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Adaptive stabilization of non-linear oscillators using direct adaptive control

JEONGHO HONG† and DENNIS S. BERNSTEIN†*

Direct adaptive controllers developed for linear systems are applied to non-linear oscillators. A wide range of non-linearities are considered, including stiffness non-linearities, input non-linearities, limit cycle oscillations and friction. Numerical results suggest that by increasing the speed of adaptation, these direct adaptive controllers are highly effective when applied to non-linear plants.

1. Introduction

The goal of both robust control and adaptive control is to achieve system performance without excessive reliance on plant models. While robust control seeks to desensitize a control system to plant uncertainty, the gains of a robust controller are fixed. On the other hand, an adaptive controller seeks to adjust controller gains during operation in order to permit greater uncertainty levels than can be tolerated by robust control and to improve system performance during operation, which is not possible with robust control.

This paper considers an output feedback adaptive stabilization problem with unknown constant disturbance rejection. Our results are closely related to those of Åström and Wittenmark (1995), Krstic *et al.* (1995), Ioannou and Sun (1996) and Kaufman *et al.* (1998) which focus on model reference adaptive control. The adaptive controller given by Theorem 1 requires that the disturbance satisfy a matching condition and that an output range condition be satisfied. This range condition is related to a positive real condition for the closed-loop system. Next we specialise this result in Corollary 1 and Corollary 2 to the case of full-state feedback, in which case the range condition is satisfied. By representing the system in controllable canonical form, we show that adaptive stabilization is possible without additional knowledge of the plant dynamics. However, this approach assumes that the sign of the input coefficient is known. If this assumption is violated then universal stabilization techniques are required (Ilchmann 1993).

The primary objective of the present paper is to apply the adaptive controller of Corollary 2 to non-linear systems. In particular, we consider non-linear oscillators possessing various non-linearities including stiffness non-linearities, input non-linearities, limit cycle oscillations and friction. As shown in the paper, the direct adaptive controller is remarkably effective in adaptively stabilizing these plants in spite of the broad range of non-linearities.

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* Author for correspondence. e-mail: dsbaero@umich.edu

† Department of Aerospace Engineering, Department of Electrical Engineering and Computer Science, The University of Michigan, Ann Arbor, MI 48109-2140, USA.

2. Adaptive stabilization with constant disturbance rejection

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + d \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

$$z(t) = Ex(t) \tag{3}$$

where $x(t) \in \Re^{n_x}, u(t) \in \Re^{n_u}, d \in \Re^{n_x}$, $y(t) \in \Re^{n_y}$ and $z(t) \in \Re^{n_z}$.

Theorem 1: *Assume there exists $K_s \in \Re^{n_u \times n_y}$ such that $A_s \triangleq A + BK_sC$ is asymptotically stable and assume there exists $\phi_s \in \Re^{n_u}$ such that $B\phi_s = d$. Let $R \in \Re^{n_x \times n_x}$ be positive semidefinite and assume (A_s, R) is controllable. Let $P \in \Re^{n_x \times n_x}$ be the positive-definite solution to the Lyapunov equation $0 = A_s^T P + PA_s + R$, and assume there exists $M \in \Re^{n_u \times n_z}$ such that $B^T P = ME$. Finally, let $\Gamma \in \Re^{n_u \times n_u}$ and $\Lambda \in \Re^{n_y \times n_y}$ be positive definite, and let $\lambda > 0$. Then* (2.1)–(2.3) *with the control law*

$$u(t) = K(t)y(t) + \phi(t) \tag{4}$$

where

$$\dot{K}(t) = -\Gamma M z(t) y^T(t) \Lambda \tag{5}$$

$$\dot{\phi}(t) = -\lambda M z(t) \tag{6}$$

yields $Rx(t) \to 0$ as $t \to \infty$.

Proof: Define

$$\hat{K}(t) \triangleq K(t) - K_s$$

$$\hat{\phi}(t) \triangleq \phi(t) + \phi_s$$

so that (5) and (6) imply

$$\dot{\hat{K}}(t) = -\Gamma M z(t) y^T(t) \Lambda \tag{7}$$

$$\dot{\hat{\phi}}(t) = -\lambda M z(t) \tag{8}$$

Then the closed-loop system consists of (7) and (8) and

$$\dot{x}(t) = (A_s + B\hat{K}(t)C)x(t) + B\hat{\phi}(t) \tag{9}$$

Next, consider the positive-definite Lyapunov candidate

$$V(x, \hat{K}, \hat{\phi}) = x^T P x + \operatorname{tr} \Gamma^{-1}\hat{K}\Lambda^{-1}\hat{K}^T + \operatorname{tr} \hat{\phi}\lambda^{-1}\hat{\phi}^T$$

The derivative of V along trajectories of the closed-loop system is given by

$$\begin{aligned}
\dot{V}(x, \hat{K}, \hat{\phi}) &= x^T(A_s^T P + PA_s)x + 2x^T PB\hat{K}Cx \\
&\quad + 2x^T PB\hat{\phi} + 2\operatorname{tr} \Gamma^{-1}\hat{K}\Lambda^{-1}\dot{\hat{K}}^T \\
&\quad + 2\operatorname{tr} \hat{\phi}\lambda^{-1}\dot{\hat{\phi}}^T \\
&= -x^T Rx + 2\operatorname{tr} \hat{K}(\Lambda^{-1}\dot{\hat{K}}^T\Gamma^{-1} + Cxx^T PB) \\
&\quad + 2\operatorname{tr} \hat{\phi}(\lambda^{-1}\dot{\hat{\phi}}^T + x^T PB) \\
&= -x^T Rx + 2\operatorname{tr} \hat{K}(\Lambda^{-1}\dot{\hat{K}}^T\Gamma^{-1} + Cxx^T E^T M^T) \\
&\quad + 2\operatorname{tr} \phi(\lambda^{-1}\dot{\hat{\phi}}^T + x^T E^T M^T) \\
&= -x^T Rx + 2\operatorname{tr} \hat{K}(\Lambda^{-1}\dot{\hat{K}}^T\Gamma^{-1} + yz^T M^T) \\
&\quad + 2\operatorname{tr} \phi(\lambda^{-1}\dot{\hat{\phi}}^T + z^T M^T) \\
&= -x^T Rx
\end{aligned}$$

It now follows from Theorem 4.4 in Khalil (1996) that, for every initial condition $x(0)$, $\hat{K}(0)$ and $\hat{\phi}(0)$, the states of the closed-loop system are bounded, and $x^T(t)Rx(t) \to 0$ as $t \to \infty$. Since R is positive semidefinite, it follows that $Rx(t) \to 0$ as $t \to \infty$. □

Theorem 1 requires that there exist K_s and ϕ_s such that $A_s = A + BK_sC$ is asymptotically stable and $B\phi_s = d$. However, the control law (4)–(6) does not require explicit knowledge of K_s, ϕ_s and d. On the other hand, implementation of (5) and (6) requires that there exist a known matrix M such that $B^T P = ME$. This condition and the Lyapunov equation $0 = A_s^T P + PA_s + R$ are KYP conditions that are equivalent to the assumption that (A_s, B, ME) is the realization of a positive real transfer function.

Note that (6) is an integrator state which serves to reject the constant disturbance d.

Next, we specialise Theorem 1 to the full-state feedback case. In this case $C = E = I$ so that the assumptions of Theorem 1 are satisfied with $M = B^T P$.

Corollary 1: *Assume there exists $K_s \in \Re^{n_u \times n_x}$ such that $A_s \triangleq A + BK_s$ is asymptotically stable and assume there exists $\phi_s \in \Re^{n_u}$ such that $B\phi_s = d$. Let $R \in \Re^{n_x \times n_x}$ be positive semidefinite and assume (A_s, R) is controllable. Let $P \in \Re^{n_x \times n_x}$ be the positive-definite solution to the Lyapunov equation $0 = A_s^T P + PA_s + R$. Finally, let $\Gamma \in \Re^{n_u \times n_u}$ and $\Lambda \in \Re^{n_x \times n_x}$ be positive definite, and let $\lambda > 0$. Then* (1) *with the control law*

$$u(t) = K(t)x(t) + \phi(t) \tag{10}$$

where

$$\dot{K}(t) = -\Gamma B^T Px(t)x^T(t)\Lambda \tag{11}$$

$$\dot{\phi}(t) = -B^T Px(t)\lambda \tag{12}$$

yields $Rx(t) \to 0$ as $t \to \infty$.

