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Identification of Wiener Systems With Known Noninvertible Nonlinearities[1]

In this paper we develop a method for identifying SISO Wiener-type nonlinear systems, that is, systems consisting of a linear dynamic system followed by a static nonlinearity. Unlike previous techniques developed for Wiener system identification, our approach allows the identification of systems with nonlinearities that are known but not necessarily invertible, continuous, differentiable, or analytic. [DOI: 10.1115/1.1409256]

1 Introduction

Linear system identification has been extensively studied [1–4]. However, nonlinear system identification has received less attention. Since most real systems are nonlinear, techniques for nonlinear system identification are broadly applicable.

There are two basic approaches to nonlinear system identification. *Black box* identification [5–12] assumes little or no model structure. In contrast, *gray box* or *block-structured* identification [13–28], involves the interconnection of two types of input-output blocks. The first type of block is a linear dynamic system, for example $y = G(q^{-1})u$, while the second type is a static nonlinearity, for example $y = u^2$. Gray box identification provides physically meaningful engineering models of the system components but requires prior knowledge of the system structure.

Three common block-structured models are the Hammerstein, nonlinear feedback, and Wiener models. A Hammerstein nonlinear system consists of a static nonlinearity followed by a linear system; a nonlinear feedback system consists of a linear system with a nonlinearity in feedback; and a Wiener system (see Fig. 1) consists of a linear system followed by a static nonlinearity. Wiener models have been used in biological systems [28], as well as representing a linear system with a nonlinear sensor [29,30].

In this work we develop a new method for identifying SISO Wiener systems in a deterministic setting, and then demonstrate the algorithm with noise. Unlike previous work [13–23,30], we consider the identification of systems with nonlinearities that are known but not necessarily invertible, continuous, differentiable, or analytic. Typical noninvertible nonlinearities encountered in practice are polynomial, saturation, deadzone, step [29], quantization, and absolute value functions. We also consider the case where the nonlinearity is unknown, and the linear system $G(q^{-1})$ has a nonzero DC gain, that is, $G(1) \neq 0$ (Section 6).

The method we develop is based on several assumptions. Specifically, we assume the order of the linear system is known. Next, we assume that the inverse image of any point $z \in \mathbb{R}$ under the nonlinearity $\mathcal{N}^{-1}(z)$ consists of a finite number of intervals (points are considered to be intervals of zero length) in $\mathbb{R}$. If the nonlinearity does not initially satisfy this assumption, its domain can often be restricted such that it will: consider $\mathcal{N}(y) = \sin(y)$ and restrict its domain to be bounded. We are unable to identify Wiener systems with certain pathological nonlinearities, such as the rational indicator function $\mathcal{N}(y) = \{1$ if y is rational, 0 if y is irrational$\}$, which have limited engineering applicability. The only other constraint is that the inverse image of the output z must be sufficiently rich in content to permit inversion of a certain matrix. This last condition is a persistency of excitation condition involving the nonlinearity and the linear system. We do not need to assume that the linear system is stable, but we do need to assume that it is controllable and observable. Any modes that cannot be excited and observed cannot be identified.

Our method consists of minimizing a cost function that depends on the vector of unknown system parameters $\boldsymbol{\theta}$ and the intermediate signal $\mathbf{y}$, which is the not necessarily unique inverse image of the output $\mathbf{z}$, that is, $\mathrm{y} \in \mathcal{N}^{-1}(\mathbf{z})$. This cost function can be separated into the sum of two nonnegative functions, one of which involves the intermediate signal $\mathbf{y}$ and the input $\mathbf{u}$ but not $\boldsymbol{\theta}$, while the other can be set to zero by a suitable choice of $\boldsymbol{\theta}$. The advantage of this decomposition is that these functions can be minimized sequentially. Without introducing this decomposition, the minimization problem would be significantly more complex.

This paper is organized as follows. In Section 2 we present the notation used throughout the paper. In Section 3 we define the identification problem. In Section 4 we present a decomposition of the least squares method in non-conventional notation. In Section 5 we consider increasingly complex nonlinear identification problems involving systems with invertible nonlinearities, noninvertible nonlinearities without constant regions, and noninvertible nonlinearities with constant regions. In Section 6 we extend the method to systems with unknown nonlinearities and nonzero DC

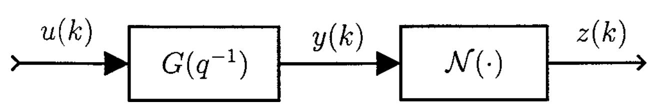

Fig. 1 Block diagram of a Wiener system

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gains. In Section 7 we present simulation results to illustrate the method. Finally, Section 8 contains concluding remarks.

2 Notation

q^{-1} represents the backward shift operator, $\mathbb{R}$ represents the set of real numbers. $\|\mathbf{v}\|$ represents the standard Euclidean norm. $\mathbf{I}_n$ represents the $n\times n$ identity matrix, $\mathbf{0}_{n\times m}$ represents the $n\times m$ zero matrix, and $\mathbf{V}^T$ denotes the transpose of $\mathbf{V}$.

