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Optimal Projection Equations for Reduced-Order Modelling, Estimation, and Control of Linear Systems with Multiplicative White Noise[1,2]

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Abstract. The optimal projection equations for quadratically optimal reduced-order modelling, estimation, and control are generalized to include the effects of state, control, and measurement dependent noise.

Key Words. Feedback control, robust control, fixed-order compensation, optimal control.

1. Introduction

As is well known, LQR and LQG controllers lack guaranteed robustness with respect to arbitrary parameter variations (Refs. 1 and 2). A widely studied approach to correcting this defect involves introducing noise into the plant via the imperfectly known parameters (Refs. 3–10). Intuitively speaking, the quadratically optimal feedback controller designed in the presence of such disturbances is automatically desensitized to actual parameter variations. This was demonstrated in Ref. 11 for the example given in Ref. 1.

The contribution of the present paper is a generalization of classical steady-state LQG theory to include the effects of state, control, and measurement dependent noise. In contrast to the classical solution involving a pair

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of separated Riccati equations, the necessary conditions for quadratic optimality in the presence of multiplicative white noise consist of a system of two modified Riccati equations and two modified Lyapunov equations coupled by stochastic effects. The coupling serves as a graphic portrayal of the breakdown of the separation principle in the multiplicative noise case. When the multiplicative noise terms are set to zero, the modified Lyapunov equations drop out and the modified Riccati equations immediately reduce to the standard pair of separated LQG Riccati equations. Related results were obtained for the discrete-time, finite-interval problem in Ref. 10.

To attain further generality, a constraint is imposed on controller order as in Ref. 12. Hence, the results of the present paper also constitute a direct generalization of the coupled system of modified Riccati and Lyapunov equations which arise in characterizing reduced-order controllers.

For the special case of full-order compensation in the presence of state-dependent noise only, versions of these equations were discovered independently by Hyland (Refs. 13 and 14) and Mil'stein (Ref. 15). An interesting difference between Refs. 13–14 and Ref. 15 is that Mil'stein interpreted the plant model as an Ito stochastic differential equation, whereas Hyland utilized the Fisk–Stratonovich definition (Refs. 16–18). In earlier work on modelling flexible mechanical structures (Refs. 19 and 20), justification for this interpretation as an appropriate model for parameter uncertainty was based upon the maximum entropy principle of Jaynes (Ref. 21) and the theory of stochastic approximation (Ref. 22). A summary of this approach and its relationship to Refs. 23 and 24 can be found in Ref. 25. Rigorous guarantees of robustness over a prescribed range of parameter variations have been obtained using Lyapunov functions (Refs. 26–29). Although the present paper utilizes an Ito model for simplicity, results based on Stratonovich models are readily obtained by means of standard transformations.

An immediate practical benefit of the structured form of the necessary conditions is the means for constructing numerical algorithms which differ fundamentally from gradient search techniques. One such iterative algorithm, proposed in Refs. 30–32, exploits the characterization of the oblique projection as the sum of rank-1 eigenprojections of the product of the rank-deficient pseudogramians satisfying the modified Lyapunov equations. As discussed in Ref. 32, the necessary conditions fail to specify which eigenprojections comprise the oblique projection; indeed, each choice may correspond to a local extremal. In practice, judicious choice of the eigenprojections can eliminate extremals with high cost and hence efficiently identify the global minimum. These issues are a result of the reduced-order constraint only; the stochastic effects alone do not appear to introduce extremal multiplicity.

The scope of the present paper involves deriving the optimal projection equations for reduced-order modelling, estimation, and control obtained in Refs. 32, 33, and 12 to include state, control, and measurement dependent noise. The main results, Theorems 2.1–2.3, present the necessary conditions for optimality as systems of two, three, and four matrix equations (modified Riccati and Lyapunov equations), coupled by both the optimal projection and stochastic effects. The necessary conditions in this generality are presented here for the first time. The dynamic compensation result supports the numerical results obtained in Refs. 11 and 34. Appendix D contains the proof of Theorem 2.3; the proofs of Theorems 2.1 and 2.2 are similar and hence are omitted. Although the derivations in Refs. 32, 33, and 12 utilizing Lagrange multipliers could have been adapted to the present case, we have devised a new proof based upon Kronecker products, which is thought to be more direct.

2. Problem Statement and Main Results

The following notation and definitions will be used throughout the paper.

$\mathbb{E}$ = expected value;
$\mathbb{R}$ = real numbers;
$\mathbb{R}^{\alpha\times\beta} = \alpha\times\beta$ real matrices;
$\mathbb{R}^{\alpha} = \mathbb{R}^{\alpha\times 1}$;
$I_\alpha = \alpha\times\alpha$ identity matrix;
$Z_{(i)}$ = ith element of vector Z;
$Z_{(i,j)} = (i, j)$ element of matrix Z;
Z^T = transpose of vector of matrix Z;
$Z^{-T} = (Z^T)^{-1}$ or $(Z^{-1})^T$;
$\rho(Z)$ = rank of matrix Z;
$\operatorname{tr} Z$ = trace of square matrix Z;
$\|Z\| = (\operatorname{tr} ZZ^T)^{1/2}$, Frobenius norm;
$\operatorname{diag}(a_1, \ldots, a_\alpha) = \alpha\times\alpha$ diagonal matrix with listed diagonal elements;
E_i = matrix with unity in the (i, i) position and zeros elsewhere;
$\Pi_i(\Psi) = \Psi E_i \Psi^{-1}$;
$X\otimes Y = \begin{bmatrix} X_{(1,1)}Y & \cdots & X_{(1,\beta)}Y \\ \vdots & & \vdots \\ X_{(\alpha,1)}Y & \cdots & X_{(\alpha,\beta)}Y \end{bmatrix}$, $X\in\mathbb{R}^{\alpha\times\beta}$, $Y\in\mathbb{R}^{\gamma\times\delta}$ (Kronecker product, Refs. 35 and 36);
$X\oplus Y = X\oplus I_\beta + I_\alpha\otimes Y$, $X\in\mathbb{R}^{\alpha\times\alpha}$, $Y\in\mathbb{R}^{\beta\times\beta}$ (Kronecker sum);
$Z^{\#}$ = group generalized inverse (Ref. 37);