3. State feedback for uncertain systems

Consider the linear system (1) with

$$A = \begin{bmatrix} A_0 \\ a \end{bmatrix} \qquad B = \begin{bmatrix} 0_{(n_x-1)\times 1} \\ b \end{bmatrix} \qquad d = \begin{bmatrix} 0_{(n_x-1)\times 1} \\ d_0 \end{bmatrix} \tag{13}$$

where $x(t) \in \Re^{n_x}$, $u(t) \in \Re$, $d \in \Re^{n_x}$, $A_0 \in \Re^{(n_x-1)\times n_x}$, $a \in \Re^{1\times n_x}$, b, $d_0 \in \Re$ and $b \neq 0$. Define

$$B_0 \triangleq \begin{bmatrix} 0_{(n_x-1)\times 1} \\ \operatorname{sign} b \end{bmatrix}$$

Corollary 2: *Assume there exists $K_s \in \Re^{1\times n_x}$ such that $A_s \triangleq A + BK_s$ is asymptotically stable. Let $R \in \Re^{n_x \times n_x}$ be positive semidefinite and assume (A_s, R) is controllable. Let $P \in \Re^{n_x \times n_x}$ be the positive-definite solution to the Lyapunov equation $0 = A_s^T P + PA_s + R$. Finally, let $\Gamma > 0$ and $\lambda > 0$ and let $\Lambda \in \Re^{n_x \times n_x}$ be positive definite. Then* (1) *with the control law*

$$u(t) = K(t)x(t) + \phi(t) \tag{14}$$

where

$$\dot{K}(t) = -\Gamma B_0^T Px(t)x^T(t)\Lambda \tag{15}$$

$$\dot{\phi}(t) = -B_0^T Px(t)\lambda \tag{16}$$

yields $Rx(t) \to 0$ as $t \to \infty$.

Proof: First, note that because of the structure of B and d, it follows that $\phi_s = d_0/b$ satisfies $B\phi_s = d$. Second, since Λ and λ are arbitrary, Λ in (11) and λ in (12) can be replaced by $|b|^{-1}\Lambda$ and $|b|^{-1}\lambda$, respectively. Thus, (11) and (12) imply (15) and (16). □

Note that (15) and (16) require the solution P of the Lyapunov equation $0 = A_s^T P + PA_s + R$. Since $b \neq 0$, let $K_s = (1/b)(a_s - a)$, where $a_s \in \Re^{1\times n_x}$. It thus follows that

$$A_s = A + BK_s = \begin{bmatrix} A_0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \frac{1}{b}(a_s - a) = \begin{bmatrix} A_0 \\ a_s \end{bmatrix}$$

Since a_s can be chosen to stabilize A_s without knowledge of either a or b, it follows that P can be determined without knowledge of either a or b. However, sign b must be known in order to implement (15) and (16).

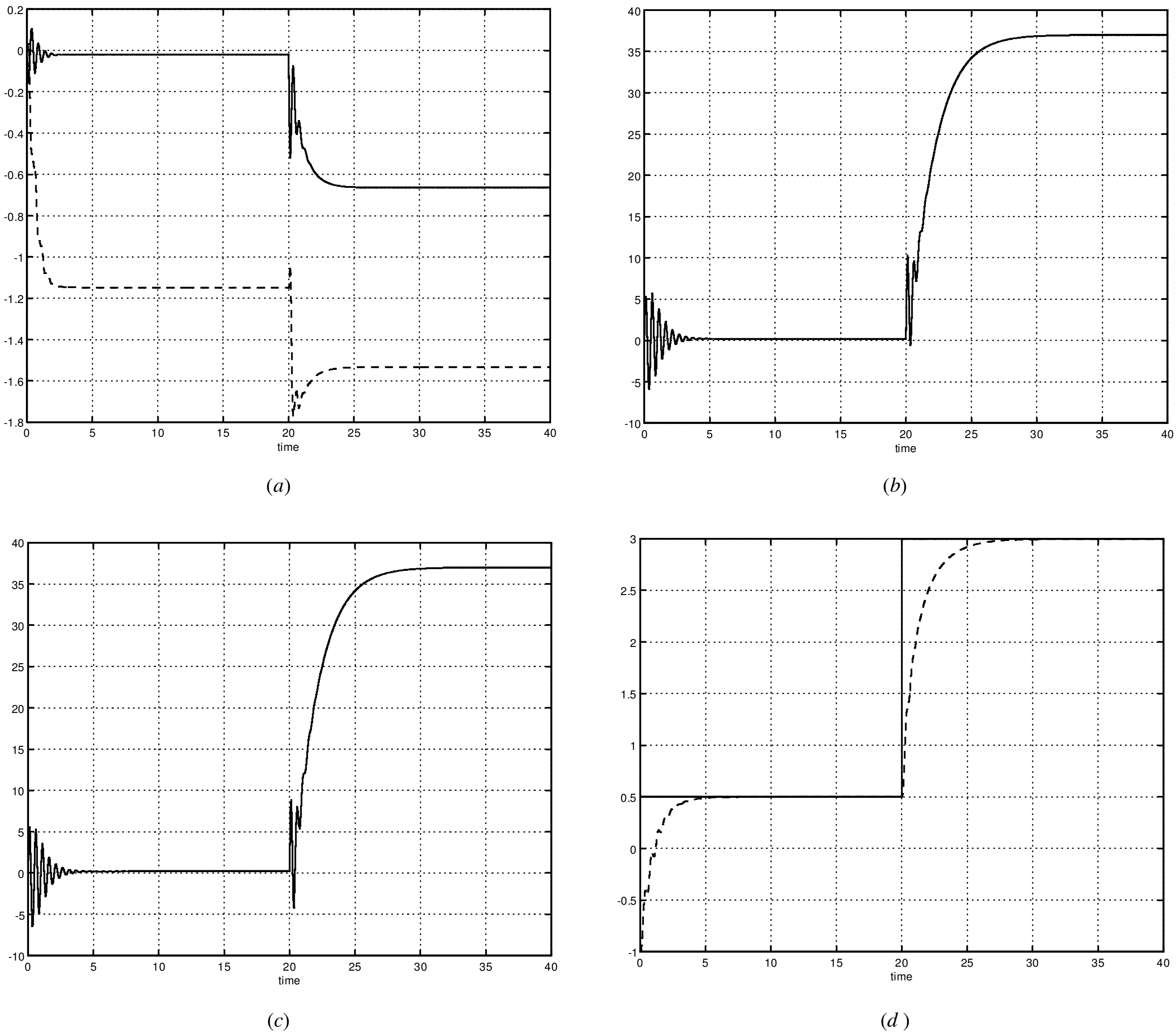

Figure 1. Adaptive control of a non-linear oscillator with hardening spring: (*a*) K_1 (solid), K_2 (dashed), (*b*) ϕ, (*c*) u, (*d*) $r_{\rm des}$ (solid), r (dashed).

To illustrate Corollary 2, consider the case $n_x = 1$ and let $a_{\rm s} < 0$ and $R = -2a_{\rm s}$. Then $P = 1$, and (15) and (16) are given by

$$\dot{K}(t) = -(\text{sign } b)\lambda_1 x^2(t) \tag{17}$$

$$\dot{\phi}(t) = -(\text{sign } b)\lambda_2 x(t) \tag{18}$$

where $\lambda_1 \triangleq \Lambda/\Gamma$ and $\lambda_2 \triangleq \lambda$. Note that (17) and (18) yield $x(t) \to 0$ as $t \to \infty$ for all $\lambda_1, \lambda_2 > 0$.

Next, consider the case $n_x = 2$, and let $A_0 = [0 \;\; 1]$, $p > 0$, $a_{\rm s1} < 0$, $a_{\rm s2} < -p$ and

$$R = \begin{bmatrix} -2pa_{\rm s1} & 0 \\ 0 & -2p - 2a_{\rm s2} \end{bmatrix}$$

Then

$$P = \begin{bmatrix} -pa_{\rm s2} - a_{\rm s1} & p \\ p & 1 \end{bmatrix}$$

satisfies $0 = A_{\rm s}^{\rm T} P + PA_{\rm s} + R$ and (15) and (16) are given by

$$\dot{K}_1(t) = -(\text{sign } b)[\lambda_1 p x_1^2(t) + (\lambda_1 + \lambda_{12} p) x_1(t) x_2(t) + \lambda_{12} x_2^2(t)] \tag{19}$$

$$\dot{K}_2(t) = -(\text{sign } b)[\lambda_{12} p x_1^2(t) + (\lambda_{12} + \lambda_2 p) x_1(t) x_2(t) + \lambda_2 x_2^2(t)] \tag{20}$$

$$\dot{\phi}(t) = -(\text{sign } b)\lambda_3 [p x_1(t) + x_2(t)] \tag{21}$$

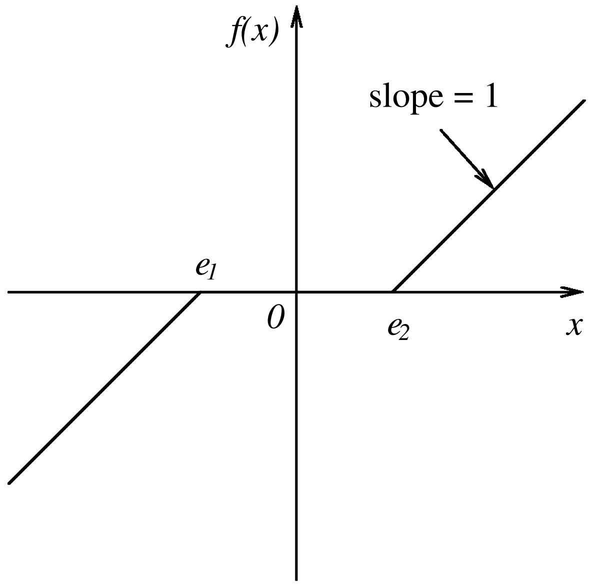

Figure 2. Deadzone non-linearity.

