Letting $\boldsymbol{x}_c(v) = (\boldsymbol{x}(v), z(v))$, then (3.21) can be written as

$$\frac{\partial \boldsymbol{x}(v)}{\partial v} a(v) = f(\boldsymbol{x}(v), K_1\boldsymbol{x}(v) + K_2 z(v), v),$$
$$\frac{\partial z(v)}{\partial v} a(v) = G_1\boldsymbol{x}(v) + G_2\boldsymbol{y}(v) \tag{3.22}$$

where

$$\boldsymbol{y}(v) = h(\boldsymbol{x}(v), K_1\boldsymbol{x}(v) + K_2 z(v), v). \tag{3.23}$$

In terms of notation (3.11), $\boldsymbol{x}(v)$, $z(v)$, and $\boldsymbol{y}(v)$ can be uniquely expressed as

$$\boldsymbol{x}(v) = \sum_{l=1}^{k} \phi_l v^{[l]} + O(v^{(k+1)}),$$
$$z(v) = \sum_{l=1}^{k} \theta_l v^{[l]} + O(v^{(k+1)}),$$
$$\boldsymbol{y}(v) = \sum_{l=1}^{k} Y_l v^{[l]} + O(v^{(k+1)}). \tag{3.24}$$

Substituting (3.24) into (3.22) and (3.23), expanding (3.22) and (3.23) as power series in $v^{[l]}$, and identifying the coefficients of $v^{[l]}$, $l = 1, 2, \cdots, k$, yield the result

$$\phi_l A_l = (A + BK_1)\phi_l + BK_2\theta_l + U_l,$$
$$\theta_l A_l = G_1\theta_l + G_2((C + DK_1)\phi_l + DK_2\theta_l + V_l) \tag{3.25}$$

and

$$Y_l = (C + DK_1)\phi_l + DK_2\theta_l + V_l \tag{3.26}$$

where $(U_1, V_1) = (E, F)$, and, for $l = 2, 3, \cdots, k$, (U_l, V_l) depends only on $\phi_1, \cdots, \phi_{l-1}$ and $\theta_1, \cdots, \theta_{l-1}$. Since, for each l, all the eigenvalues of A_l have zero real parts, ϕ_l, θ_l are the unique solution of (3.23). Therefore by Lemma 3.9, we have, for $l = 1, \cdots, k$,

$$Y_l = 0. \tag{3.27}$$

That is, we have

$$h(\boldsymbol{x}(v), K_1\boldsymbol{x}(v) + K_2 z(v), v) = O(v^{(k+1)}).$$

Note that by Remark 3.10, the above argument still holds for the case that there are regular perturbations in the plant and the control law. This completes the sufficiency part.

To show necessity, assume that there exist a kth-order robust controller of the form (2.3). Let $\boldsymbol{x}(v)$ and $z(v)$ be smooth functions satisfying

$$\frac{\partial \boldsymbol{x}(v)}{\partial v} a(v) = f(\boldsymbol{x}(v), \boldsymbol{u}(v), v)$$
$$\frac{\partial z(v)}{\partial v} a(v) = g(\boldsymbol{x}(v), \boldsymbol{y}(v))$$
$$h_c(\boldsymbol{x}_c(v), v) = O(v^{(k+1)})$$

where $\boldsymbol{u}(v) = k(\boldsymbol{x}(v), z(v))$. Now expand $\boldsymbol{x}(v)$ as in (3.24), and let

$$\boldsymbol{u}(v) = \sum_{l=1}^{k} \psi_l v^{[l]} + O(v^{(k+1)})$$

then, for $l = 1, \cdots, k$, (ϕ_l, ψ_l) will necessarily be a solution of the linear matrix equation

$$\phi_l A_l = A\phi_l + B\psi_l + U_l,$$
$$0 = C\phi_l + D\psi_l + V_l. \tag{3.28}$$

Note that kth-order robustness implies that regardless of the perturbations, in particular, of U_l, V_l, (3.28) is solvable for (ϕ_l, ψ_l). This is possible only if (3.18) holds.

Remark 3.13: When $k = 1$, (3.20) is nothing but a linear robust servo-regulator as given in [6]. Applying this controller to a nonlinear plant only annihilates the coefficient Y_1 of the error mapping. This explains why a linear robust servo-regulator cannot tolerate parameter perturbation in a nonlinear plant as shown in the example of [2].