3 Problem Definition

Consider the SISO Wiener system with nth-order linear dynamics

$$y(k)\triangleq G(q^{-1},\boldsymbol{\theta})u(k)=\frac{b_0+b_1q^{-1}+\ldots+b_nq^{-n}}{1+a_1q^{-1}+\ldots+a_nq^{-n}}u(k), \quad (1)$$

where we define the vector of system parameters

$$\boldsymbol{\theta}\triangleq[a_1\ \cdots\ a_n\ \ b_0\ \cdots\ b_n]^T\in\mathbb{R}^{2n+1}, \quad (2)$$

and the known, but not necessarily invertible, output nonlinearity

$$z(k)\triangleq\mathcal{N}(y(k)), \quad (3)$$

where $\mathcal{N}:\mathcal{D}\to\mathbb{R}$, $\mathcal{D}\subset\mathbb{R}$. To identify this system we determine an estimate $\hat{\boldsymbol{\theta}}\triangleq[\hat{a}_1\cdots\hat{a}_n\,\hat{b}_0\cdots\hat{b}_n]^T$ of $\boldsymbol{\theta}$ such that the system

$$\hat{y}(k)\triangleq G(q^{-1},\hat{\boldsymbol{\theta}})u(k)=\frac{\hat{b}_0+\hat{b}_1q^{-1}+\ldots+\hat{b}_nq^{-n}}{1+\hat{a}_1q^{-1}+\ldots+\hat{a}_nq^{-n}}u(k), \quad (4)$$

$$\hat{z}(k)\triangleq\mathcal{N}(\hat{y}(k)), \quad (5)$$

approximates the true system. For identification, we assume knowledge of the signals u and z, but we do not assume access to the intermediate signal y.

4 A Decomposition of Least Squares Identification

Here we rederive least squares identification for linear systems. In doing so we introduce nonstandard notation that will be useful in formulating the Wiener system identification method. Let $p\geqslant 2n+1$. For an $n+p$ sequence of inputs $u(1),u(2),\ldots,u(n+p)$ and intermediate variables $y(1),y(2),\ldots,y(n+p)$ we define

$$\mathbf{u}\triangleq[u(1)\ \ u(2)\ \ \ldots\ \ u(n+p)]^T\in\mathbb{R}^{n+p},$$

$$\mathbf{y}\triangleq[y(1)\ \ y(2)\ \ \ldots\ \ y(n+p)]^T\in\mathbb{R}^{n+p}.$$

Next, let

$$\mathbf{P}\triangleq[\mathbf{I}_p\ \ \mathbf{0}_{p\times n}]\in\mathbb{R}^{p\times n+p},$$

$$\mathbf{R}\triangleq\begin{bmatrix}\mathbf{0}_{n+p-1\times 1} & \mathbf{I}_{n+p-1}\\ 1 & \mathbf{0}_{1\times n+p-1}\end{bmatrix}\in\mathbb{R}^{n+p\times n+p},$$

and

$$\mathbf{e}_{ij}\triangleq[\mathbf{0}_{1\times i-1}\ \ 1\ \ \mathbf{0}_{1\times j-i}]^T\in\mathbb{R}^{j}, \quad (6)$$

where $\mathbf{P}$ removes the last n components from a vector, $\mathbf{R}$ moves the first component of a vector to the last position, and $\mathbf{e}_{i,j}$ is the ith column of $\mathbf{I}_j$. Now, define the input part of the regression matrix $\mathbf{U}:\mathbb{R}^{n+p}\to\mathbb{R}^{p\times 2n+1}$ and the measurement part of the regression matrix $\mathbf{Y}:\mathbb{R}^{n+p}\to\mathbb{R}^{p\times 2n+1}$ by

$$\mathbf{U}(u)\triangleq\mathbf{P}\sum_{i=1}^{n+1}\mathbf{R}^{i-1}\mathbf{u}\mathbf{e}^T_{2(n+1)-i,2n+1}$$

$$=\begin{bmatrix}\mathbf{0}_{1\times n} & u(n+1) & \cdots & u(1)\\ \mathbf{0}_{1\times n} & u(n+2) & \cdots & u(2)\\ \vdots & \vdots & & \vdots\\ \mathbf{0}_{1\times n} & u(n+p) & \cdots & u(p)\end{bmatrix}, \quad (7)$$

$$\mathbf{Y}(\hat{\mathbf{y}})\triangleq\mathbf{P}\sum_{i=1}^{n}\mathbf{R}^{i-1}\hat{\mathbf{y}}\mathbf{e}^T_{n+1-i,2n+1}$$

$$=\begin{bmatrix}\hat{y}(n) & \cdots & \hat{y}(1) & \mathbf{0}_{1\times n+1}\\ \hat{y}(n+1) & \cdots & \hat{y}(2) & \mathbf{0}_{1\times n+1}\\ \vdots & & \vdots & \vdots\\ \hat{y}(n+p-1) & \cdots & \hat{y}(p) & \mathbf{0}_{1\times n+1}\end{bmatrix}. \quad (8)$$

We then define the regression matrix $\boldsymbol{\Phi}:\mathbb{R}^{n+p}\times\mathbb{R}^{n+p}\to\mathbb{R}^{p\times 2n+1}$ as

$$\boldsymbol{\Phi}(\mathbf{u},\hat{\mathbf{y}})\triangleq\mathbf{U}(\mathbf{u})-\mathbf{Y}(\hat{\mathbf{y}})$$

$$=\begin{bmatrix}-\hat{y}(n) & \cdots & -\hat{y}(1) & u(n+1) & \cdots & u(1)\\ -\hat{y}(n+1) & \cdots & -\hat{y}(2) & u(n+2) & \cdots & u(2)\\ \vdots & & & \vdots & & \vdots\\ -\hat{y}(n+p-1) & \cdots & -\hat{y}(p) & u(n+p) & \cdots & u(p)\end{bmatrix}. \quad (9)$$

Thus

$$\mathbf{Q}\mathbf{y}=\boldsymbol{\Phi}(\mathbf{u},\mathbf{y})\,\boldsymbol{\theta}, \quad (10)$$

where $\mathbf{Q}\in\mathbb{R}^{p\times n+p}$ defined by

$$\mathbf{Q}\triangleq[\mathbf{0}_{p\times n}\ \ \mathbf{I}_p] \quad (11)$$

removes the first n components of a vector.