where

$$\begin{bmatrix} \lambda_1 & \lambda_{12} \\ \lambda_{12} & \lambda_2 \end{bmatrix} \triangleq (1/\Gamma)\Lambda$$

is positive definite and $\lambda_3 \triangleq \lambda > 0$. Note that (19)–(21) yield $x(t) \to 0$ as $t \to \infty$ for all $p, \lambda_1, \lambda_2, \lambda_3 > 0$ and for all λ_{12} such that $\lambda_{12}^2 < \lambda_1\lambda_2$. Setting $\lambda_{12} = 0$ for simplicity yields

$$\dot{K}_1(t) = -(\text{sign } b)\lambda_1[px_1^2(t) + x_1(t)x_2(t)] \quad (22)$$

$$\dot{K}_2(t) = -(\text{sign } b)\lambda_2[x_2^2(t) + px_1(t)x_2(t)] \quad (23)$$

$$\dot{\phi}(t) = -(\text{sign } b)\lambda_3[px_1(t) + x_2(t)] \quad (24)$$

Corollary 2 can be applied to the following step command problem. Consider the nth-order linear system

$$r^{(n)}(t) - a_n r^{(n-1)}(t) - \cdots - a_2\dot{r}(t) - a_1 r(t) = bu(t) \quad (25)$$

with the requirement that $r(t)$ approach r_{des} without knowledge of $a = [a_1 \ \cdots \ a_n]$ and b, except the sign of b. Defining the error signal $x_1(t) \triangleq r(t) - r_{\text{des}}$ and the state $x = [x_1 \ \dot{x}_1 \ \cdots \ x_1^{(n-1)}]^{\text{T}}$, (25) becomes

$$\dot{x}(t) = Ax(t) + Bu(t) + \begin{bmatrix} 0_{(n-1)\times 1} \\ d_0 \end{bmatrix} \quad (26)$$

where

$$A = \begin{bmatrix} A_0 \\ a \end{bmatrix}$$

and $d_0 = -a_1 r_{\text{des}}$, where $A_0 = [0_{(n_x-1)\times 1} \ I_{n_x-1}]$, $a = [a_1 \ \cdots \ a_n]$, and I_{n_x-1} is the $(n_x - 1) \times (n_x - 1)$ identity matrix. Thus (26) has the form (1) and (13). Note that d_0 is unknown since a_1 is unknown. Thus the controller (14)–(16) can be used for this problem.

4. Non-linear stiffness

Consider an oscillator with non-linear stiffness modelled by

$$\ddot{r}(t) + c\dot{r}(t) + f(r(t)) = bu(t) \quad (27)$$

where $f: \Re \to \Re$. The control objective is to require $r(t)$ to approach r_{des} without knowledge of $c, f(\cdot)$, and b, except the sign of b which is taken to be positive. If $c > 0$ and $rf(r) > 0$ for all $r \in \Re$, then (27) is a stable oscillator. However, we do not invoke these assumptions. Defining the error signal $x_1(t) \triangleq r(t) - r_{\text{des}}$ and the state $x \triangleq [x_1 \ \dot{x}_1]^{\text{T}}$, equation (27) becomes

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\hat{f}(x_1(t)) & -c \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ d_0 \end{bmatrix} \quad (28)$$

where $\hat{f}(x_1)x_1 = f(x_1 + r_{\text{des}}) - f(r_{\text{des}})$ and $d_0 = -f(r_{\text{des}})$. Equation (28) is of the form (26) with a_1 replaced by the unknown state-dependent coefficient $-\hat{f}(x_1)$ and $a_2 = -c$. The controller (14) with (22)–(24) is applied to this problem.

First we let $f(\cdot)$ be a hardening spring modelled by

$$f(r) = k_1 r + k_3 r^3 \quad (29)$$

where $k_1 > 0$ and $k_3 > 0$. In this case $\hat{f}(\cdot)$ and d_0 are given by

$$\hat{f}(x_1) = k_1 + k_3(x_1^2 + 3r_{\text{des}}x_1 + 3r_{\text{des}}^2)$$

$$d_0 = -k_1 r_{\text{des}} - k_3 r_{\text{des}}^3$$

We apply controller (14) with (22)–(24) and $k_1 = 1$, $c = 0.2$, $k_3 = 1$, $b = 3$, and $r_{\text{des}} = 0.5$. Let $r(0) = -1$, $\dot{r}(0) = 0$, $K_1(0) = 0$, $K_2(0) = 0$, $\phi(0) = 0$ and choose adaptation weights $p = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 50$. Furthermore, at $t = 20$, k_3 is changed from 1 to 4, and r_{des} is changed from 0.5 to 3.

Next, we let $f(\cdot)$ be the deadzone function shown in figure 2. We apply controller (14) with (22)–(24) and with $e_1 = -0.5$, $e_2 = 0.5$, $c = 1$, $b = 5$, and $r_{\text{des}} = 0$. Let $r(0) = 0.1$, $\dot{r}(0) = 0.2$, $K_1(0) = 0$, $K_2(0) = 0$, $\phi(0) = 0$ and choose adaptation weights $p = 1$, $\lambda_1 = 10$, $\lambda_2 = 10$, $\lambda_3 = 10$. Initially, the adaptation is stopped. As can be seen from figure 3, $r(t)$ approaches 0.3 due to the deadzone. At $t = 10$, the adaptation is started, and, as can be seen from figure 3, $r(t)$ approaches 0.

Next, we let $f(\cdot)$ be the relay function shown in figure 4. We apply controller (14) with (22)–(24) and with $c = 1$, $b = 5$, and $r_{\text{des}} = 0$. Let $r(0) = 0.6$, $\dot{r}(0) = 0.9$, $K_1(0) = 0$, $K_2(0) = 0$, $\phi(0) = 0$ and choose adaptation weights $p = 1$, $\lambda_1 = 100$, $\lambda_2 = 100$, $\lambda_3 = 100$. Initially, the adaptation is stopped. As can be seen from the dashed line in figure 5, the phase plot

