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An implicit small gain condition and an upper bound for the real structured singular value[1]

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Abstract

In this paper we develop an upper bound for the real structured singular value that has the form of an implicit small gain theorem. The implicit small gain condition involves a shifted plant whose dynamics depend upon the uncertainty set bound and, unlike prior bounds, does not require frequency-dependent scales or multipliers. Numerical results show that the implicit small gain bound compares favorably with real-μ bounds that involve frequency-dependent scales and multipliers.

Keywords: Robust stability; Robust performance; Guaranteed cost bounds; Real-μ bounds

1. Introduction

The classical small gain theorem [17], along with its multivariable extension and quadratic stability interpretation [7, 10], provides the essential foundation for modern robust control theory. The extension of the small gain theorem in terms of the structured singular value [11] provides reduced conservatism in the case of complex block-structured uncertainty, while providing a bound on worst-case H_∞ performance. Further extension to the case of real and mixed uncertainty is considered in [5, 12].

The problem of worst-case H_2 performance with constant real uncertainty has been addressed by guaranteed cost bounds. A variety of such bounds have been developed; see [1] for a unified discussion. An alternative approach to worst-case H_2 performance is given in [6] which characterizes the exact worst-case performance. As in [1] we seek guaranteed cost bounds that can be used for fixed-structure controller synthesis [8].

The goal of the present paper is to develop a guaranteed cost bound that is particularly effective for skew-symmetric uncertainty. For motivation, consider the case of a dissipative nominal dynamics matrix A (that is, $A + A^{\mathrm{T}}$ is negative definite) with perturbation ΔA, where $\Delta A = \sum_{i=1}^{r} \delta_i A_i$, δ_i is a real uncertain parameter and A_i is skew symmetric, $i = 1,\dots,r$. The perturbed system $\dot{x}(t) = (A + \Delta A)x(t)$ is quadratically stable for $\delta_i \in (-\infty,\infty)$, $i = 1,\dots,r$, with common Lyapunov function

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$V(x) = x^{\mathrm{T}}x$. Hence this uncertain system is asymptotically stable for arbitrary time-varying functions $\delta_i(t)$, $t \in [0, \infty)$.

Although quadratic stability analysis of this case is transparent, this problem is not easily addressed by the quadratic stability bound of [10, 13]. Specifically, in [10, 13] the perturbation ΔA has the form $\Delta A = DFE$, where F is a dense uncertain matrix assumed to satisfy $F^{\mathrm{T}}F \leqslant \gamma^{-2}I$ or, in the complex case, $F^*F \leqslant \gamma^{-2}I$. However, for assessing robust stability and bounding worst-case H_2 performance, this structure does not account for the internal structure of skew-symmetric matrices. The resulting conservatism is shown by Example 4.2 of the present paper. Skew-symmetric uncertainty is more appropriately treated by real structured singular value bounds that allow repeated parameters.

The guaranteed cost bound developed herein involves a shifted dynamics matrix and thus is most closely related to the bounds obtained in [2, 15, 16]. In fact, this new bound is shown to be less conservative than the bounds of [2, 15, 16]. An interesting aspect of this new bound is its interpretation in terms of an *implicit small gain theorem*, where the system matrices are functions of the uncertainty set bound. Furthermore, the implicit small gain condition provides a novel upper bound for the real structured singular value that does not require frequency-dependent scales or multipliers. Several numerical examples are provided to demonstrate the utility of the proposed bound.

Notation

$\mathbb{R}$, $\mathbb{R}^{n\times m}$, $\mathbb{C}^{n\times m}$	real numbers, $n \times m$ real matrices, $n \times m$ complex matrices
$\mathbb{S}^n$, $\mathbb{N}^n$	$n \times n$ symmetric matrices, $n \times n$ nonnegative-definite matrices
$\boldsymbol{E}$, $\mathscr{R}(A)$, I_n	expectation, range space of matrix A, $n \times n$ identity

2. Robust stability and performance analysis

Let $\mathscr{U} \subset \mathbb{R}^{n\times n}$ denote a set of perturbations ΔA of a given nominal dynamics matrix $A \in \mathbb{R}^{n\times n}$, where A is asymptotically stable and $0 \in \mathscr{U}$.

Robust Stability Problem. Determine whether the linear system

$$\dot{x}(t) = (A + \Delta A)x(t), \quad t \geqslant 0, \tag{1}$$

is asymptotically stable for all $\Delta A \in \mathscr{U}$.

Robust Performance Problem. For the uncertain linear system

$$\dot{x}(t) = (A + \Delta A)x(t) + Dw(t), \quad t \geqslant 0, \tag{2}$$

$$z(t) = Ex(t), \tag{3}$$

where $w(\cdot)$ is zero-mean d-dimensional white-noise with intensity I_d and $E \in \mathbb{R}^{q\times n}$, determine a performance bound $\mathscr{J}$ satisfying

$$J(\mathscr{U}) \triangleq \sup_{\Delta A \in \mathscr{U}} \limsup_{t\to\infty} \boldsymbol{E}\{\|z(t)\|_2^2\} \leqslant \mathscr{J}. \tag{4}$$

If $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathscr{U}$, then

$$J(\mathscr{U}) = \sup_{\Delta A \in \mathscr{U}} \operatorname{tr} P_{\Delta A} V, \tag{5}$$

where $P_{\Delta A} \in \mathbb{R}^{n\times n}$ is the unique, nonnegative-definite solution to

$$0 = (A + \Delta A)^{\mathrm{T}} P_{\Delta A} + P_{\Delta A}(A + \Delta A) + R, \tag{6}$$

and $R \triangleq E^{\mathrm{T}}E$ and $V \triangleq DD^{\mathrm{T}}$. We now state a sufficient condition for robust stability and performance.