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Nonlinear Controllers for Positive Real Systems with Arbitrary Input Nonlinearities

Dennis S. Bernstein and Wassim M. Haddad

***Abstract*—Input nonlinearities such as saturation can severely degrade closed-loop performance due to integrator windup and other effects. For positive real plants with positive real controllers, we propose a nonlinear controller modification that effectively counteracts the effects of arbitrary input nonlinearities. For this class of problems, we prove global asymptotic stability of the closed-loop system and demonstrate closed-loop performance by means of system simulation.**

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I. Introduction

In certain applications, such as the control of flexible structures, the plant transfer function is known to be positive real. This property arises if the sensor and actuator are colocated and also dual, for example, force actuator and velocity sensor, or torque actuator and angular rate sensor. In practice, the prospects for controlling such systems are quite good since, if sensor and actuator dynamics are negligible, stability is unconditionally guaranteed as long as the controller is strictly positive real [1], [2]. Although there is no general theory yet available for designing positive real controllers, a variety of techniques have been proposed based upon H_2 theory [3]–[5] and H_∞ theory [6]–[8].

The purpose of this note is to address the following question: Given a positive real plant and strictly positive real compensator, how can the compensator be modified if the plant is found to possess an input nonlinearity? For exmaple, proof mass and piezoelectric actuators have force constraints that lead to saturation nonlinearities [9]. There exists an extensive literature devoted to the control saturation and associated windup problem (see, for example, [10]–[14] and the numerous references cited therein).

Our main result (Theorem 1) implies that closed-loop stability is guaranteed so long as the compensator is modified to include a suitable input nonlinearity. Although this result is limited to positive real plants, it turns out that it is not limited to saturation nonlinearity, but rather applies to a large class of input nonlinearities. We require only that the nonlinearity be memoryless and that either its characteristics be known or its output be measurable. The proof of this result is based upon Lyapunov function theory. An alternative proof based upon dissipative system theory [15, 16] shows that the nonlinear controller modification counteracts the effects of the input nonlinearity by recovering the passivity of the plant.

Since our results focus on positive real plants, it is natural to suspect that our results are related to classical absolute stability criteria such as the circle or Popov criterion. Such results are often used to verify stability of closed-loop systems involving saturation nonlinearities [10], [11]. Such criteria, however, require a gain or phase constraint on the linear portion of the loop transfer function. Such constraints are not satisfied in our formulation since both the plant and compensator are positive real, and hence the loop gain need not possess either a gain or phase constraint. In addition, the approach of [10], [11] assumes beforehand that only a finite portion of the nonlinearity is used in closed-loop operation or, equivalently, that the state is confined to a finite region of the state space. Our approach, however, guarantees unconditional global asymptotic stability.

In certain special cases, absolute stability criteria can be used to guarantee closed-loop stability in the presence of an input nonlinearity and without modifying the compensator [17]. Specifically, if the input nonlinearity is sector-bounded and either monotonic or odd monotonic, then closed-loop stability is guaranteed if the product $G(s)G_c(s)Z(s)$ is positive real, where $G(s)$ and $G_c(s)$ denote the linear portion of the plant and the linear compensator, respectively, and $Z(s)$ denotes a stability multiplier of a specified class [18]. If $G_c^{-1}(s)$ belongs to this class of multipliers, then by choosing $Z(s) = G_c^{-1}(s)$, it follows that the closed loop is stable. Our results, however, are valid for nonlinearities that are not necessarily either sector-bounded or odd or monotonic and for positive real compensators that are otherwise arbitrary. Closed-loop stability for such systems is guaranteed by employing the modified nonlinear compensator introduced herein.

II. Input Nonlinearities

Consider the positive real plant

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

$$y(t) = Cx(t) + Du(t) \tag{2}$$

with the positive real feedback compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{3}$$

$$u(t) = -[C_c x_c(t) + D_c y(t)] \tag{4}$$

where $x(t) \in \mathbf{R}^n$, $u(t), y(t) \in \mathbf{R}^m$, $x_c(t) \in \mathbf{R}^{n_c}$, and all matrices are real with appropriate dimensions. In (4), the minus sign denotes the fact that the positive real plant (A, B, C, D) and positive real compensator (A_c, B_c, C_c, D_c) are interconnected in a negative feedback configuration. As discussed in Section I, such compensators can be designed by means of a variety of techniques [3]–[8]. Also, by standard theory [1] the closed-loop system is guaranteed to be stable in the sense of Lyapunov and, furthermore, is asymptotically stable if either the plant or the compensator is strictly positive real.