With this notation we define the standard least squares cost function $J_{\mathrm{LS}}:\mathbb{R}^{n+p}\times\mathbb{R}^{n+p}\times\mathbb{R}^{2n+1}\to\mathbb{R}$ as

$$J_{\mathrm{LS}}(\mathbf{u},\hat{\mathbf{y}},\hat{\boldsymbol{\theta}})\triangleq\|\mathbf{Q}\hat{\mathbf{y}}-\boldsymbol{\Phi}(\mathbf{u},\hat{\mathbf{y}})\,\hat{\boldsymbol{\theta}}\|. \quad (12)$$

Now we decompose the cost function (12) by expanding and completing the square

$$\begin{aligned}J^2_{\mathrm{LS}}(\mathbf{u},\hat{\mathbf{y}},\hat{\boldsymbol{\theta}})&=\hat{\mathbf{y}}^T\mathbf{Q}^T\mathbf{Q}\hat{\mathbf{y}}+\hat{\boldsymbol{\theta}}^T\boldsymbol{\Phi}^T(\mathbf{u},\hat{\mathbf{y}})\boldsymbol{\Phi}(\mathbf{u},\hat{\mathbf{y}})\,\hat{\boldsymbol{\theta}}\\ &\quad-\hat{\boldsymbol{\theta}}^T\boldsymbol{\Phi}^T(\mathbf{u},\hat{\mathbf{y}})\mathbf{Q}\hat{\mathbf{y}}-\hat{\mathbf{y}}^T\mathbf{Q}^T\boldsymbol{\Phi}(\mathbf{u},\hat{\mathbf{y}})\,\hat{\boldsymbol{\theta}}\\ &=\hat{\mathbf{y}}^T\mathbf{Q}^T\mathbf{Q}\hat{\mathbf{y}}+\hat{\boldsymbol{\theta}}^T\boldsymbol{\Phi}^T(\mathbf{u},\hat{\mathbf{y}})\boldsymbol{\Phi}(\mathbf{u},\hat{\mathbf{y}})\,\hat{\boldsymbol{\theta}}-\hat{\boldsymbol{\theta}}^T\boldsymbol{\Phi}^T(\mathbf{u},\hat{\mathbf{y}})\mathbf{Q}\hat{\mathbf{y}}\\ &\quad-\hat{\mathbf{y}}^T\mathbf{Q}^T\boldsymbol{\Phi}(\mathbf{u},\hat{\mathbf{y}})\,\hat{\boldsymbol{\theta}}+\hat{\mathbf{y}}^T\mathbf{Q}^T\boldsymbol{\Phi}(\mathbf{u},\hat{\mathbf{y}})\\ &\quad\times(\boldsymbol{\Phi}^T(\mathbf{u},\hat{\mathbf{y}})\boldsymbol{\Phi}(\mathbf{u},\hat{\mathbf{y}}))^{-1}\boldsymbol{\Phi}^T(\mathbf{u},\hat{\mathbf{y}})\mathbf{Q}\hat{\mathbf{y}}-\hat{\mathbf{y}}^T\mathbf{Q}^T\boldsymbol{\Phi}(\mathbf{u},\hat{\mathbf{y}})\\ &\quad\times(\boldsymbol{\Phi}(\mathbf{u},\hat{\mathbf{y}})^T\boldsymbol{\Phi}(u,\hat{y}))^{-1}\boldsymbol{\Phi}^T(\mathbf{u},\hat{\mathbf{y}})\mathbf{Q}\hat{\mathbf{y}}\\ &=\hat{\mathbf{y}}^T\mathbf{Q}^T(\mathbf{I}_p-\boldsymbol{\Phi}(\mathbf{u},\hat{y})(\boldsymbol{\Phi}^T(\mathbf{u},\hat{\mathbf{y}})\boldsymbol{\Phi}(\mathbf{u},\hat{\mathbf{y}}))^{-1}\\ &\quad\times\boldsymbol{\Phi}^T(\mathbf{u},\hat{\mathbf{y}}))\mathbf{Q}\hat{\mathbf{y}}+(\hat{\boldsymbol{\theta}}-\boldsymbol{\Psi}(\mathbf{u},\hat{\mathbf{y}})\mathbf{Q}\hat{\mathbf{y}})^T\boldsymbol{\Phi}^T(\mathbf{u},\hat{\mathbf{y}})\boldsymbol{\Phi}(\mathbf{u},\hat{\mathbf{y}})\\ &\quad\times(\hat{\boldsymbol{\theta}}-\boldsymbol{\Psi}(\mathbf{u},\hat{\mathbf{y}})\mathbf{Q}\hat{\mathbf{y}})\\ &=J^2_y(\mathbf{u},\hat{\mathbf{y}})+J^2_\theta(\mathbf{u},\hat{\mathbf{y}},\hat{\boldsymbol{\theta}}),\end{aligned} \quad (13)$$

where

$$J_{\mathbf{y}}(\mathbf{u},\hat{\mathbf{y}}) \triangleq \|\mathbf{\Pi}^{\perp}_{\Phi(u,\hat{y})}\mathbf{Q}\hat{\mathbf{y}}\|,$$

$$J_{\theta}(\mathbf{u},\hat{\mathbf{y}},\hat{\boldsymbol{\theta}}) \triangleq \|\mathbf{\Phi}(\mathbf{u},\hat{\mathbf{y}})(\hat{\boldsymbol{\theta}}-\mathbf{\Psi}(\mathbf{u},\hat{\mathbf{y}})\mathbf{Q}\hat{\mathbf{y}})\|,$$

and $\mathbf{\Psi}:\mathbb{R}^{n+p}\times\mathbb{R}^{n+p}\to\mathbb{R}^{2n+1\times p}$ is the left inverse of $\mathbf{\Phi}$ given by

$$\mathbf{\Psi}(\mathbf{u},\hat{\mathbf{y}}) \triangleq (\mathbf{\Phi}^T(\mathbf{u},\hat{\mathbf{y}})\mathbf{\Phi}(\mathbf{u},\hat{\mathbf{y}}))^{-1}\mathbf{\Phi}^T(\mathbf{u},\hat{\mathbf{y}}). \tag{14}$$