Lemma 2.1 (Bernstein and Haddad [1]). *Let $\Omega : \mathbb{N}^n \to \mathbb{N}^n$ be such that*

$$\Delta A^{\mathrm{T}}P + P\Delta A \leqslant \Omega(P), \quad \Delta A \in \mathcal{U}, \ P \in \mathbb{N}^n, \tag{7}$$

and suppose there exists $P \in \mathbb{N}^n$ satisfying

$$0 = A^{\mathrm{T}}P + PA + \Omega(P) + R. \tag{8}$$

Then $(A + \Delta A, E)$ is detectable for all $\Delta A \in \mathcal{U}$ if and only if $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. In this case,

$$P_{\Delta A} \leqslant P, \qquad \Delta A \in \mathcal{U}, \tag{9}$$

where $P_{\Delta A}$ satisfies (6), and

$$J(\mathcal{U}) \leqslant \operatorname{tr} PV. \tag{10}$$

Remark 2.1. Lemma 2.1 provides sufficient conditions for robust stability and performance for real parameter uncertainty $\Delta A \in \mathcal{U} \subset \mathbb{R}^{n\times n}$. For complex uncertainty $\Delta A \in \mathcal{U}_{\mathrm{c}} \subset \mathbb{C}^{n\times n}$ Lemma 2.1 can be extended to provide robust stability by replacing (7) with

$$\Delta A^{*}P + P\Delta A \leqslant \Omega(P), \quad \Delta A \in \mathcal{U}_{\mathrm{c}}, \ P \in \mathbb{N}^n, \tag{11}$$

where P is nonnegative definite Hermitian and $\Omega(P)$ is Hermitian.

Next we specialize to the case in which $\mathcal{U}$ is given by

$$\mathcal{U} \triangleq \left\{ \Delta A \in \mathbb{R}^{n\times n}\colon \Delta A = \sum_{i=1}^{r} \delta_i A_i, \ |\delta_i| \leqslant \gamma^{-1}, \ i = 1,\ldots,r \right\}, \tag{12}$$

where γ is a positive number and, for $i = 1,\ldots,r$, $A_i \in \mathbb{R}^{n\times n}$ is a fixed matrix denoting the structure of the parametric uncertainty and δ_i is an uncertain real parameter.

Remark 2.2. The set $\mathcal{U}$ defined by (12) includes repeated parameters without loss of generality. For example, if $\delta_1 = \delta_2$ then discard δ_2 and replace A_1 by $A_1 + A_2$. In addition, $\mathcal{U}$ includes real block uncertainty. For example, if

$$\Delta A = \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{bmatrix},$$

then $\Delta A = \sum_{i=1}^{4} \delta_i A_i$, where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and likewise for A_2, A_3, and A_4.

We now introduce a specific choice of $\Omega(P)$. For $i = 1,\ldots,r$, let $S_i \in \mathbb{R}^{n\times n}$ and define

$$Z_i \triangleq [(S_i + S_i^{\mathrm{T}})^2]^{1/2}.$$

Note that $-Z_i \leqslant \alpha(S_i + S_i^{\mathrm{T}}) \leqslant Z_i$ for all $\alpha \in [-1, 1]$. If S_i is skew symmetric then $Z_i = 0$. Furthermore, for $i = 1,\ldots,r$, define

$$\hat{I}_i \triangleq [S_i \ A_i^{\mathrm{T}}][S_i \ A_i^{\mathrm{T}}]^{\dagger},$$

where $(\ )^{\dagger}$ denotes the Moore–Penrose generalized inverse. Note that $\hat{I}_i$ is symmetric and idempotent, that is, $\hat{I}_i = \hat{I}_i^{\mathrm{T}} = \hat{I}_i^2$. Furthermore, since $\hat{I}_i[S_i \ A_i^{\mathrm{T}}] = [S_i \ A_i^{\mathrm{T}}]$, it follows that $\hat{I}_i S_i = S_i$ and $A_i \hat{I}_i = A_i$. If $S_i = A_i$ and A_i

is an EP matrix [3], that is, $\mathscr{R}(A_i) = \mathscr{R}(A_i^{\mathrm{T}})$, then $\hat{I}_i = A_i^{\dagger}A_i$. Recall that normal matrices (and thus symmetric and skew-symmetric matrices) are EP.