Now suppose that the plant is found to possess an input nonlinearity so that, in reality, (1) is not valid. Rather, in place of (1) a more accurate plant model is

$$\dot{x}(t) = Ax(t) + B\sigma(u(t)), \tag{5}$$

$$y(t) = Cx(t) + D\sigma(u(t)) \tag{6}$$

where σ: $\mathbf{R}^m \to \mathbf{R}^m$ denotes the input nonlinearity. We shall require the following assumption concerning $\sigma(\cdot)$. Let $u = [u_1 \cdots u_m]^T$ and $\sigma(u) = [\sigma_1(u) \cdots \sigma_m(u)]^T$ denote the components of u and σ.

Assumption 1: For all $i = 1, \cdots, m$, if $u_i = 0$, then $\sigma_i(u) = 0$. That is, the ith component of $\sigma(u)$ vanishes whenever the ith component of u vanishes.

To illustrate the allowable input nonlinearities, consider first the special case $\sigma(u) = [\hat{\sigma}_1(u_1) \cdots \hat{\sigma}_m(u_m)]^T$ of decoupled nonlinearities. In this case, the ith component $\hat{\sigma}_i(u_i)$ of $\sigma(\cdot)$ depends only upon the ith component u_i of u. Now $\hat{\sigma}_i(\cdot)$ can represent an arbitrary scalar nonlinearity that vanishes at the origin. For example, the saturation nonlinearity $\hat{\sigma}_1(u_1) = \mathrm{sat}(u_1)$ is allowable as well as deadzone, quantization, and relay nonlinearities. Note that different types of nonlinearities are permissible. For example, $\sigma(u) = [\mathrm{sat}(u_1)\,\mathrm{sgn}(u_2)]^T$ is allowed, where $\mathrm{sgn}(0) = 0$.

More generally, $\sigma(u)$ may also denote a nonlinearity whose coordinates are not necessarily decoupled. For exmaple, the nonlinearity

$$\begin{aligned} \sigma(u) &= u, \qquad \|u\|_2 \le 1, \\ &= 1, \qquad \|u\|_2 > 1 \end{aligned}$$

where $\|u\|_2 = \sqrt{u^T u}$, satisfies Assumption 1 and has the form of a radial saturation function on $\mathbf{R}^m$.

In the presence of such nonlinearities, closed-loop stability and performance can be affected. In the next section we modify the controller (3), (4) to account for the input nonlinearity to guarantee closed-loop stability.

III. Nonlinear Controller Modification

To counteract the effect of the input nonlinearity $\sigma(u)$ in (5), (6) we modify the controller by replacing the compensator dynamics (3) and control inputs (4) by

$$\dot{x}_c(t) = A_c x_c(t) + B_c \beta(u(t)) y(t), \tag{7}$$

$$u(t) = -[C_c x_c(t) + D_c \beta(u(t)) y(t)] \tag{8}$$

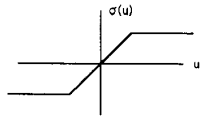
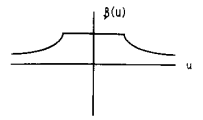
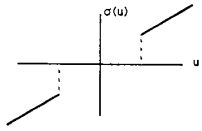
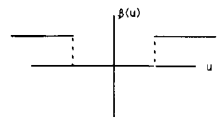
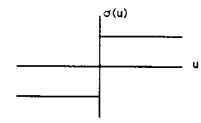
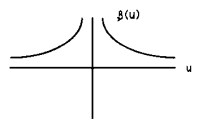
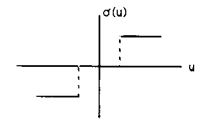
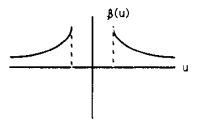
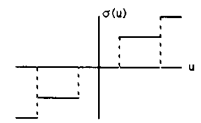
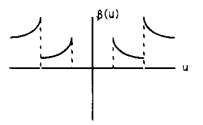
where the controller nonlinearity $\beta(u)$ is the diagonal matrix

$$\beta(u) = \begin{bmatrix} \beta_1(u) & & 0 \\ & \ddots & \\ 0 & & \beta_m(u) \end{bmatrix} \tag{9}$$

where, for $i = 1, \cdots, m$

$$\begin{aligned} \beta_i(u) &= \sigma_i(u)/u_i, \qquad u_i \neq 0, \\ &= \text{arbitrary}, \qquad u_i = 0. \end{aligned} \tag{10}$$