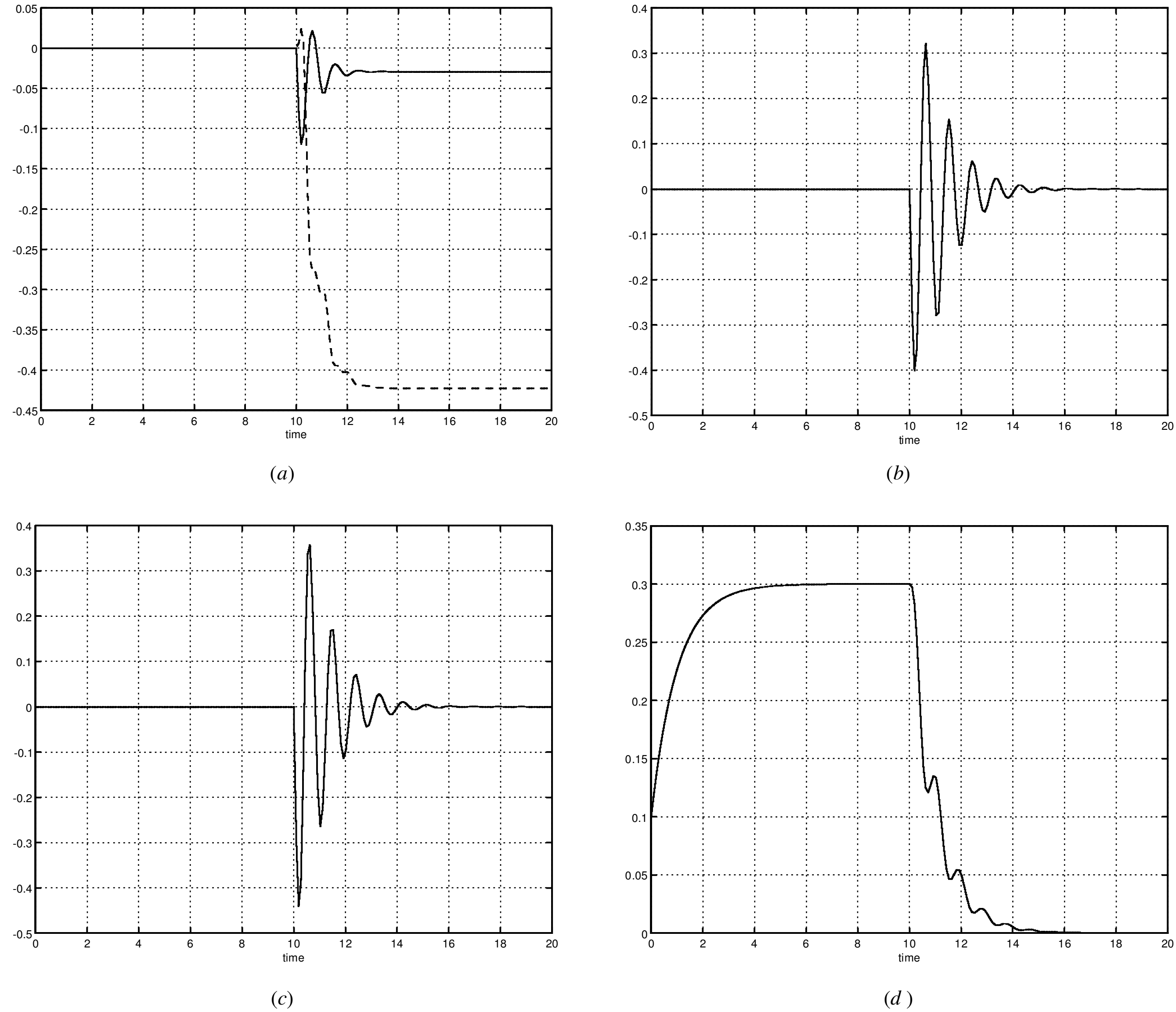

Figure 3. Adaptive control of a non-linear oscillator with deadzone in stiffness: (*a*) K_1 (solid), K_2 (dashed), (*b*) ϕ, (*c*) u, (*d*) r.

shows a limit cycle. At $t = 30$, the adaptation is started, and, as can be seen from the solid line in figure 5, $r(t)$ and $\dot{r}(t)$ approach 0.

Finally, we let $f(\cdot)$ be the backlash/hysteresis function shown in figure 6. We apply controller (14) with (22)–(24) and with $h = 1$, $c = 1$, $b = 5$ and $r_{\rm des} = 0$. Let $r(0) = 0.6$, $\dot{r}(0) = 0.9$, $K_1(0) = 0$, $K_2(0) = 0$, $\phi(0) = 0$ and choose adaptation weights $p = 1$, $\lambda_1 = 100$, $\lambda_2 = 100$, $\lambda_3 = 100$. Initially, the adaptation is stopped. As can be seen from the dashed line in figure 7, the phase plot shows a limit cycle. At $t = 30$, the adaptation is started and, as can be seen from the solid line in figure 7, $r(t)$ and $\dot{r}(t)$ approach 0.

As can be seen from figures 1, 3, 5 and 7, $r(t)$ tracks $r_{\rm des}$, that is, the controller (14) with (22)–(24) is able to compensate for the non-linear stiffness in (27).

5. Non-linear damping

Consider an oscillator with position-dependent damping modelled by

$$\ddot{r}(t) + g(r(t))\dot{r}(t) + kr(t) = bu(t) \tag{30}$$

where $g: \Re \to \Re$. The control objective is to require $r(t)$ to approach $r_{\rm des}$ without knowledge of $g(\cdot), k$ and b, except the sign of b, which is taken to be positive. Defining the error signal $x_1(t) \triangleq r(t) - r_{\rm des}$ and the state $x \triangleq [x_1 \ \dot{x}_1]^{\rm T}$, equation (30) becomes

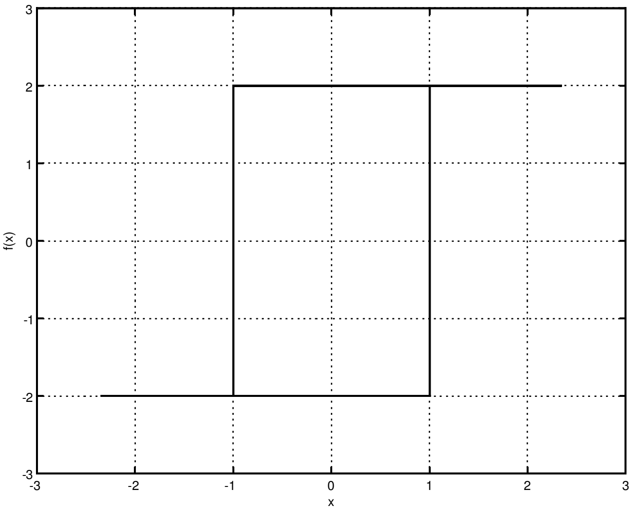

Figure 4. Relay non-linearity.

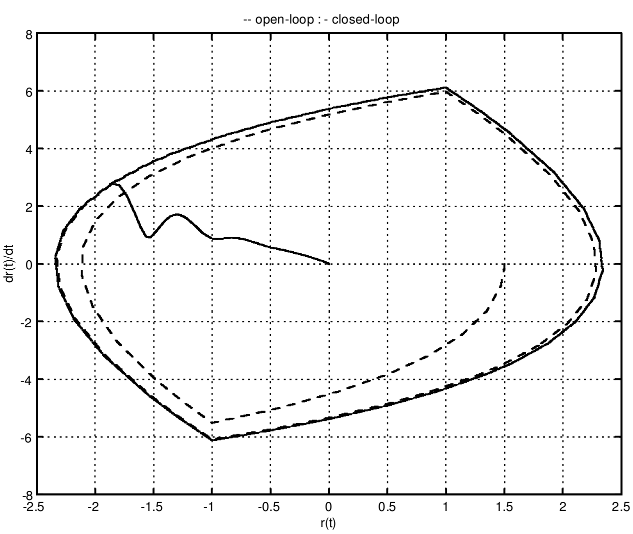

Figure 5. Adaptive control of a non-linear oscillator with relay in stiffness.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k & -g(x_1(t) + r_{\mathrm{des}}) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -k r_{\mathrm{des}} \end{bmatrix} \quad (31)$$

Equation (31) has the form (26) with $a_1 = -k$, a_2 replaced by the unknown state-dependent coefficient $-g(x_1 + r_{\mathrm{des}})$, and d_0 replaced by the unknown constant $-k r_{\mathrm{des}}$. The controller (14) with (22)–(24) is applied to this problem.

As a special case, consider the Van der Pol equation modelled by

$$g(r) = \varepsilon(1 - r^2) \quad (32)$$

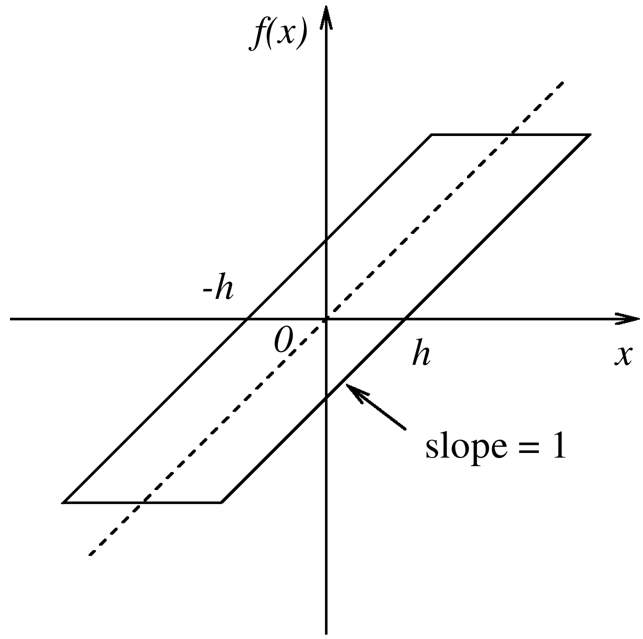

Figure 6. Backlash/hysteresis non-linearity.

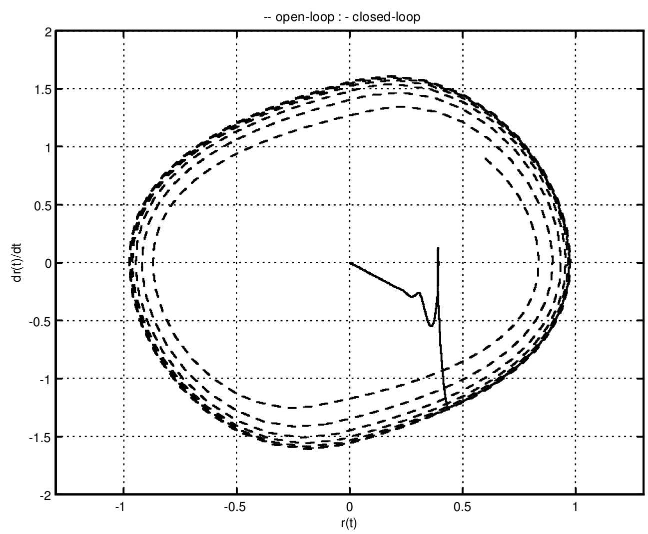

Figure 7. Adaptive control of a non-linear oscillator with backlash/hysteresis in stiffness.

with $k = 1$ and $b = 1$. We apply controller (14) with (22)–(24) and with $\varepsilon = 0.4$ and $r_{\mathrm{des}} = 0$. Let $r(0) = 1$, $\dot{r}(0) = 0$, $K_1(0) = 0$, $K_2(0) = 0$, $\phi(0) = 0$ and choose adaptation weights $p = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$. Initially, the adaptation is stopped. As can be seen from the dashed line in figure 8, the phase plot shows a limit cycle. At $t = 20$, the adaptation is started and, as can be seen from the solid line in figure 8, $r(t)$ and $\dot{r}(t)$ approach 0.