Lemma 2.2. *For $i = 1,\dots,r$, let $\alpha_i \in \mathbb{R}$, $\beta_i > 0$, and $S_i \in \mathbb{R}^{n\times n}$. Then (7) is satisfied with $\Omega(P)$ given by*

$$\Omega(P) = \sum_{i=1}^{r}[\gamma^{-2}(\alpha_i S_i + \beta_i A_i^{\mathrm{T}}P)^{\mathrm{T}}(\alpha_i S_i + \beta_i A_i^{\mathrm{T}}P) + \gamma^{-1}\beta_i^{-1}|\alpha_i|Z_i + \beta_i^{-2}\hat{I}_i]. \tag{13}$$

Proof. Let $\Delta A \in \mathscr{U}$ and $P \in \mathbb{N}^n$. Then

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^{r}[\alpha_i\delta_i S_i + \beta_i\delta_i A_i^{\mathrm{T}}P - \beta_i^{-1}\hat{I}_i]^{\mathrm{T}}[\alpha_i\delta_i S_i + \beta_i\delta_i A_i^{\mathrm{T}}P - \beta_i^{-1}\hat{I}_i] \\ &= \sum_{i=1}^{r}[\delta_i^2(\alpha_i S_i + \beta_i A_i^{\mathrm{T}}P)^{\mathrm{T}}(\alpha_i S_i + \beta_i A_i^{\mathrm{T}}P) - \beta_i^{-1}\alpha_i\delta_i(S_i + S_i^{\mathrm{T}}) + \beta_i^{-2}\hat{I}_i - \delta_i(A_i^{\mathrm{T}}P + PA_i)] \\ &\leqslant \sum_{i=1}^{r}[\gamma^{-2}(\alpha_i S_i + \beta_i A_i^{\mathrm{T}}P)^{\mathrm{T}}(\alpha_i S_i + \beta_i A_i^{\mathrm{T}}P) + \gamma^{-1}\beta_i^{-1}|\alpha_i|Z_i + \beta_i^{-2}\hat{I}_i - \delta_i(A_i^{\mathrm{T}}P + PA_i)] \\ &= \Omega(P) - (\Delta A^{\mathrm{T}}P + P\Delta A). \qquad \square \end{aligned}$$

Remark 2.3. If δ_i is assumed to be complex for some i, then it can be shown that $\Omega(P)$ given by (13) does *not* satisfy (11). Hence, unlike the quadratic stability bound of [10, 13], the bound (13) can distinguish between real and complex uncertainty.

Combining Lemma 2.1 with Lemma 2.2 yields the following result. For convenience define the shifted dynamics matrix $A_{s\gamma} \triangleq A + \gamma^{-2}\sum_{i=1}^{r}\alpha_i\beta_i A_i S_i$.

Theorem 2.1. *For $i = 1,\dots,r$, let $\alpha_i \in \mathbb{R}$, $\beta_i > 0$, and $S_i \in \mathbb{R}^{n\times n}$. Furthermore, suppose there exists a non-negative-definite matrix P satisfying*

$$0 = A_{s\gamma}^{\mathrm{T}}P + PA_{s\gamma} + \sum_{i=1}^{r}[\gamma^{-2}(\alpha_i^2 S_i^{\mathrm{T}}S_i + \beta_i^2 PA_iA_i^{\mathrm{T}}P) + \gamma^{-1}\beta_i^{-1}|\alpha_i|Z_i + \beta_i^{-2}\hat{I}_i] + R. \tag{14}$$

Then $(A + \Delta A, E)$ is detectable for all $\Delta A \in \mathscr{U}$ if and only if $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathscr{U}$. In this case,

$$P_{\Delta A} \leqslant P, \qquad \Delta A \in \mathscr{U}, \tag{15}$$

where $P_{\Delta A}$ satisfies (6), and

$$J(\mathscr{U}) \leqslant \operatorname{tr} PV. \tag{16}$$

Remark 2.4. If only robust stability is required, then E can be chosen such that $(A + \Delta A, E)$ is detectable for all $\Delta A \in \mathscr{U}$. For example, choosing $E = \varepsilon I_n$ with $\varepsilon > 0$ implies the detectability condition, where ε is chosen so that (14) has a solution P with $R = \varepsilon^2 I_n$. For robust performance, however, (14) must be solved with E specified by the robust performance problem.

Remark 2.5. To draw connections of Theorem 2.1 with traditional Lyapunov theory, let R be positive definite, assume there exists a positive-definite solution to (14), and define the Lyapunov function candidate $V(x) \triangleq x^{\mathrm{T}}Px$. Then the Lyapunov derivative is given by

$$\dot{V}(x) = -x^{\mathrm{T}}[\Omega(P) - \{\Delta A^{\mathrm{T}}P + P\Delta A\} + R]x,$$

where $\Omega(P)$ is given by (13). Now, using (7), it follows that $\dot{V}(x(t)) < 0$ for $x(\cdot)$ satisfying (1) and for all $\Delta A \in \mathcal{U}$. Thus $V(\cdot)$ is a Lyapunov function for (1) that guarantees robust asymptotic stability over $\mathcal{U}$. A unified framework for constructing other Lyapunov functions satisfying (7) for $\mathcal{U}$ given by (12) is given in [1].