Note that $\mathbf{\Phi}(\mathbf{u},\hat{\mathbf{y}})$ is assumed to have full rank which is a persistency of excitation condition. The orthogonal projector $\mathbf{\Pi}^{\perp}_{\mathbf{V}} \in \mathbb{R}^{n\times n}$ is defined as

$$\mathbf{\Pi}^{\perp}_{\mathbf{V}} \triangleq \mathbf{I}-\mathbf{V}(\mathbf{V}^T\mathbf{V})^{-1}\mathbf{V}^T. \tag{15}$$

Let

$$\hat{\boldsymbol{\theta}}(\mathbf{u},\hat{\mathbf{y}}) \triangleq \mathbf{\Psi}(\mathbf{u},\hat{\mathbf{y}})\mathbf{Q}\hat{\mathbf{y}}. \tag{16}$$

Thus $\mathbf{J}_{\theta}(\mathbf{u},\hat{\mathbf{y}},\hat{\boldsymbol{\theta}}(\mathbf{u},\hat{\mathbf{y}}))=0$, leaving $J_{\mathrm{LS}}(\mathbf{u},\hat{\mathbf{y}},\hat{\boldsymbol{\theta}}(\mathbf{u},\hat{\mathbf{y}}))=J_{\mathbf{y}}(\mathbf{u},\hat{\mathbf{y}}) = \|\mathbf{\Pi}^{\perp}_{\mathbf{\Phi}(\mathbf{u},\hat{\mathbf{y}})}\mathbf{Q}\hat{\mathbf{y}}\|$, which is the standard least squares result.

We now proceed to describe the Wiener system identification method, where the nonstandard notation introduced in this section will prove useful in calculating gradients for numerical optimization routines.

5 Wiener Identification Method

First we consider identification of Wiener systems with invertible nonlinearities, then noninvertible nonlinearities with no constant regions, and, finally, noninvertible nonlinearities with constant regions.

In the case of an invertible nonlinearity $\mathcal{N}:\mathcal{D}\to\mathbb{R}$, the inverse image $\mathcal{N}^{-1}(z)$ of z is single valued. Hence we set

$$\hat{\mathbf{y}}=\mathcal{N}^{-1}(\mathbf{z}), \tag{17}$$

where the vector of measurements $\mathbf{z}\in\mathbb{R}^{n+p}$ is given by

$$\mathbf{z} \triangleq [z(1)\ \ z(2)\ \ \cdots\ \ z(n+p)]^T, \tag{18}$$

and, for

$$\hat{\mathbf{y}} \triangleq [\hat{y}(1)\ \ \hat{y}(2)\ \ \cdots\ \ \hat{y}(n+p)]^T, \tag{19}$$

we adopt the vector notation

$$\mathcal{N}(\hat{\mathbf{y}}) \triangleq [\mathcal{N}(\hat{y}(1))\ \ \cdots\ \ \mathcal{N}(\hat{y}(n+p))]^T. \tag{20}$$

and

$$\mathcal{N}^{-1}(\mathbf{z}) \triangleq \{\hat{\mathbf{y}}\in\mathcal{D}^{n+p} \,|\, \mathcal{N}(\hat{\mathbf{y}})=\mathbf{z}\}\subset\mathcal{D}^{n+p}. \tag{21}$$

Then we can proceed as in the previous section by obtaining an estimate $\hat{\boldsymbol{\theta}}(\mathbf{u},\hat{\mathbf{y}})$ with $\hat{\mathbf{y}}$ given by (17).

Next, we no longer assume that $\mathcal{N}$ is invertible, but rather we assume that the inverse image of every point in the range space $\mathbb{R}^{n+p}$ of $\mathcal{N}$ is a finite set in the domain $\mathcal{D}$ of $\mathcal{N}$. Hence $\mathcal{N}^{-1}(\mathbf{z})$ is also finite for all $\mathbf{z}\in\mathbb{R}^{n+p}$. The least squares decomposition described in Section 4 can then be applied to each element of $\mathcal{N}^{-1}(\mathbf{z})$. We then evaluate $J_y(\mathbf{u},\hat{\mathbf{y}})$ at each point $\hat{\mathbf{y}}\in\mathcal{N}^{-1}(\mathbf{z})$, and let $\hat{\mathbf{y}}^*$ denote a minimizer of $J_y(\mathbf{u},\hat{\mathbf{y}})$. Our optimal estimate of the system parameters is thus given by $\hat{\boldsymbol{\theta}}(\mathbf{u},\hat{\mathbf{y}}^*)$.

Since $\mathcal{N}$ is not necessarily invertible in this case, it follows that $\hat{\mathbf{y}}^*$, and thus $\hat{\boldsymbol{\theta}}(\mathbf{u},\hat{\mathbf{y}}^*)$ need not be unique. For example, consider the quadratic nonlinearity, $\mathcal{N}(y)=y^2$, where $\mathcal{N}(y)=\mathcal{N}(-y)$. In this case there are multiple solutions that minimize J_{LS} and multiple estimates of the system parameters, all of which are consistent with the measured data.