Next, we consider a time-varying command $r_{\mathrm{des}}(t)$ and define the error signal $x_1(t) \triangleq r(t) - r_{\mathrm{des}}(t)$ and the state $x \triangleq [x_1 \;\; \dot{x}_1]^{\mathrm{T}}$. Then equation (30) becomes

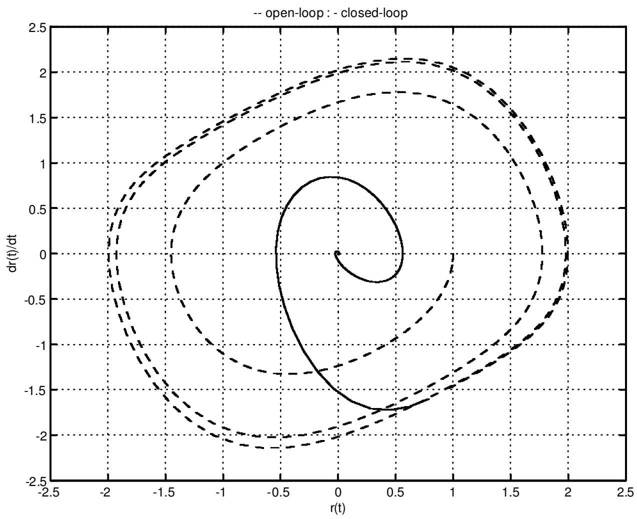

Figure 8. Adaptive control of Van der Pol's oscillator with $r_{\text{des}} = 0$.

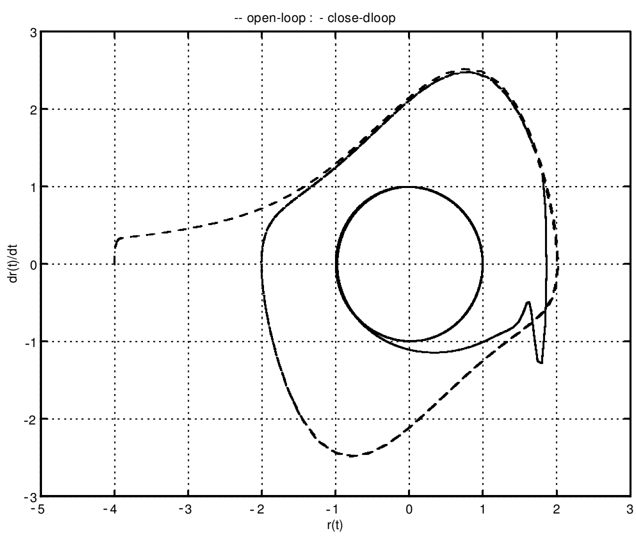

Figure 9. Adaptive command following for Van der Pol's oscillator with $r_{\text{des}}(t) = \sin \omega t$.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -(k + \hat{g}(x_1(t))\dot{r}_{\text{des}}(t)) & -g(x_1(t) + r_{\text{des}}(t)) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ d_0(t) \end{bmatrix} \quad (33)$$

where $\hat{g}(x_1)x_1 = g(x_1 + r_{\text{des}}) - g(r_{\text{des}})$ with $d_0(t) = -\ddot{r}_{\text{des}}(t) - g(r_{\text{des}}(t))\dot{r}_{\text{des}}(t) - kr_{\text{des}}(t)$. Equation (33) has the form (26) with a_1 and a_2 replaced by unknown state-dependent, time-varying coefficients and with time-varying $d_0(t)$. The controller (14) with (22)–(24) is applied to this problem.

As a special case, consider the Van der Pol equation (32) with the time-varying command $\sin \omega t$ and $k = 1$ and $b = 1$. Then $\hat{g}(\cdot)$ and d_0 are given by

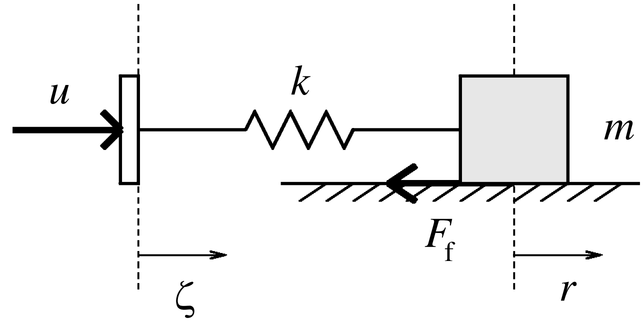

Figure 10. Mass-spring system with stick-slip friction.

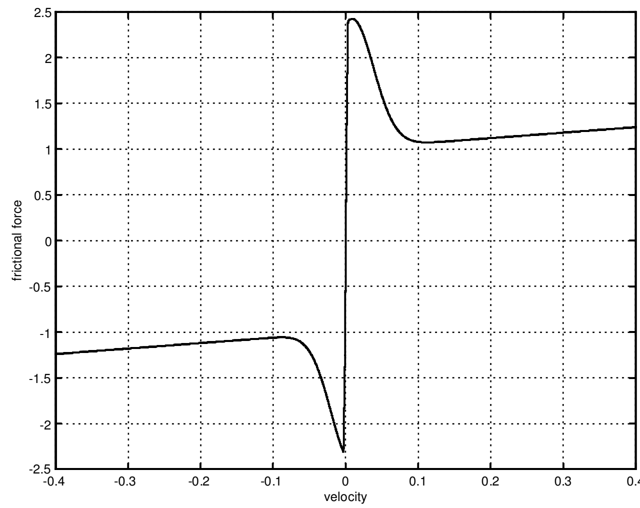

Figure 11. Stick-slip friction versus velocity.

$$\hat{g}(x_1(t)) = -\varepsilon(x_1(t) + 2\sin \omega t)$$

$$d_0(t) = \omega^2 \sin \omega t - \varepsilon\omega \cos^3 \omega t - k \sin \omega t$$

We apply controller (14) with (22)–(24) and with $\varepsilon = 0.8$ and $\omega = 1$. Let $r(0) = -5$, $\dot{r}(0) = 0$, $K_1(0) = 0$, $K_2(0) = 0$, $\phi(0) = 0$ and choose adaptation weights $p = 1$, $\lambda_1 = 100$, $\lambda_2 = 100$, $\lambda_3 = 100$. Initially, the adaptation is stopped. As can be seen from the dashed line in figure 9, the phase plot shows the Van der Pol limit cycle. At $t = 20$, the adaptation is started and, as can be seen from the solid line in figure 9, the phase plot shows a change from the Van der Pol limit cycle to a circle.

6. Stick-slip friction

The equations of motion for the mass-spring system shown in figure 10 are given by

$$m\ddot{r}(t) + kr(t) = k\zeta(t) + kL - F_f(t) \quad (34)$$

$$k\zeta(t) = kr(t) - kL + u(t) \quad (35)$$

where m, $k > 0$, L is the distance between the mass and the massless bar when the spring is relaxed and the stick-