To improve the performance bound (16) we can optimize $\mathcal{J} \triangleq \operatorname{tr} PV$ with respect to α_i and β_i. The simplest case to consider is the case where S_i is skew symmetric or, equivalently, $Z_i = 0$. In this case

$$\frac{\partial \mathcal{J}}{\partial \alpha_i} = 2\beta_i \gamma^{-2} \operatorname{tr} PA_i S_i Q + 2\alpha_i \gamma^{-2} \operatorname{tr} S_i^{\mathrm{T}} S_i Q = 0 \tag{17}$$

and

$$\frac{\partial \mathcal{J}}{\partial \beta_i} = 2\alpha_i \gamma^{-2} \operatorname{tr} PA_i S_i Q + 2\beta_i \gamma^{-2} \operatorname{tr} PA_i A_i^{\mathrm{T}} PQ - 2\beta_i^{-3} \operatorname{tr} \hat{I}_i Q = 0, \tag{18}$$

where Q satisfies

$$0 = \left(A_{s\gamma} + \sum_{i=1}^{r} \gamma^{-2}\beta_i^2 A_i A_i^{\mathrm{T}} P\right) Q + Q \left(A_{s\gamma} + \sum_{i=1}^{r} \gamma^{-2}\beta_i^2 A_i A_i^{\mathrm{T}} P\right)^{\mathrm{T}} + V. \tag{19}$$

If (19) has a solution then (17) and (18) imply

$$\alpha_i = \frac{\operatorname{tr} PA_i S_i Q}{\operatorname{tr} S_i^2 Q} \beta_i, \qquad \beta_i = \gamma^{1/2} \left[\frac{\operatorname{tr} \hat{I}_i Q \operatorname{tr} S_i Q S_i^{\mathrm{T}}}{\operatorname{tr} S_i Q S_i^{\mathrm{T}} \operatorname{tr} A_i^{\mathrm{T}} PQPA_i - (\operatorname{tr} PA_i S_i Q)^2}\right]^{1/4}. \tag{20}$$

Furthermore, in this case,

$$\frac{\partial^2 \mathcal{J}}{\partial \alpha_i^2} = 2\gamma^{-2} \operatorname{tr} S_i Q S_i^{\mathrm{T}} \geqslant 0, \tag{21}$$

$$\frac{\partial^2 \mathcal{J}}{\partial \beta_i^2} = 2\gamma^{-2} \operatorname{tr} A_i^{\mathrm{T}} PQPA_i + 6\beta_i^{-4} \operatorname{tr} \hat{I}_i Q \geqslant 0, \tag{22}$$

and

$$\frac{\partial^2 \mathcal{J}}{\partial \alpha_i^2} \frac{\partial^2 \mathcal{J}}{\partial \beta_i^2} - \left(\frac{\partial^2 \mathcal{J}}{\partial \alpha_i \partial \beta_i}\right)^2 = 16\gamma^{-2}\beta_i^{-4} \operatorname{tr} \hat{I}_i Q \operatorname{tr} S_i Q S_i^{\mathrm{T}} \geqslant 0 \tag{23}$$

which imply that (20) provides necessary conditions for a local minimum.

In the case $Z_i \neq 0$ we need to consider the cases $\alpha_i = 0$ and $\alpha_i \neq 0$ since $\mathcal{J}$ is not differentiable at $\alpha_i = 0$. First, let $\alpha_i = 0$. In this case

$$\frac{\partial \mathcal{J}}{\partial \beta_i} = 2\beta_i \gamma^{-2} \operatorname{tr} PA_i A_i^{\mathrm{T}} PQ - 2\beta_i^{-3} \operatorname{tr} \hat{I}_i Q = 0, \tag{24}$$

where Q satisfies (19) with $\alpha_i = 0$. If (19) has a solution with $\alpha_i = 0$ then (24) implies

$$\beta_i = \gamma^{1/2} \left[\frac{\operatorname{tr} \hat{I}_i Q}{\operatorname{tr} A_i^{\mathrm{T}} PQPA_i}\right]^{1/4}. \tag{25}$$

Furthermore, in this case (22) holds which implies that (25) provides necessary conditions for a local minimum. Next, consider the case where $\alpha_i \neq 0$. In this case

$$\frac{\partial \mathcal{J}}{\partial \alpha_i} = 2\beta_i \gamma^{-2} \operatorname{tr} PA_i S_i Q + 2\alpha_i \gamma^{-2} \operatorname{tr} S_i^{\mathrm{T}} S_i Q + \gamma^{-1}\beta_i^{-1} \operatorname{sgn} \alpha_i \operatorname{tr} Z_i Q = 0 \tag{26}$$

and

$$\frac{\partial \mathscr{J}}{\partial \beta_i} = 2\alpha_i\gamma^{-2}\operatorname{tr} PA_iS_iQ + 2\beta_i\gamma^{-2}\operatorname{tr} PA_iA_i^{\mathrm{T}}PQ - \gamma^{-1}\beta_i^{-2}|\alpha_i|\operatorname{tr} Z_iQ - 2\beta_i^{-3}\operatorname{tr}\hat{I}_iQ = 0, \tag{27}$$

where Q satisfies (19) and $\operatorname{sgn}\alpha_i \triangleq |\alpha_i|/\alpha_i$. If (19) has a solution then (27) implies

$$\alpha_i = -\frac{\operatorname{tr} PA_iS_iQ}{\operatorname{tr} S_iQS_i^{\mathrm{T}}}\beta_i - \frac{\operatorname{tr} Z_iQ}{2\operatorname{tr} S_iQS_i^{\mathrm{T}}}\gamma\beta_i^{-1}\operatorname{sgn}\alpha_i. \tag{28}$$

Next, we obtain a frequency-domain condition for robust stability in terms of an implicit small gain condition.