Now we consider a more general case in which we assume $\mathcal{N}$ satisfies the property that, for all $z\in\mathbb{R}$, the primage $\mathcal{N}^{-1}(z)$ of z consists of a finite union of intervals. Points are considered to be intervals of zero length. This assumption allows us to consider nonlinearities that have constant regions such as deadzone or saturation. The inverse image $\mathcal{N}^{-1}(\mathbf{z})$ of $\mathbf{z}$ will consist of a union of (perhaps lower dimensional) cubes in $\mathbb{R}^{n+p}$. Since $\mathcal{N}^{-1}(\mathbf{z})$ consists of a finite number of disjoint, convex regions in $\mathbb{R}^{n+p}$, we minimize $J_y(\mathbf{u},\hat{\mathbf{y}})$ over each region by using a nonlinear optimization routine, obtaining a (possibly local) minimum and minimizer for each region. Since the number of regions is finite, if the minima obtained are global then we can determine the global minimizer $\hat{\mathbf{y}}^*$, which yields the estimate of the system parameters $\hat{\boldsymbol{\theta}}(\mathbf{u},\hat{\mathbf{y}}^*)$. Of course, J_y is in general not convex and local minima may be obtained.

We can apply a variety of numerical optimization routines to minimize a nonlinear cost function over a convex region. To use a gradient-based method, we differentiate $J_y^2(\mathbf{u},\hat{\mathbf{y}})$ with respect to $\hat{\mathbf{y}}$ to find the gradient $\mathbf{G}\in\mathbb{R}^{n+p}$ of $J_y^2(\mathbf{u},\hat{\mathbf{y}})$ given by

$$\begin{aligned}\mathbf{G}(\mathbf{u},\hat{\mathbf{y}}) &\triangleq \frac{\partial J_y^2(\mathbf{u},\hat{y})}{\partial\hat{\mathbf{y}}}\\ &= 2\mathbf{Q}^T\mathbf{\Pi}^{\perp}_{\Phi(u,\hat{y})}\mathbf{Q}\hat{\mathbf{y}}\\ &\quad + 2\sum_{i=1}^{n}\mathbf{R}^{Ti-1}\mathbf{P}^T\mathbf{\Pi}^{\perp}_{\Phi(u,\hat{y})}\mathbf{Q}\hat{\mathbf{y}}\hat{\mathbf{y}}^T\mathbf{Q}^T\mathbf{\Psi}^T(\mathbf{u},\hat{\mathbf{y}})\mathbf{e}_{n+1-i,2n+1}.\end{aligned} \tag{22}$$

We now compute the Hessian $\mathbf{H}\in\mathbb{R}^{n+p\times n+p}$ of $J_y^2(\mathbf{u},\hat{\mathbf{y}})$. The jth column of $\mathbf{H}$ is given by

$$\begin{aligned}\mathbf{H}(\mathbf{u},\hat{\mathbf{y}})(:,j) &= \frac{\partial}{\partial\hat{\mathbf{y}}}\left(\frac{\partial J_y(\mathbf{u},\hat{\mathbf{y}})^2}{\partial y(j)}\right)\\ &= \frac{\partial}{\partial\hat{\mathbf{y}}}\mathbf{e}^T_{j,n+p}\mathbf{G} = 2\mathbf{Q}^T\mathbf{\Pi}^{\perp}_{\Phi(u,\hat{y})}\mathbf{Q}\mathbf{e}_{j,n+p}\\ &\quad + 2\sum_{i=1}^{n}\Bigg\{\mathbf{Q}^T\mathbf{V}(i,j)\mathbf{Q}\hat{\mathbf{y}}\\ &\quad + \mathbf{R}^{Ti-1}\mathbf{P}^T\mathbf{\Pi}^{\perp}_{\Phi(u,\hat{y})}\mathbf{Q}\mathbf{U}(j)\mathbf{Q}^T\mathbf{\Psi}^T(\mathbf{u},\hat{\mathbf{y}})\mathbf{e}_{n+1-i,2n+1}\\ &\quad + \sum_{\ell=1}^{n}[\mathbf{R}^{T\ell-1}\mathbf{P}^T(\mathbf{\Psi}^T(\mathbf{u},\hat{\mathbf{y}})\mathbf{W}(i,j)\\ &\quad + \mathbf{\Pi}^{\perp}_{\Phi(u,\hat{y})}\mathbf{X}(i,j)\mathbf{\Phi}(\mathbf{u},\hat{\mathbf{y}})\\ &\quad - \mathbf{Q}\hat{\mathbf{y}}\hat{\mathbf{y}}^T\mathbf{Q}^T\mathbf{\Pi}^{\perp}_{\Phi(u,\hat{y})}\mathbf{P}\mathbf{R}^{i-1}\mathbf{e}_{j,n+p}\mathbf{e}^T_{n+1-i,2n+1})\\ &\quad \times(\mathbf{\Phi}^T(\mathbf{u},\hat{\mathbf{y}})\mathbf{\Phi}(\mathbf{u},\hat{\mathbf{y}}))^{-1}\mathbf{e}_{n+1-\ell,2n+1}]\Bigg\},\end{aligned} \tag{23}$$

where the symmetric matrices $\mathbf{U}(i)\in\mathbb{R}^{p\times p}$, $\mathbf{V}(i,j)\in\mathbb{R}^{p\times p}$, $\mathbf{W}(i,j)\in\mathbb{R}^{2n+1\times 2n+1}$, and $\mathbf{X}(i,j)\in\mathbb{R}^{p\times p}$ are given by

$$\mathbf{U}(i) \triangleq \mathbf{e}_{i,n+p}\hat{\mathbf{y}}^T + \hat{\mathbf{y}}\mathbf{e}_{i,n+p}^T, \tag{24}$$

$$\mathbf{V}(i,j) \triangleq \mathbf{\Pi}_{\Phi(u,\hat{y})}^{\perp}\mathbf{P}\mathbf{R}^{i-1}e_{j,n+p}e_{n+1-i,2n+1}^T\mathbf{\Psi}(\mathbf{u},\hat{\mathbf{y}}) + \mathbf{\Psi}^T(\mathbf{u},\hat{\mathbf{y}})\mathbf{e}_{n+1-i,2n+1}\mathbf{e}_{j,n+p}^T\mathbf{R}^{Ti-1}\mathbf{P}^T\mathbf{\Pi}_{\Phi(u,\hat{y})}^{\perp}, \tag{25}$$