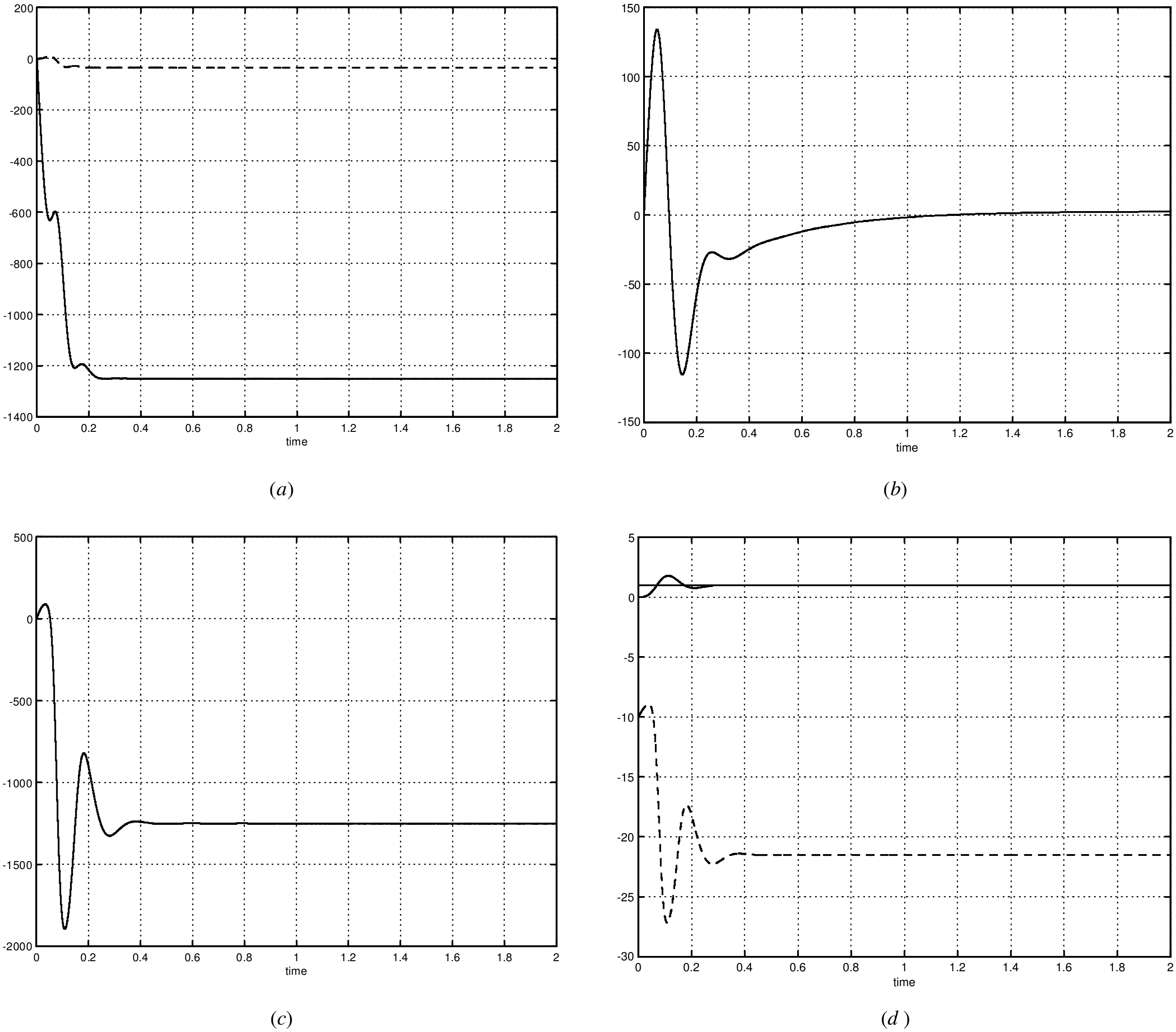

Figure 12. Adaptive control of a mass-spring system with stick-slip friction: (*a*) K_1 (solid), K_2 (dashed), (*b*) ϕ, (*c*) u, (*d*) command = 1, r (solid), ζ (dashed).

slip frictional force $F_f(t)$ is given by (Canudas de Wit *et al.* 1993)

$$F_f(t) = (1 - \kappa(t))F_s(t) + \kappa(t)F_d(t) \tag{36}$$

The stick friction $F_s(t)$ is given by

$$F_s(t) = \text{sat}_{(\alpha_0+\alpha_1)}\left(k_s\eta(t) + d_s\dot{r}(t)\right) \tag{37}$$

and the dynamic friction $F_d(t)$ is given by

$$F_d(t) = \alpha_0 \,\text{sgn}\left(\dot{r}(t)\right) + \alpha_2\dot{r}(t) \tag{38}$$

where α_0, α_1, α_2, k_s, $d_s > 0$

$$\tau_\kappa\dot{\kappa}(t) = -\kappa(t) + 1 - \mathrm{e}^{-(\dot{r}(t)/\dot{r}(0))^2} \tag{39}$$

and

$$\dot{\eta}(t) = (1 - \kappa(t))\dot{x}(t) - \kappa(t)\frac{1}{\tau_r}\eta(t) \tag{40}$$

where τ_κ, $\tau_r > 0$. The sat function is defined by

$$\text{sat}_\alpha(\eta) = \begin{cases} \alpha & \text{if} \quad \eta \geq \alpha \\ \eta & \text{if} \quad |\eta| < \alpha \\ -\alpha & \text{if} \quad \eta \leq \alpha \end{cases} \tag{41}$$

As can be seen in figure 11, the magnitude of stick friction, which affects the initial movement of the mass, is greater than the magnitude of the slip friction, which is the frictional force when the mass is moving. By

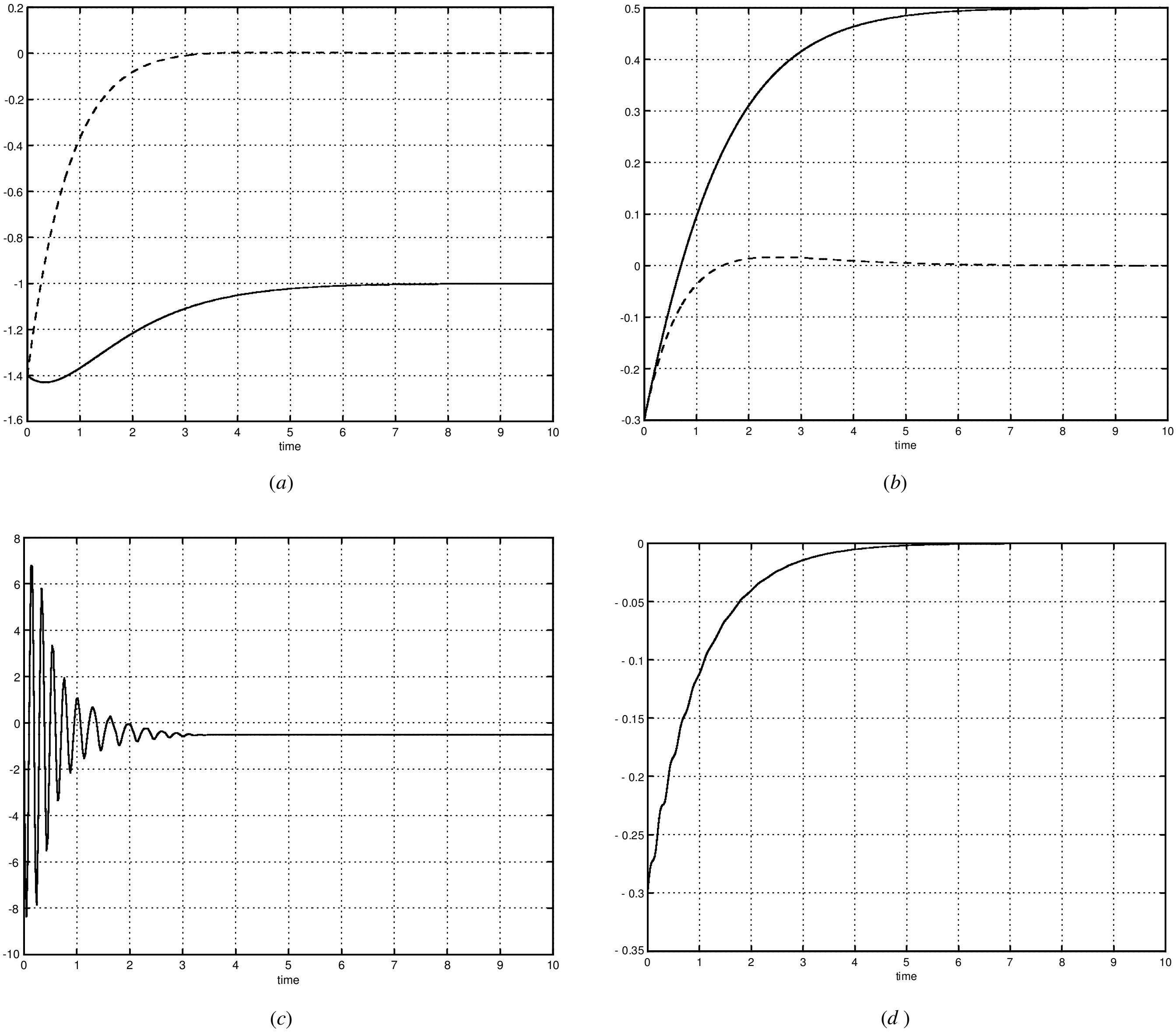

Figure 13. Adaptive control of a non-linear oscillator with deadzone in the input path: linear controller: (*a*) u, (*b*) r without deadzone (dashed), with deadzone (solid); adaptive controller: (*c*) u, (*d*) r.

defining the error signal $x_1(t) \triangleq r(t) - r_{\rm des}$ and eliminating the internal physical variable $\zeta(t)$, (34) becomes

$$\dot{x}_2(t) = \ddot{x}_1(t) = \frac{1}{m}u(t) - \frac{1}{m}F_{\rm f}(t) \qquad (42)$$

The controller (14) with (22)–(24) is applied to this problem.