Corollary 2.1. *Let $\varepsilon > 0$, let $C_{s\gamma}$ satisfy $C_{s\gamma}^{\mathrm{T}}C_{s\gamma} = \varepsilon^2 I_n + \sum_{i=1}^{r}[\gamma^{-2}\alpha_i^2 S_i^{\mathrm{T}}S_i + \gamma^{-1}\beta_i^{-1}|\alpha_i|Z_i + \beta_i^{-2}\hat{I}_i]$, and define $B_s \triangleq [\beta_1A_1 \cdots \beta_rA_r]$. If $A_{s\gamma}$ is asymptotically stable and*

$$\|C_{s\gamma}(sI_n - A_{s\gamma})^{-1}B_s\|_\infty < \gamma, \tag{29}$$

then (1) *is asymptotically stable for all $\Delta A \in \mathscr{U}$.*

Proof. If $A_{s\gamma}$ is asymptotically stable and (29) holds, then it follows from Theorem 2.7 of [10] that there exists a unique nonnegative-definite matrix P satisfying (14). Now, asymptotic stability of (1) for all $\Delta A \in \mathscr{U}$ follows from Lemma 2.1 with $\Omega(P)$ given by (13). □

Remark 2.6. Note that (29) is an implicit small gain condition since $A_{s\gamma}$ and $C_{s\gamma}$ depend on the uncertainty bound γ.

Remark 2.7. If S_i, $i=1,\ldots,r$, is skew symmetric then $C_{s\gamma}$ can be chosen to be

$$C_{s\gamma} = [\gamma^{-1}\alpha_1S_1 + \beta_1^{-1}\hat{I}_1 \cdots \gamma^{-1}\alpha_rS_r + \beta_r^{-1}\hat{I}_r \;\; \varepsilon I_n]^{\mathrm{T}}.$$

Remark 2.8. Let $\alpha_i = 0$, $\beta_i = 1$, and $\hat{I}_i = I_n$, $i = 1,\ldots,r$, let $\varepsilon = 0$, and define $B_0 \triangleq [B_1 \cdots B_r]$ and $C_0 \triangleq [C_1^{\mathrm{T}} \cdots C_r^{\mathrm{T}}]^{\mathrm{T}}$. Then (29) becomes

$$\|C_0(sI_n - A)^{-1}B_0\|_\infty < \gamma, \tag{30}$$

which is a necessary and sufficient condition for robust stability with respect to complex uncertainty. In contrast, as noted in Remark 2.3, (29) is sufficient for real uncertainty. In Section 4 we show by means of a counterexample that (29) is not sufficient for complex uncertainty.

3. A peak upper bound for real-μ

In this section we obtain an upper bound for the real structured singular value. For $i = 1,\ldots,r$, let $A_i = B_iC_i$, where $B_i \in \mathbb{R}^{n\times q_i}$, $C_i \in \mathbb{R}^{q_i\times n}$, and $q_i \leqslant n$. Defining $B_0 \triangleq [B_1 \cdots B_r]$ and $C_0 \triangleq [C_1^{\mathrm{T}} \cdots C_r^{\mathrm{T}}]^{\mathrm{T}}$, $\mathscr{U}$ can be written as

$$\mathscr{U} = \{\Delta A \in \mathbb{R}^{n\times n}\colon \Delta A = B_0\Delta C_0,\ \Delta = \text{block–diag}(\delta_1 I_{q_1},\ldots,\delta_r I_{q_r}), |\delta_i| \leqslant \gamma^{-1},\ i = 1,\ldots,r\}, \tag{31}$$

which is the real parameter uncertainty set considered in [5]. Conversely, an uncertainty set of the form (31) can always be written in the form (12) by partitioning B_0, C_0 as above and defining $A_i = B_iC_i$, $i = 1,\ldots,r$.