TABLE I

	Input Nonlinearity $\sigma(u)$	Controller Nonlinearity $\beta(u) = \sigma(u)/u$
Saturation		
Deadzone		
Relay		
Relay with Deadzone		
Quantization		

Because of Assumption 1, it can be seen that $\beta_i(u)u_i = \sigma_i(u)$, for all $i = 1, \cdots, m$ and $u \in \mathbb{R}^m$. Since $\sigma_i(u) = 0$ whenever $u_i = 0$, it can also be seen that $\beta_i(u)u_i = \sigma_i(u)$ is satisfied for arbitrary $\beta_i(u)$ whenever $u_i = 0$. Consequently, it follows that

$$\beta(u)u = \sigma(u), \qquad u \in \mathbb{R}^m. \tag{11}$$

By using (11), it thus turns out that the value of $\beta_i(u)$ when $u_i = 0$ plays no role in the subsequent stability analysis.

The form of the controller nonlinearity $\beta(u)$ to be implemented in (7) and (8) is quite simple, requiring only knowledge of $\sigma(u)$ and division by u_i. For the case $m = 1$ and several common nonlinearities, the required controller nonlinearity $\beta(u)$ is illustrated in Table I. It can be seen that a relay nonlinearity $\sigma(u) = \operatorname{sgn}(u)$ leads to unbounded $\beta(u)$ for u near zero. Hence in this case it may be desirable to artificially implement a deadzone so that $\beta(u)$ is bounded. Finally, although all of the input nonlinearities shown in Table I are sector-bounded and odd monotonic, our results are valid for nonlinearities that are not necessarily either sector-bounded or odd or monotonic.

The modified nonlinear controller (7), (8) can be implemented in two different ways. If the model $\sigma(u)$ of the input nonlinearity is known, then $\beta(u)$ can be constructed from (10) by evaluating $\sigma(u)$ in real time for each value of u. If, however, the model $\sigma(u)$ is not available but $\sigma(u(t))$ can be measured during closed-loop operation, then $\beta(u(t))$ can be formed from $u(t)$ and $\sigma(u(t))$ by implementing (10) with $u = u(t)$. This scheme is illustrated in Fig. 1. If, however, neither a model of $\sigma(u)$ nor a measurement of $\sigma(u(t))$ is available, then $\beta(u(t))$ cannot be formed, and our approach does not apply. We assume, however, that either an accurate model of $\sigma(u)$ is available or that the signal $\sigma(u(t))$ is available for feedback.

In the case in which the controller is proper but not strictly proper, that is, $D_c \neq 0$, then the controller output equation contains an

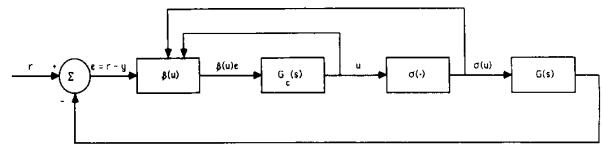


Fig. 1.

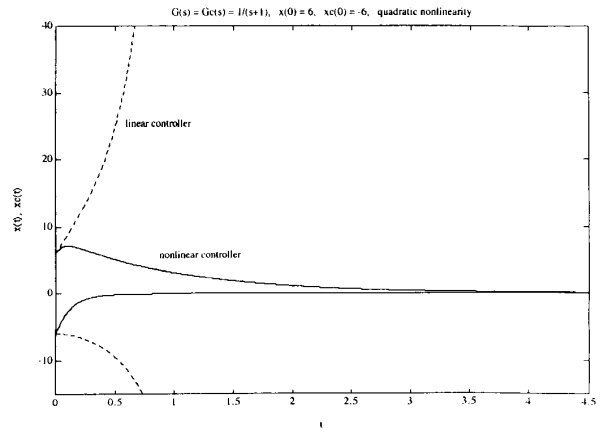


Fig. 2.

algebraic constraint on u. For each choice of D_c and $\beta(u)$ this equation must be examined for solvability in terms of u. For the PI controller with saturation nonlinearity considered in Section VI, it can be shown that (8) has a unique solution u for each x.