$$\mathbf{W}(i,j) \triangleq \mathbf{\Phi}^T(\mathbf{u},\hat{\mathbf{y}})\mathbf{Q}\hat{\mathbf{y}}\hat{\mathbf{y}}^T\mathbf{Q}^T\mathbf{\Pi}_{\Phi(u,\hat{y})}^{\perp}\mathbf{P}\mathbf{R}^{i-1}e_{j,n+p}e_{n+1-i,2n+1}^T + \mathbf{e}_{n+1-i,2n+1}\mathbf{e}_{j,n+p}^T\mathbf{R}^{Ti-1}\mathbf{P}^T\mathbf{\Pi}_{\Phi(u,\hat{y})}^{\perp}\mathbf{Q}\hat{\mathbf{y}}\hat{\mathbf{y}}^T\mathbf{Q}^T\mathbf{\Phi}(\mathbf{u},\hat{\mathbf{y}}), \tag{26}$$

$$\mathbf{X}(i,j) \triangleq \mathbf{Q}\hat{\mathbf{y}}\hat{\mathbf{y}}^T\mathbf{Q}^T\mathbf{\Psi}^T(\mathbf{u},\hat{\mathbf{y}})\mathbf{e}_{n+1-i,2n+1}\mathbf{e}_{j,n+p}^T\mathbf{R}^{Ti-1}\mathbf{P}^T + \mathbf{P}\mathbf{R}^{i-1}\mathbf{e}_{j,n+p}\mathbf{e}_{n+1-i,2n+1}^T\mathbf{\Psi}(\mathbf{u},\hat{\mathbf{y}})\mathbf{Q}\hat{\mathbf{y}}\hat{\mathbf{y}}^T\mathbf{Q}^T. \tag{27}$$

The minima for the examples in Section 7 were computed using a subspace trust region method in the Optimization Toolbox in MATLAB. Let R denote the number of disjoint regions comprising $\mathcal{N}^{-1}(\mathbf{z})$, let q_r denote the dimension of the rth region. The method involves R minimizations, each one a minimization over a q_r-cube, involving q_r variables. However, there are a few simple special cases:

1. $\mathcal{N}$ invertible, $R=1$, and $q_1=0$ (for example, $\mathcal{N}(y) = \arctan(y)$). In this case the problem reduces to a single function evaluation.
2. $\mathcal{N}$ noninvertible, but monotonic, $R=1$ (for example, $\mathcal{N}(y) = \mathrm{sat}(y)$). In this case the problem reduces to a single minimization over a q_1-cube in q_1 variables.
3. $\mathcal{N}$ noninvertible, but contains no constant regions, $q_r=0$ (for example, $\mathcal{N}(y)=y^2$). In this case the problem reduces to R function evaluations.

6 Unknown Nonlinearity With $G(1)\neq 0$

The first step of the method is to obtain a representation of the nonlinearity, then we proceed as above, with a few modifications. We assume that the linear system has a nonzero DC gain, $G(1) \neq 0$.

We begin by applying a step input to the system. We then measure the output $z=\mathcal{N}(G(1)u)$. We repeat this experiment with step inputs of different amplitude until we have sufficiently many points to characterize the nonlinearity $\mathcal{N}$ over the range of interest. In the examples that follow, we have arbitrarily chosen a uniform distribution of step inputs, but the density of data can be increased in regions where the nonlinearity has high variation.

Note that the system $H(q^{-1})=\alpha G(q^{-1})$, $\mathcal{M}(y)=\mathcal{N}(y/\alpha)$ has the same input-output map for all nonzero $\alpha\in\mathbb{R}$. Hence without loss of generality, we normalize the DC gain of the linear system to one, that is $G(1)=1$.

We then apply a function approximation technique to obtain $\hat{\mathcal{N}}:\mathbb{R}\to\mathbb{R}, \hat{\mathcal{N}}\approx\mathcal{N}$. In the examples that follow we use piecewise linear interpolation.

Using the approximate nonlinearity $\hat{\mathcal{N}}$, we turn to the problem of identifying the linear system. Since the DC gain of the system has been normalized, we can remove one of the unknowns from the parameter vector. Hence we set

$$b_0 = 1 + \sum_{i=1}^{n} a_i - b_i, \tag{28}$$

and the previous development is modified accordingly.

7 Numerical Examples

In this section we apply the identification algorithm to several numerical examples. In all cases we use a discretized second-order spring-mass-dashpot system given by

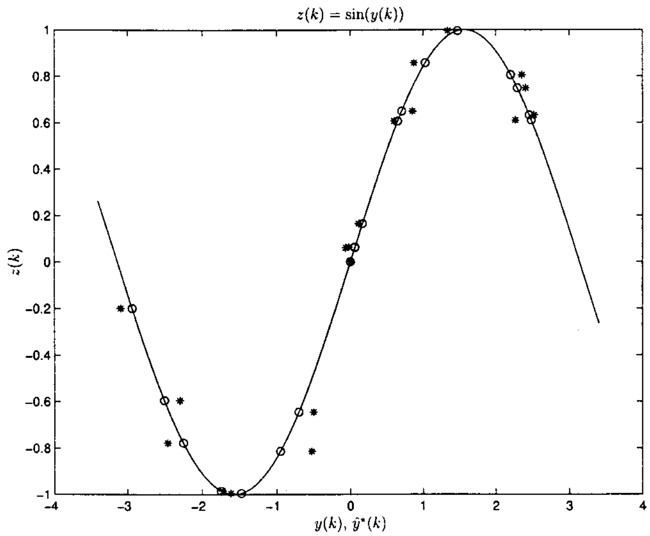

Fig. 2 Sin nonlinearity. *'s represent identification data points (*y* and *z*), o's represent optimal estimates ($\hat{y}$* and *z*)