Next, to obtain an upper bound for the real structured singular value, we note that the robust stability of (1) for all $\Delta A \in \mathscr{U}$ is equivalent to the robust stability of the feedback interconnection of $G(s)$ and Δ, where $G(s) \triangleq C_0(sI_n - A)^{-1}B_0$ and $\Delta \in \mathbf{\Delta}_\gamma$, where $\mathbf{\Delta}_\gamma \triangleq \{\Delta \in \mathbb{R}^{q\times q}\colon \Delta = \text{block–diag}(\delta_1 I_{q_1},\ldots,\delta_r I_{q_r}), |\delta_i| \leqslant \gamma^{-1},\ i =$

$1,\ldots,r\}$ and $q \triangleq \sum_{i=1}^{r} q_i$. Stability of this interconnection can be analyzed in terms of the real structured singular value defined by [5]

$$\mu(G(\jmath\omega)) \triangleq \left(\min_{\Delta\in\boldsymbol{\Delta}}\{\sigma_{\max}(\Delta)\colon \det(I_n + G(\jmath\omega)\Delta) = 0\}\right)^{-1},$$

and, if $\det(I_n + G(\jmath\omega)\Delta) \neq 0$ for all $\Delta \in \boldsymbol{\Delta}$, then $\mu(G(\jmath\omega)) \triangleq 0$ where $\boldsymbol{\Delta} \triangleq \{\Delta \in \mathbb{R}^{q\times q}\colon \Delta = \text{block-diag}(\delta_1 I_{q_1},\ldots,\delta_r I_{q_r}),\ \delta_i \in \mathbb{R},\ i = 1,\ldots,r\}$. A necessary and sufficient condition for robust stability of the feedback interconnection of $G(s)$ and Δ for all $\Delta \in \boldsymbol{\Delta}_\gamma$ is given by $\mu(G(\jmath\omega)) < \gamma$ for all $\omega \in \mathbb{R}$ [9].

Using the implicit small gain condition (29) we provide a peak upper bound for $\mu(G(\jmath\omega))$, that is, a bound for the peak value of $\mu(G(\jmath\omega))$ over frequency. For the statement of this result define

$$\begin{aligned}\mu_{\text{isg}}(G(s)) \triangleq \inf\{\hat\gamma > 0\colon & \text{ there exist } \alpha_i \in \mathbb{R},\ \varepsilon,\beta_i > 0,\ S_i \in \mathbb{R}^{n\times n},\ i = 1,\ldots,r, \text{ such that}\\ & A_{s\hat\gamma} \text{ is asymptotically stable, and } \|C_{s\hat\gamma}(sI_n - A_{s\hat\gamma})^{-1}B_s\|_\infty < \hat\gamma\}.\end{aligned} \tag{32}$$

Corollary 3.1. *Let* $\mu_{\text{isg}}(G(s))$ *be given by* (32). *Then*

$$\sup_{\omega\in\mathbb{R}} \mu(G(\jmath\omega)) \leqslant \mu_{\text{isg}}(G(s)). \tag{33}$$

Proof. Suppose $\mu_{\text{isg}}(G(s)) < \sup_{\omega\in\mathbb{R}} \mu(G(\jmath\omega))$ and let $\hat\gamma > 0$ be such that $\mu_{\text{isg}}(G(s)) < \hat\gamma \leqslant \sup_{\omega\in\mathbb{R}} \mu(G(\jmath\omega))$. Then it follows from Corollary 2.1 that the feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \boldsymbol{\Delta}_{\hat\gamma}$. However, since $\sup_{\omega\in\mathbb{R}} G(\jmath\omega) \geqslant \hat\gamma$ it follows from Theorem 4.1 of [9] that there exists $\Delta \in \boldsymbol{\Delta}_{\hat\gamma}$ such that the feedback interconnection of $G(s)$ and Δ is not asymptotically stable, which is a contradiction. □

4. Numerical examples

Example 4.1. Let

$$A = \begin{bmatrix} -\eta & \omega_d \\ -\omega_d & -\eta \end{bmatrix},\quad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\quad S_1 = A_1 \text{ and } R = \rho I_2,$$

where $\eta, \omega_d, \rho > 0$. Since $P_{\Delta A} = (\rho/2\eta)I_2$ solves (6) for all $\Delta A \in \mathscr{U}$, it follows that $J(\mathscr{U}) = (\rho/2\eta)\operatorname{tr} V$. Next, note that $P = pI_2$ solves (14) with $Z_1 = 0$ and $\hat I_1 = I_2$, where p is given by

$$p = \frac{1}{\beta_1^2}\left(\alpha_1\beta_1 + \eta\gamma^2 - \gamma\sqrt{2\alpha_1\beta_1\eta - (1+\rho\beta_1^2) + \eta^2\gamma^2}\right), \tag{34}$$

which is positive for all $\alpha_1 \geqslant (1+\rho\beta_1^2)/2\eta\beta_1$ and $\gamma, \beta_1 > 0$. With $\alpha_1 = (1+\rho\beta_1^2)/2\eta\beta_1$ satisfying (20), it follows that $p = (1+\rho\beta_1^2)/2\eta\beta_1^2$ and hence $\operatorname{tr} PV = [(1+\rho\beta_1^2)/2\eta\beta_1^2]\operatorname{tr} V$ which implies that $J(\mathscr{U}) \leqslant \inf_{\beta_1\in\mathbb{R}}[(1+\rho\beta_1^2)/2\eta\beta_1^2]\operatorname{tr} V = (\rho/2\eta)\operatorname{tr} V$. Finally, since $J(\mathscr{U}) = (\rho/2\eta)\operatorname{tr} V$ it follows that Theorem 2.1 is nonconservative with respect to both robust stability and worst-case H_2 performance.