Figure 12 shows the response of the mass-spring system with stick-slip friction with $m = 1$, $k = 100$, $L = 10$, $r_{\rm des} = 1$, $\alpha_0 = 1$, $\alpha_1 = 1.5$, $\alpha_2 = 0.6$, $\tau_\kappa = 0.01$, $\tau_r = 0.001$, $k_{\rm s} = 10\,000$, and $d_{\rm s} = 1100$. Let $r(t) = 0$, $\dot{r}(0) = 0.04$, $K_1(0) = 0$, $K_2(0) = 0$, $\phi(0) = 0$ and choose adaptation weights $p = 40$, $\lambda_1 = 500$, $\lambda_2 = 1$, $\lambda_3 = 100$. As can be seen in figure 12(*d*), $r(t)$ approaches the commanded position. However, due to stick friction, figure 12(*d*) shows overshoot at the beginning of control. A critical aspect is the distance between the mass and the massless bar, which has increased due to the control action.

7. Input non-linearity

Consider an oscillator with input non-linearity modelled by the Hammerstein system

$$\ddot{r}(t) + c\dot{r}(t) + kr(t) = bf(u(t)) \qquad (43)$$

The control objective is to require $r(t)$ to approach $r_{\rm des}$ without knowledge of c, k, $f(\cdot)$ and b, except the sign of

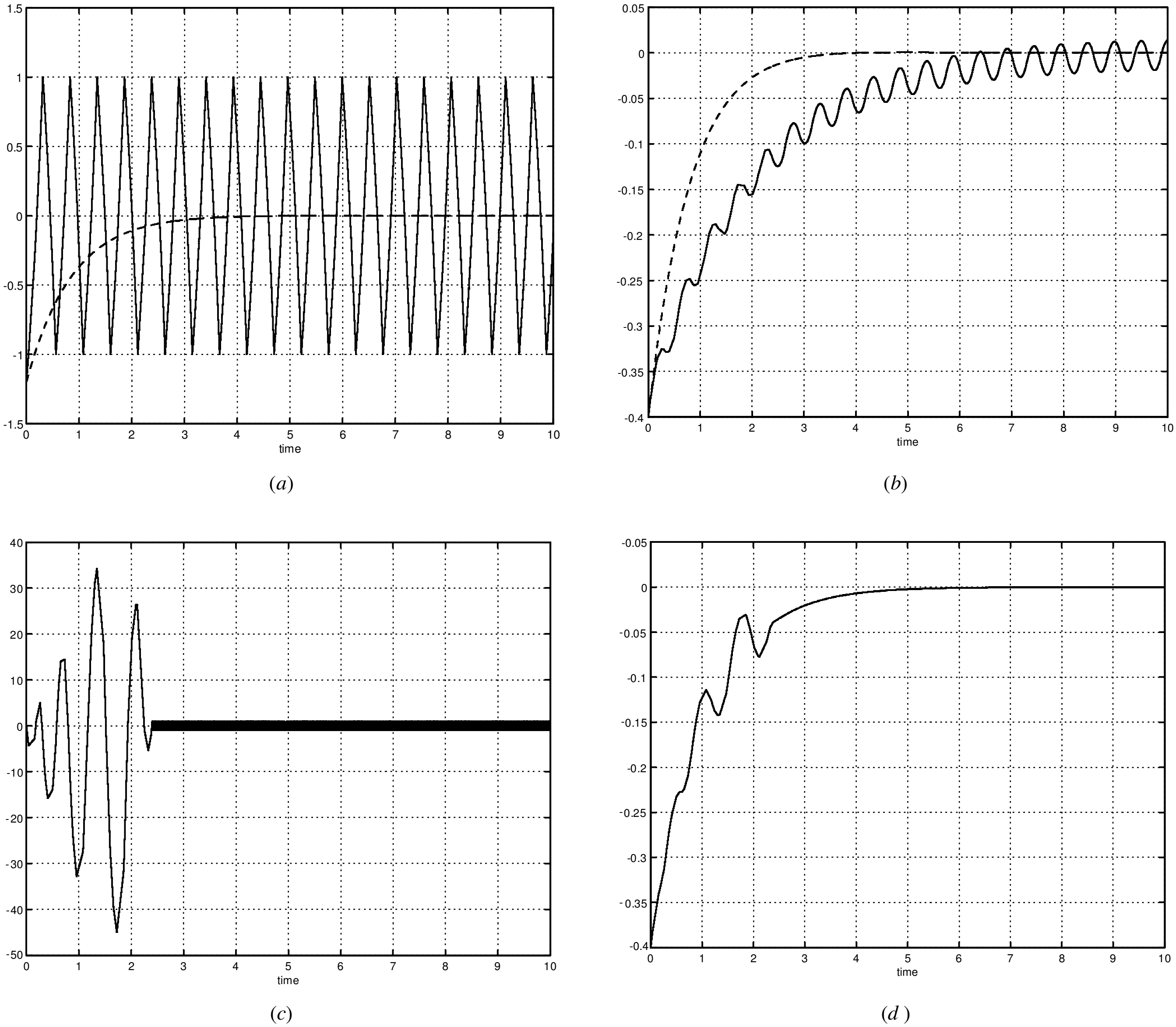

Figure 14. Adaptive control of a non-linear oscillator with relay in the input path: linear controller: (*a*) *u*, (*b*) *r*, without relay (dashed), with relay (solid); adaptive controller: (*c*) *u*, (*d*) *r*.

b which is taken to be positive. Defining the error signal $x_1(t) \triangleq r(t) - r_{\mathrm{des}}$, equation (7.1) becomes

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ bf(u(t)) \end{bmatrix} \qquad (44)$$

which has the form (26) with $a_1 = -k$ and $a_2 = -c$ and with $bu(t)$ replaced by $bf(u(t))$. The controller (14) with (22)–(24) is applied to this problem.

First, we let $f(\cdot)$ be the deadzone non-linearity shown in figure 2. In this case, $uf(u) > 0$, and thus u and $f(u)$ have the same sign. We apply controller (14) with (22)–(24) and with $e_1 = -0.5$, $e_2 = 0.5$, $c = -2$, $k = -1$, $b = 1$, $r_{\mathrm{des}} = 0$, $r(0) = -0.3$, and $\dot{r}(0) = 0.5$. For comparison, a stabilizing linear controller is designed for the system (44) with $f(u) = u$, which is $u(t) = -2x_1(t) - 4x_2(t)$. This controller is applied to the system (44) with the deadzone non-linearity $f(\cdot)$. It can be seen from the solid line in figure 13(*b*) that $r(t)$ does not approach r_{des} when a linear controller is used. However, by choosing adaptation weights $p = 1$, $\lambda_1 = 10^3$, $\lambda_2 = 10^3$, $\lambda_3 = 10^3$ and letting $K_1(0) = 0$, $K_2(0) = 0$, and $\phi(0) = 0$, figure 13(*d*) shows that $r(t)$ approaches r_{des} when the adaptive controller is used.

Next, we let $f(\cdot)$ be the relay non-linearity shown in figure 4. Note that in this case u and $f(u)$ do not always have the same sign. We apply controller (14) with (22)–(24) and with $c = -2$, $k = -1$, $b = 1$, $r_{\mathrm{des}} = 0$, $r(0) = -0.4$, and $\dot{r}(0) = 0.5$. For comparison, a