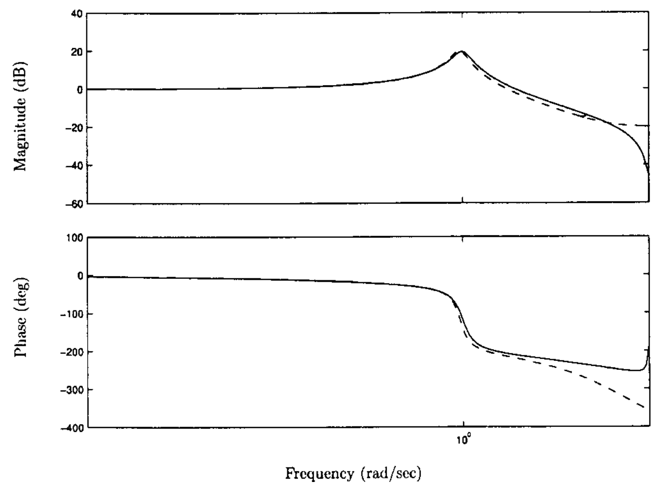

Fig. 3 Frequency response of actual (—) and estimated (--) systems of example 3

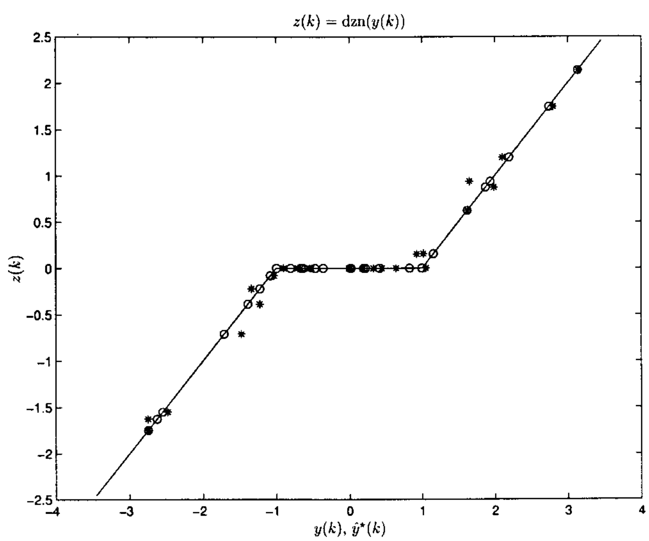

Fig. 4 Deadzone nonlinearity, *'s represent identification data points (*y* and *z*), o's represent optimal Estimates ($\hat{y}$* and *z*)

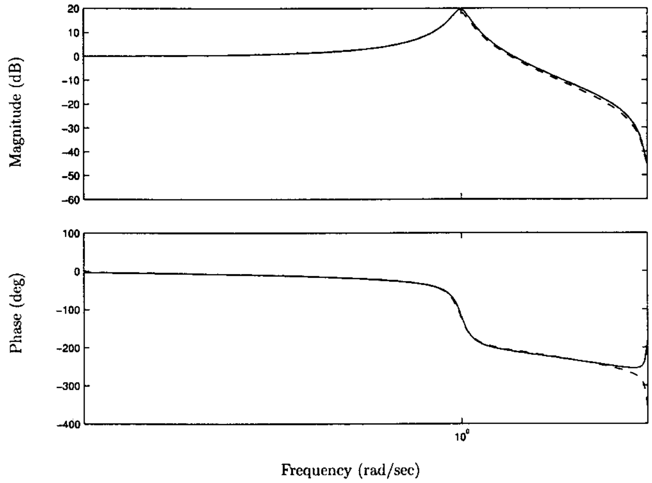

Fig. 5 Frequency response of actual (—) and estimated (--) systems of example 5

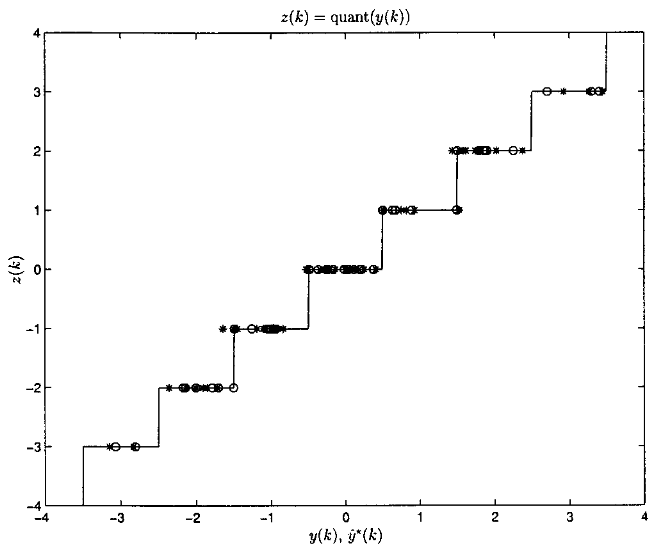

Fig. 6 Quantization nonlinearity. *'s represent identification data points (*y* and *z*), o's represent optimal estimates (*ŷ and *z*)**