Example 4.2. To compare the new bound (16) with the results given in [2, 4, 16] let

$$A = \begin{bmatrix} -0.005 & 1 \\ -1 & -0.005 \end{bmatrix},\quad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\quad R = \begin{bmatrix} 0.25 & 0.12 \\ 0.12 & 2.5 \end{bmatrix},\quad V = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix},$$

and $S_1 = A_1$. This example, which was considered in [2, 16], differs from Example 4.1 since R is not of the form ρI_2. For this example, $A + \delta_1 A_1$ is asymptotically stable for all $\delta_1 \in (-\infty, \infty)$ while Fig. 1 shows the worst-case H_2 performance using the implicit small gain bound (16) for $\delta_1 \in (-2, 2)$. For $\gamma^{-1} = 2$, α_1 and β_1 satisfying (20) are given by 553.92 and 2.12, respectively. Applying the guaranteed cost bound of [4], robust stability is predicted only for $\delta_1 \in (-1, 1)$, while the small gain theorem predicts stability only

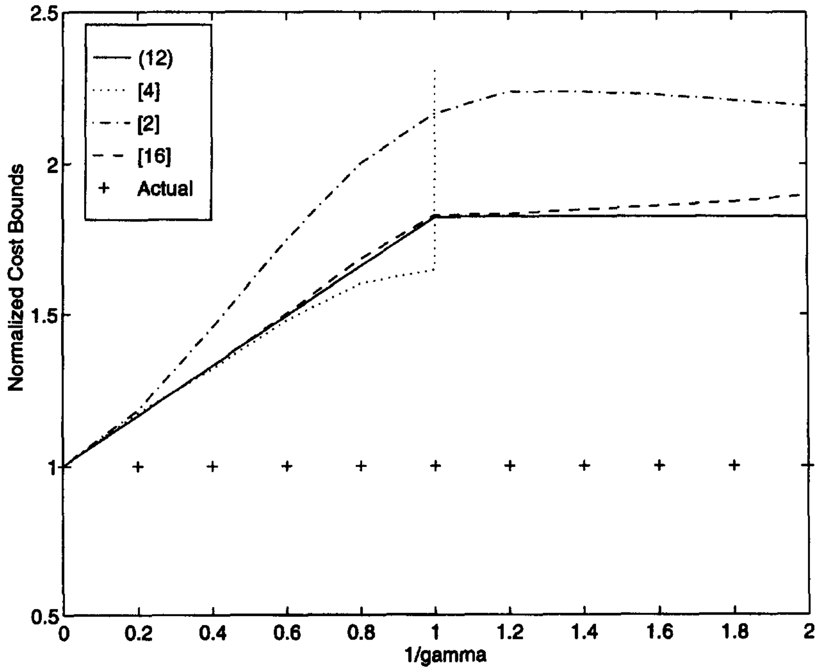

Fig. 1. Performance bounds versus uncertainty bound.

for $\delta_1 \in (-0.005, 0.005)$. Furthermore, as shown in Fig. 1, the implicit small gain bound (16) provides a less conservative estimate of robust performance than the robust performance bounds obtained in [2, 4, 16]. The cost bounds in Fig. 1 have been normalized with respect to the worst-case cost $J(\mathcal{U})$.

Next, we only consider robust stability so that $R = \varepsilon^2 I_2$. Since in this case the uncertainty structure consists of a repeated scalar uncertainty, complex-μ can be computed exactly [11] and gives the same stability predictions as the small gain theorem. With $\alpha_1 = 1.702$, $\beta_1 = 483.2$, and $\varepsilon = 0$, the implicit small gain bound (16) was computed for $\delta_1 \in (-10^5, 10^5)$ which shows that $\mu_{\mathrm{isg}}(G(s))$ is not an upper bound for complex-μ.

Example 4.3. Consider a pair of coupled oscillators with uncertain modal frequencies modeled by

$$A = \begin{bmatrix} -0.01 & 1 & 0.01 & 0 \\ -1 & -0.01 & 0 & 0.01 \\ 0.01 & 0 & -0.01 & 4 \\ 0 & 0.01 & -4 & -0.01 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Let $S_1 = A_1$ and $S_2 = A_2$. For this example, the standard small gain condition (30) guarantees robust stability for all $\Delta A \in \mathcal{U}$ with $\gamma^{-1} = 0.0071$, while peak structured singular value analysis using fixed-structure Popov multipliers [14] guarantees robust stability for $\Delta A \in \mathcal{U}$ with $\gamma^{-1} = 0.01$. Furthermore, the real-μ bound with frequency-dependent multipliers [5] predicts robust stability for all $\Delta A \in \mathcal{U}$ with $\gamma^{-1} = 1.493$. With $\alpha_i = 5.0$, $\beta_i = 35.96$, $i = 1, 2$, obtained from (20) and $\varepsilon = 10^{-3}$, the implicit small gain condition (29) guarantees robust stability for $\Delta A \in \mathcal{U}$ with $\gamma^{-1} = 1.499$.