IV. Closed-Loop Stability

Our goal now is to show that in spite of the input nonlinearity, closed-loop stability is guaranteed if the modified controller (7), (8) is implemented in place of (3), (4). To do this we invoke the positive real lemma [19], [20] which states that there exist a positive integer p and matrices $P \in \mathbb{R}^{n \times n}$, $L \in \mathbb{R}^{n \times p}$, and $W \in \mathbb{R}^{m \times p}$, where P is positive definite, such that

$$0 = A^T P + PA + L^T L, \tag{12}$$

$$0 = B^T P - C + W^T L, \tag{13}$$

$$0 = D + D^T - W^T W. \tag{14}$$

Furthermore, since the compensator is positive real there exist a positive integer p_c and matrices $P_c \in \mathbb{R}^{n_c \times n_c}$, $L_c \in \mathbb{R}^{n_c \times p_c}$, and $W_c \in \mathbb{R}^{m \times p_c}$, where P_c is positive definite, such that

$$0 = A_c^T P_c + P_c A_c + L_c^T L_c, \tag{15}$$

$$0 = B_c^T P_c - C_c + W_c^T L_c, \tag{16}$$

$$0 = D_c + D_c^T - W_c^T W_c. \tag{17}$$

If the plant or compensator is strictly positive real [21], [22], then (L, A) or (L_c, A_c) is observable, respectively.

Theorem 1: Consider the closed-loop system consisting of the nonlinear plant (5), (6) and the nonlinear controller (7), (8), where the input nonlinearity $\sigma(\cdot)$ satisfies Assumption 1. If the linear plant (1), (2) and the linear compensator (3), (4) are both positive real, then the nonlinear closed-loop system (5)–(8) is stable in the sense of Lyapunov. Furthermore, if the linear plant (1), (2) and the linear compensator (3), (4) are both strictly positive real, then the nonlinear closed-loop system (5)–(8) is globally asymptotically stable.

Proof: Using (5) and (7) we can form the closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} Ax + B\sigma(u) \\ A_c x_c + B_c \beta(u) y \end{bmatrix} \tag{18}$$

where $u = -C_c x_c - D_c \beta(u) y$, $y = Cx + D\sigma(u)$, and consider the Lyapunov function candidate

$$V(x, x_c) = x^T P x + x_c^T P_c x_c \tag{19}$$

where P and P_c are given by (12)–(17). Since $V(x, x_c)$ is positive definite, it remains to examine $\dot{V}(x, x_c)$ to determine closed-loop stability. Using the identities (11)–(17) it thus follows that

$$\begin{aligned}
\dot{V}(x, x_c) &= 2x^T P[Ax + B\sigma(u)] + 2x_c^T P_c[A_c x_c + B_c \beta(u) y] \\
&= -x^T L^T L x - x_c^T L_c^T L_c x_c + 2x^T P B \sigma(u) \\
&\quad + 2x_c^T P_c B_c \beta(u) y \\
&= -x^T L^T L x - x_c^T L_c^T L_c x_c + 2x^T [C^T - L^T W]\sigma(u) \\
&\quad + 2x_c^T [C_c^T - L_c^T W_c]\beta(u) y \\
&= -x^T L^T L x - x_c^T L_c^T L_c x_c + 2x^T C^T \sigma(u) \\
&\quad - 2x^T L^T W \sigma(u) + 2x_c^T C_c^T \beta(u) y - 2x_c^T L_c^T W_c \beta(u) y \\
&= -x^T L^T L x - x_c^T L_c^T L_c x_c + 2[y^T - \sigma^T(u) D^T]\sigma(u) \\
&\quad - 2x^T L^T W \sigma(u) - 2u^T \beta(u) y - 2y^T \beta^T(u) D_c^T \beta(u) y \\
&\quad - 2x_c^T L_c^T W_c \beta(u) y \\
&= -x^T L^T L x - 2x^T L^T W \sigma(u) - \sigma^T(u)[D + D^T]\sigma(u) \\
&\quad - x_c^T L_c^T L_c x_c - 2x_c^T L_c^T W_c \beta(u) y \\
&\quad - y^T \beta^T(u)[D_c + D_c^T]\beta(u) y + 2y^T[\sigma(u) - \beta(u) u] \\
&= -[Lx + W\sigma(u)]^T [Lx + W\sigma(u)] \\
&\quad - [L_c x_c + W_c \beta(u) y]^T [L_c x_c + W_c \beta(u) y] \\
&\le 0
\end{aligned}$$

which proves stability in the sense of Lyapunov for the nonlinear closed-loop system (5)–(8).