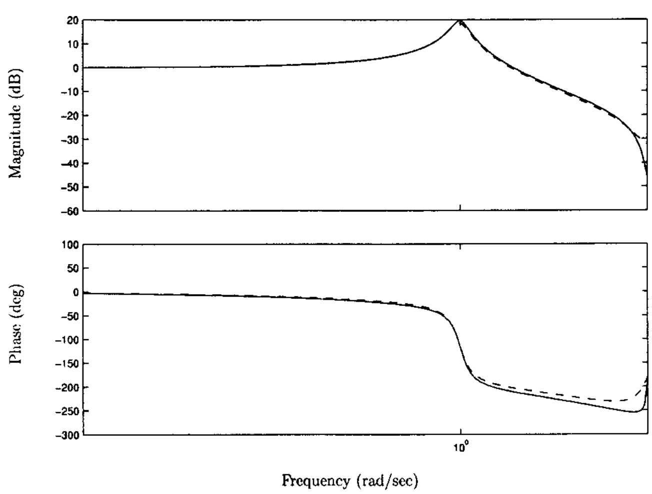

Fig. 7 Frequency response of actual (—) and estimated (--) systems of example 6

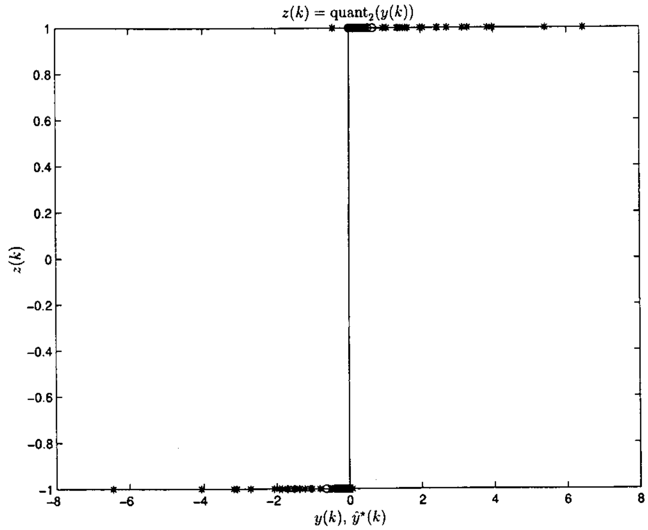

Fig. 8 Signum nonlinearity. *'s represent identification data points (*y* and *z*), o's represent optimal estimates (*ŷ and *z*)**

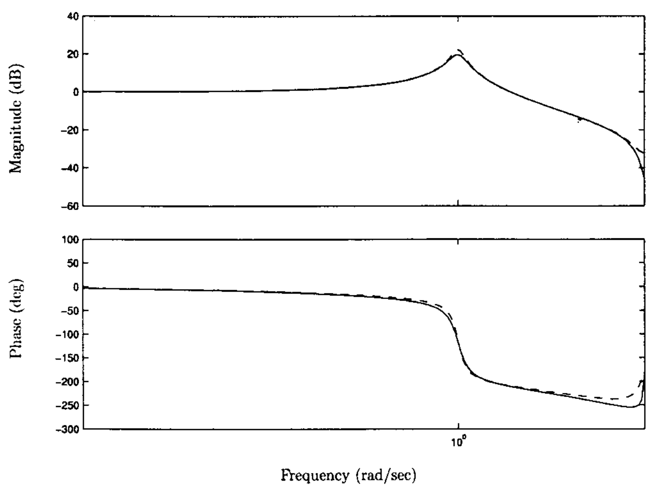

Fig. 9 Frequency response of actual (—) and estimated (--) systems of example 7

$$G(q^{-1}) \triangleq \frac{0.4450q^{-1}+0.4299}{q^{-2}-1.0300q^{-1}+0.9048}. \tag{29}$$

For each example $u(k)$ is a realization of a unit variance normally distributed random variable. In addition, we introduce normally distributed, zero mean random measurement noise $w(k)$ such that

$$z(k)=\mathcal{N}(y(k)+w(k)), \tag{30}$$

scaled so that the signal to noise ratio

$$\mathrm{S/N} \triangleq \frac{\|\mathbf{y}-\overline{\mathbf{y}}\|}{\|\mathbf{w}\|}=10, \tag{31}$$

where we define the average signal $\overline{\mathbf{y}} \triangleq 1/n+p\Sigma_{i=1}^{n+p}y(i)$.

z data from systems with additive output noise need not lie in the range of $\mathcal{N}$. If we collect a measurement of z corrupted by additive noise such that it lies outside the range of $\mathcal{N}$, we replace it with the nearest point in the range of $\mathcal{N}$, and then proceed with the identification method.

Example 1: Sine Let $\mathcal{N}(y)=\sin(y)$, which is noninvertible. A sequence of 20 input-output pairs were simulated. Figure 2 shows the nonlinearity along with the true and estimated data points, while Figure 3 compares the frequency response of the true and estimated linear systems.

Example 2: Deadzone Let $\mathcal{N}(y) = \mathrm{dzn}(y)$, which is a noninvertible nonlinearity with a constant region. A sequence of 30 input-output pairs were simulated. Figure 4 shows the nonlinearity along with the true and estimated data points, while Fig. 5 compares the frequency response of the true and estimated linear systems.

Example 3: Quantization Let $\mathcal{N}(y) = \mathrm{quant}(y)$, where quant is a quantization function similar to an analog-to-digital converter, a noninvertible nonlinearity composed exclusively of constant regions. A sequence of 50 input-output pairs were simulated. Figure 6 shows the nonlinearity along with the true and estimated data points, while Fig. 7 compares the frequency response of the true and estimated linear systems.

Example 4: Signum Let $\mathcal{N}(y) = \mathrm{sign}(y)$, where sign is the signum function that maps positive values to $+1$ and negative values to -1, a noninvertible nonlinearity composed of only two constant regions. A sequence of 60 input-output pairs were simulated. Figure 8 shows the nonlinearity along with the true and estimated data points, while Fig. 9 compares the frequency response of the true and estimated linear systems.

8 Conclusion

We developed a method for identifying Wiener nonlinear systems with known noninvertible nonlinearities. We presented an extension to Wiener systems with nonzero DC gains and unknown nonlinearities. We presented several numerical examples to illustrate the effectiveness of the method. Future work will focus on identification techniques for multivariable nonlinear systems and Wiener systems with unknown nonlinearities.

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