Example 4.4. Finally, we consider a system with two distinct uncertainties corresponding to uncertainty in the diagonal elements and the nondiagonal elements. The nominal system and uncertainty structure matrices are given by

$$A = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let $S_1 = S_2 = A_2$ so that $Z_1 = Z_2 = 0$. For this example, both the standard small gain condition (30) and the peak structured singular value analysis using fixed-structure Popov multipliers [14] guarantee robust stability for all $\Delta A \in \mathscr{U}$ with $\gamma^{-1} = 0.151$, and the real-μ bound with frequency-dependent multipliers [5] predicts robust stability for all $\Delta A \in \mathscr{U}$ with $\gamma^{-1} = 0.219$. With $\alpha_1 = 2.22 \times 10^{-3}$, $\alpha_2 = 1.3 \times 10^{-3}$, $\beta_1 = 2.84 \times 10^4$, $\beta_2 = 2.36 \times 10^4$, and $\varepsilon = 10^{-8}$ the implicit small gain condition (29) guarantees robust stability for $\Delta A \in \mathscr{U}$ with $\gamma^{-1} = 0.219$.

5. Conclusions

A novel peak upper bound for the real structured singular value that has the form of an implicit small gain theorem was developed. Specifically, this upper bound involves a small gain condition for a shifted plant whose dynamics depend upon the uncertainty set bound and, unlike prior bounds, does not require frequency-dependent scales or multipliers. The implicit small gain condition is equivalently characterized by a Riccati equation with a shifted dynamics matrix that additionally guarantees robust H_2 performance. Numerical examples were given to demonstrate the bound.

References

[1] D.S. Bernstein and W.M. Haddad, Robust stability and performance analysis for state space systems via quadratic Lyapunov bounds, *SIAM J. Matrix Anal. Appl.* **11** (1990) 236–271.

[2] D.S. Bernstein, W.M. Haddad, D.C. Hyland and F. Tyan, A maximum entropy-type Lyapunov function for robust stability and performance analysis, *Systems Control Lett.* **21** (1993) 73–87.

[3] S.L. Campbell and C.D. Meyer, *Generalized Inverses of Linear Transformations* (Pitman, London, 1979; reprinted by Dover, New York, 1991).

[4] S.S.L. Chang and T.K.C. Peng, Adaptive guaranteed cost control of systems with uncertain parameters, *IEEE Trans. Automat. Control* **17** (1972) 474–483.

[5] M.K.H. Fan, A.L. Tits and J.C. Doyle, Robustness in the presence of mixed parametric uncertainty and unmodelled dynamics, *IEEE Trans. Automat. Control* **36** (1991) 25–38.

[6] J.H. Friedman, P.T. Kabamba and P.P. Khargonekar, Worst-case and average H_2 performance analysis against real constant parametric uncertainty, *Automatica* **31** (1986) 649–657.

[7] W.M. Haddad and D.S. Bernstein, Explicit construction of quadratic Lyapunov functions for small gain, positivity, circle, and Popov theorems and their application to robust stability. Part I: Continuous-time theory, *Internat. J. Robust Nonlinear Control* **3** (1993) 313–339.

[8] W.M. Haddad and D.S. Bernstein, Parameter-dependent Lyapunov functions and the Popov criterion in robust analysis and synthesis, *IEEE Trans. Automat. Control* **40** (1995) 536–543.

[9] W.M. Haddad, D.S. Bernstein and V. Chellaboina, Generalized mixed-μ bounds for real and complex multiple-block uncertainty with internal matrix structure, in: *Proc. Amer. Control Conf.*, Seattle, WA (1995) 2843–2847; *Internat. J. Control* **64** (1996) 789–806.

[10] P.P. Khargonekar, I.R. Petersen and K. Zhou, Robust stabilization of uncertain linear systems: quadratic stabilizability and H_∞ control theory, *IEEE Trans. Automat. Control* **35** (1990) 356–361.

[11] A. Packard and J.C. Doyle, The complex structured singular value, *Automatica* **29** (1993) 71–109.

[12] A. Packard and P. Pandey, Continuity properties of the real/complex structured singular value, *IEEE Trans. Automat. Control* **38** (1993) 415–428.

[13] I.R. Petersen and C.V. Hollot, A Riccati equation approach to the stabilization of uncertain systems, *Automatica* **22** (1986) 397–411.

[14] A.G. Sparks and D.S. Bernstein, Reliable state space upper bounds for the peak structured singular value, in: *Proc. Amer. Control Conf.*, Seattle, WA (1995) 2419–2423.

[15] F. Tyan and D.S. Bernstein, Shifted quadratic guaranteed cost bounds for robust controller synthesis, in: *Proc. IFAC World Congr.*, San Francisco, CA (1996) **G**, 285–290.

[16] F. Tyan, S.R. Hall and D.S. Bernstein, A double-commutator guaranteed cost bound for robust stability and performance, *Systems Control Lett.* **25** (1995) 125–129.

[17] G. Zames, On the input–output stability of time-varying nonlinear feedback systems, part I: Conditions derived using concepts of loop gain, conicity, and positivity, *IEEE Trans. Automat. Control* **11** (1966) 228–238.