P}S\mathscr{Q} + V, \tag{5.2}$$

$$0 = \left(A + \gamma^{-2}\mathscr{Q}R_{1\infty}\right)^{\mathrm{T}}\mathscr{P} + \mathscr{P}\left(A + \gamma^{-2}\mathscr{Q}R_{1\infty}\right) + R_{1\infty} - S^{\mathrm{T}}\mathscr{P}\Sigma_\infty\mathscr{P}S, \tag{5.3}$$

where

$$S \triangleq \left(I_n + \gamma^{-2}\beta^2\mathscr{Q}\mathscr{P}\right)^{-1}, \tag{5.4}$$

$$\Sigma_\infty \triangleq BR_{2\infty}^{-1}B^{\mathrm{T}}. \tag{5.5}$$

Next, by introducing the transformation $Z = \mathscr{P}S = (\mathscr{P}^{-1} + \gamma^{-2}\beta^2\mathscr{Q})^{-1}$ and forming $Z[\mathscr{P}^{-1}(5.3)\mathscr{P}^{-1} + \beta^2\gamma^{-2}(4.2)]Z$, (5.1)–(5.3) collapse to

$$K = -R_{2\infty}^{-1}B^{\mathrm{T}}Z, \tag{5.6}$$

$$0 = A^{\mathrm{T}}Z + ZA + R_{1\infty} + \gamma^{-2}ZVZ - Z\Sigma_\infty Z, \tag{5.7}$$

which is the result given in [9]. Furthermore, it can be shown that

$$\| G_2 \|_2^2 \le I(G_\infty, \gamma) = \operatorname{tr} ZV. \tag{5.8}$$

Next, to recover the standard LQR result from Theorem 4.1, set $R_{1\infty} = 0$, $\beta = 0$ (i e., $R_{2\infty} = 0$) and $\mu = 1$ or, effectively (see Remark 2.1), $\gamma \to \infty$ In this case (4.3) and (4.5) are superfluous while (4.6) implies $\mathscr{P} = 0$. Furthermore, (4.9) becomes

$$K = -R_2^{-1} B^{\mathrm{T}} P, \tag{5.9}$$

where P satisfies the standard regulator Riccati equation

$$0 = A^{\mathrm{T}} P + PA + R_1 - P \Sigma_2 P, \tag{5.10}$$

where

$$\Sigma_2 \triangleq B R_2^{-1} B^{\mathrm{T}}. \tag{5.11}$$

Furthermore,

$$\| G_2 \|_2^2 = I(G_\infty, \infty) = \operatorname{tr} PV. \tag{5.12}$$

Note that in this case the H_∞ performance bound (2.6) is not enforced since the entropy functional is excluded from the optimality criterion.

Finally, it is important to point out a generalization of (5.1)–(5.3) Specifically, suppose as in [1] we seek to minimize an *overbound* on the H_2 cost while enforcing the disturbance attenuation constraint with performance variables $z_2 \ne z_\infty$, i.e., (3.14) replaced by tr $\mathscr{Q}\tilde{R}$ and $\mu = 0$ so that the actual H_2 cost is not considered Note that in this case tr $\mathscr{Q}\tilde{R}$ is not generally equal to $I(G_\infty, \gamma)$ and the entropy interpretation of the performance is no longer valid. In this case, (4.3)–(4.6) and (4.8) become

$$K = -\hat{R}_2^{-1} B^{\mathrm{T}} \mathscr{P} \hat{S}, \tag{5.13}$$

where $\mathscr{Q}$, $\mathscr{P}$ satisfy

$$0 = (A - \Sigma P \hat{S}) \mathscr{Q} + \mathscr{Q}(A - \Sigma \mathscr{P} \hat{S})^{\mathrm{T}} + \gamma^{-2} \mathscr{Q} R_{1\infty} \mathscr{Q} + \gamma^{-2} \beta^2 \mathscr{Q} \hat{S}^{\mathrm{T}} \mathscr{P} \Sigma \mathscr{P} \hat{S} \mathscr{Q} + V, \tag{5.14}$$

$$0 = \left(A + \gamma^{-2} \mathscr{Q} R_{1\infty}\right)^{\mathrm{T}} \mathscr{P} + \mathscr{P}\left(A + \gamma^{-2} \mathscr{Q} R_{1\infty}\right) + R_1 - \hat{S}^{\mathrm{T}} \mathscr{P} \Sigma \mathscr{P} \hat{S}, \tag{5.15}$$

and

$$\hat{S} \triangleq \left(\alpha^2 I_n + \gamma^{-2} \beta^2 \mathscr{Q} \mathscr{P}\right)^{-1}. \tag{5.16}$$

Furthermore,

$$\| G_2 \|_2^2 \le \operatorname{tr} \mathscr{Q}\left(R_1 + \hat{S}^{\mathrm{T}} \mathscr{P} \Sigma \mathscr{P} \hat{S}\right). \tag{5.17}$$

It is interesting to note that the full state feedback overbound H_2/H_∞ unequalized weights case involves two coupled equations, one modified Riccati equation, and one modified Lyapunov equation, unlike the entropy/H_∞ (equalized weights) case, which involves one modified Riccati equation given by (5 7)

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