To prove asymptotic stability of the nonlinear closed-loop system (5)–(8), we assume that the plant and compensator are both strictly positive real so that, by Lemma 5.1 of [23], there exist $\epsilon > 0$ and $\epsilon_c > 0$ such that (12) and (15) can be replaced by

$$0 = A^T P + PA + L^T L + \epsilon P, \tag{20}$$

$$0 = A_c^T P_c + P_c A_c + L_c^T L_c + \epsilon_c P_c \tag{21}$$

respectively. Using (20) and (21), it follows that $\dot{V}(x, x_c)$ is now given by

$$\begin{aligned}
\dot{V}(x, x_c) < &-\epsilon x^T P x - \epsilon_c x_c^T P_c x_c \\
&- [Lx + W\sigma(u)]^T [Lx + W\sigma(u)] \\
&- [L_c x_c + W_c \beta(u) y]^T [L_c x_c + W_c \beta(u) y] < 0
\end{aligned}$$

which proves global asymptotic stability of the nonlinear closed-loop system (5)–(8).

An alternative proof of Theorem 1 in the case $D = 0$ can be obtained by using dissipative system theory [15], [16]. Let $V_s(x) = (1/2)x^T P x$ be a storage function and consider the supply rate $r(u, \hat{y}) = u^T \hat{y}$, where $\hat{y}$ is obtained by rewriting the closed-loop system (5)–(8) as

$$\dot{x}(t) = Ax(t) + B\sigma(u(t)), \tag{22}$$

$$\hat{y}(t) = \beta(u(t)) C x(t), \tag{23}$$

$$\dot{x}_c(t) = A_c x_c(t) + B_c \hat{y}(t), \tag{24}$$

$$u(t) = -[C_c x_c(t) + D_c \hat{y}(t)]. \tag{25}$$

It thus follows that

$$\begin{aligned}
\dot{V}_s(x) &= \frac{1}{2} x^T (A^T P + PA) x + \sigma^T(u) B^T P x \\
&= -\frac{1}{2} x^T L^T L x + \sigma^T(u) C x \\
&\le u^T \beta(u) C x \\
&= \hat{y}^T u
\end{aligned}$$

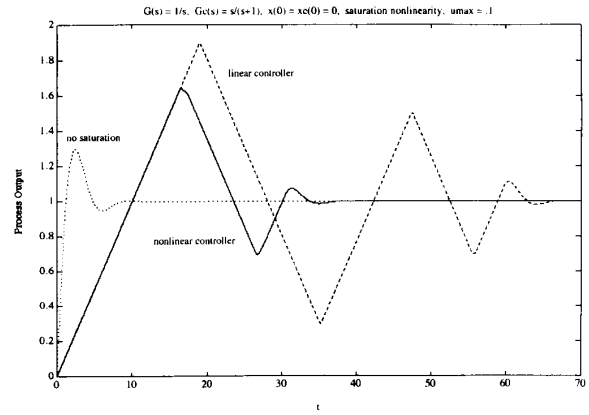


Fig. 3.

which shows that the modified plant (22), (23) is dissipative. Consequently, by dissipative system theory [15], [16] the closed-loop system is Lyapunov stable. It can thus be seen that the nonlinear controller modification counteracts the effects of the input nonlinearity by recovering the passivity of the plant.

V. An Illustrative Example Involving a Quadratic Nonlinearity

As a first example we consider the quadratic nonlinearity $\sigma(u) = u^2$, and, for simplicity, we set $G(s) = G_c(s) = 1/(s+1)$. As shown in Fig. 2, this nonlinearity leads to a finite escape time instability for certain initial conditions. The modified nonlinear controller, however, is guaranteed by Theorem 1 to yield global closed-loop stability. This property is confirmed by Fig. 2.