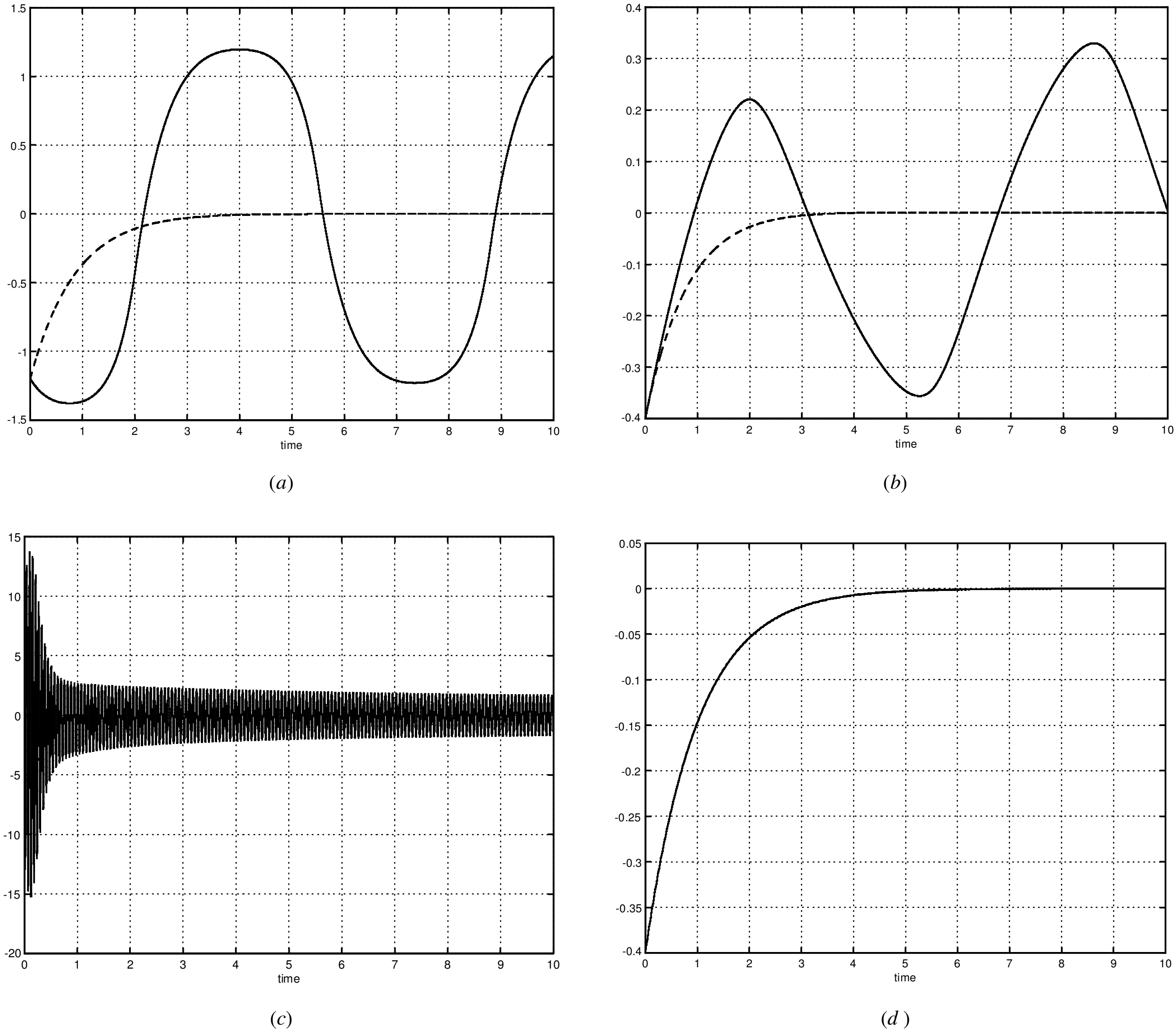

Figure 15. Adaptive control of a non-linear oscillator with backlash/hysteresis in the input path: linear controller: (*a*) u, (*b*) r, without hysteresis (dashed), with hysteresis (solid); adaptive controller: (*c*) u, (*d*) r.

stabilizing linear controller is designed for the system (44) with $f(u) = u$, which is $u(t) = -2x_1(t) - 4x_2(t)$. This linear controller is applied to the system (44) with relay $f(\cdot)$. Choose adaptation weights $p = 1$, $\lambda_1 = 10^3$, $\lambda_2 = 10^3$, $\lambda_3 = 10^3$ and let $K_1(0) = 0$, $K_2(0) = 0$, and $\phi(0) = 0$. As can be seen from the solid line in figure 14(*b*), $r(t)$ does not approach r_{des} when the linear controller is used. However, figure 14(*d*) shows that $r(t)$ approaches r_{des} when the adaptive controller is used.

Next, we let $f(\cdot)$ be the backlash/hysteresis non-linearity shown in figure 6. Note that in this case u and $f(u)$ do not always have the same sign. We apply controller (14) with (22)–(24) and with backlash/hysteresis with $h = 1$, $c = -2$, $k = -1$, $b = 1$, $r_{\mathrm{des}} = 0$, $r(0) = -0.4$, and $\dot{r}(0) = 0.5$. For comparison, the stabilizing linear controller $u(t) = -2x_1(t) - 4x_2(t)$ is designed for the system (44) with $f(u) = u$. This linear controller is applied to the system (44) with the backlash/hysteresis non-linearity $f(\cdot)$. Choose adaptation weights $p = 1$, $\lambda_1 = 10^4$, $\lambda_2 = 10^4$, $\lambda_3 = 10^4$ and let $K_1(0) = 0$, $K_2(0) = 0$, and $\phi(0) = 0$. As can be seen from the solid line in figure 15(*b*), $r(t)$ does not approach r_{des} when a linear controller is used. However, figure 15(*d*) shows that $r(t)$ approaches r_{des} when the adaptive controller is used.

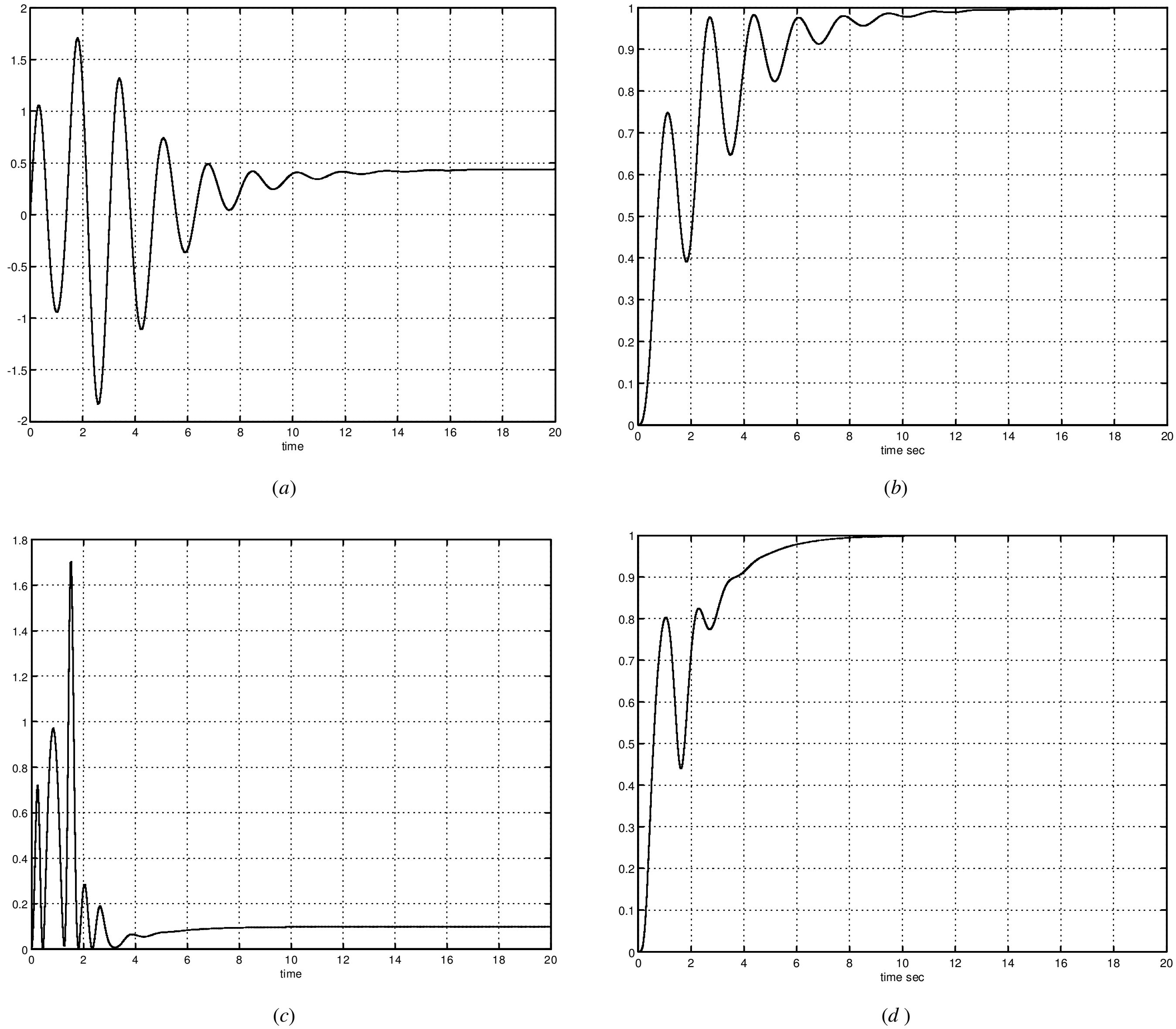

Figure 16. Adaptive control of a non-linear oscillator with linear and odd quadratic input: (*a*) u, (*b*) r for linear input; (*c*) u, (*d*) r for odd quadratic input.

Finally, we let $f(x) = \mathrm{sign}(x)x^2$. In this case, $uf(u) > 0$, and thus u and $f(u)$ have the same sign. We apply controller (14) with (22)–(24) and with $c = 0.1$, $k = 5$, $b = 1$, $r_{\mathrm{des}} = 1$, $r(0) = 0$ and $\dot{r}(0) = 0$. Choosing adaptation weights $p = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 5$ and let $K_1(0) = 0$, $K_2(0) = 0$ and $\phi(0) = 0$, figure 16 shows the response of the adaptive controller. For comparison, figure 16 shows also the response of the same system with $f(u) = u$. As can be seen from figure 16(*a*), $u(t)$ has negative values from time to time when a linear input is used. However, figure 16(*c*) shows that $u(t)$ remains positive when the odd quadratic non-linearity is present.

8. Conclusion

In this paper we applied a direct adaptive control law derived for linear systems to non-linear oscillators possessing dynamic and input non-linearities. The adaptive controller was shown to be effective in all cases considered for the problems of adaptive stabilization and command following. Finally, it was shown by Roup and Bernstein (2000) that the controller given by Theorem 1 is guaranteed to stabilize a class of non-linear systems.

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