VI. Application to Integrator Windup

In this section we apply our approach to the problem of integrator windup with a saturation nonlinearity. This problem has been extensively studied by prior researchers; see, for exmaple, [11]–[13] and the numerous references cited therein. Since our results are limited to positive real plants, we cannot make general comparisons with the results of [11]–[13] and others. We can, however, investigate the performance of the modified nonlinear controller in a situation that typically entails integrator windup. We thus consider the illustrative example considered in [12] in which $G(s) = 1/s$, $G_c(s) = (1/s)+1$, the signal $r(t)$ shown in Fig. 1 is a unit step command, and the saturation limits are set at $\pm .1$. Note that because the compensator $G_c(s)$ is not strictly proper, it follows that D_c in (8) is nonzero. Thus the algebraic constraint on u in (8) must be taken into account. It is easy to show that (8) has a unique solution u for each value of x. For the simulation results shown below, we took advantage of the MATLAB®/Simulink feature of automatically solving algebraic loops by means of a Newton–Raphson iteration. Fig. 3 shows the ideal system behavior in the absence of the saturation nonlinearity and compares the performance of the linear controller with the performance of the modified nonlinear controller. The performance improvement attained by the nonlinear controller is directly attributable to the decreased integrator windup as shown in Fig. 4. Finally, the signals u, $\sigma(u) = \mathrm{sat}(u)$, and $\beta(u) = \mathrm{sat}(u)/u$ are shown in Fig. 5. Note that, in accordance with the form of $\beta(u)$ shown in Table I, the multiplicative coefficient $\beta(u)$ is small when β is large, thus effectively "shutting down" the integrator to reduce windup.

VII. Conclusions

A new approach based upon Lyapunov stability theory has been developed for addressing the problem of input nonlinearities. The

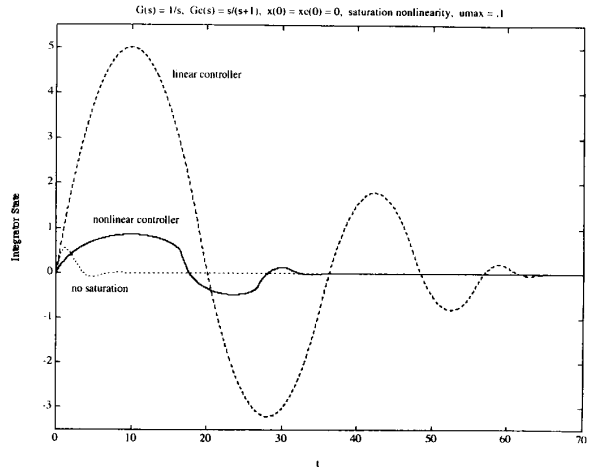


Fig. 4.

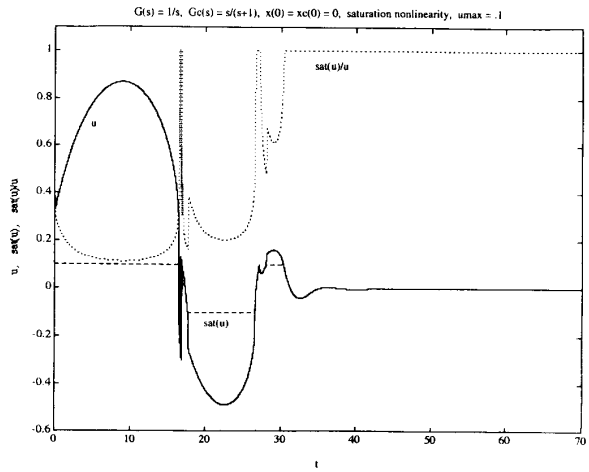


Fig. 5.

approach assumes that the linear plant and compensator are positive real, while the class of input nonlinearities that can be addressed is quite general. To guarantee global asymptotic stability, the linear compensator is modified to form a nonlinear compensator that counteracts the effects of the input nonlinearity by recovering the passivity of the plant. We demonstrated special cases of this result by simulating control systems having quadratic and saturation nonlinearities. Future extensions will focus on extending the result to larger classes of linear plants and compensators.

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Analysis on the Laguerre Formula for Approximating Delay Systems

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***Abstract*—This note provides a detailed analysis on the commonly employed Laguerre formula for approximating delay systems. Convergence proofs are provided, and error bounds are constructed with respect to the L_∞ and L_2 norms.**

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