$\text{row}_i(Z) = i$th row of matrix Z;

$\text{col}_i(Z) = i$th column of matrix Z;

$$\text{vec}(Z) = \begin{bmatrix} \text{col}_1(Z) \\ \vdots \\ \text{col}_\beta(Z) \end{bmatrix} \in \mathbb{R}^{\alpha\beta},\ Z \in \mathbb{R}^{\alpha\times\beta};$$

$$\text{vec}^{-1}_{(\alpha,\beta)}(Z) = \begin{bmatrix} Z_{(1)} & \cdots & Z_{(\alpha\beta-\alpha+1)} \\ \vdots & & \vdots \\ Z_{(\alpha)} & \cdots & Z_{(\alpha\beta)} \end{bmatrix} \in \mathbb{R}^{\alpha\times\beta},\ Z \in \mathbb{R}^{\alpha\beta};$$

stable matrix = matrix with eigenvalues in open left half plane;

nonnegative-definite matrix = symmetric matrix with nonnegative eigenvalues, $Z \geq 0$;

positive-definite matrix = symmetric matrix with positive eigenvalues, $Z > 0$;

semisimple eigenvalue = eigenvalue with equal algebraic and geometric multiplicity;

simple eigenvalue = eigenvalue with unity algebraic multiplicity;

group-invertible matrix = matrix Z satisfying $\rho(Z) = \rho(Z^2)$, i.e., matrix which is either invertible or whose zero eigenvalue is semisimple (Ref. 37);

semisimple matrix = matrix with semisimple eigenvalues, i.e., nondefective matrix;

real-semisimple matrix = semisimple matrix with real eigenvalues;

nonnegative-semisimple matrix = semisimple matrix with nonnegative eigenvalues;

positive-semisimple matrix = semisimple matrix with positive eigenvalues;

simple matrix = matrix with distinct (i.e., simple) eigenvalues;

r = generic subscript denoting m, e, or c;

$n, m, l, n_m, n_e, n_c, k, p, q$ = positive integers, $n_m \leq n$, $n_e \leq n$, $n_c \leq n$, $l \leq q$, $m \leq q$;

$\tilde{n}_r = n + n_r$;

$x, x_m, x_e, x_c = n, n_m, n_e, n_c$-dimensional vectors;

$u, y, y_e = m, l, k$-dimensional vectors;

$A, A_1, \ldots, A_p = n \times n$ matrices;

$B, B_1, \ldots, B_p = n \times m$ matrices;

$C, C_1, \ldots, C_p = l \times n$ matrices;

$A_m, B_m, C_m = n_m \times n_m,\ n_m \times m,\ l \times n_m$ matrices;

$A_e, B_e, C_e = n_e \times n_e,\ n_e \times l,\ k \times n_e$ matrices;

$A_c, B_c, C_c = n_c \times n_c,\ n_c \times l,\ m \times n_c$ matrices;

$R, N, R_2 = l \times l,\ k \times k,\ m \times m$ positive-definite matrices;

$R_1 = n \times n$ nonnegative-definite matrix;

$R_{12} = n \times m$ matrix, $R_1 - R_{12}R_2^{-1}R_{12}^T \geq 0$;

$$\tilde{R} = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix} \geq 0;$$

$L = k \times n$ matrix;

$v_{1t}, \ldots, v_{pt},\ w_{1t}, \ldots, w_{qt}$ = standard, independent Wiener processes, $t \geq 0$;

$w_t = (w_{1t}, \ldots, w_{qt})^T$;

$G, G_1, G_2 = m \times q,\ n \times q,\ l \times q$ matrices, $\rho(G) = m$, $\rho(G_2) = l$;

$$V_1 \triangleq G_1G_1^T \geq 0, \qquad V_{12} \triangleq G_1G_2^T, \qquad V_2 \triangleq G_2G_2^T > 0,$$
$$V \triangleq GG^T > 0;$$

$$\tilde{G} \triangleq \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \qquad \tilde{V} \triangleq \tilde{G}\tilde{G}^T = \begin{bmatrix} V_1 & V_{12} \\ V_{12}^T & V_2 \end{bmatrix} \geq 0;$$

$$\tilde{B}_m \triangleq \begin{bmatrix} B \\ B_m \end{bmatrix}, \qquad \tilde{B}_e \triangleq \begin{bmatrix} I_n & 0 \\ 0 & B_e \end{bmatrix}, \qquad \tilde{B}_c \triangleq \begin{bmatrix} I_n & 0 \\ 0 & B_c \end{bmatrix};$$

$$\tilde{G}_m \triangleq \tilde{B}_mG, \qquad \tilde{G}_e \triangleq \tilde{B}_e\tilde{G}, \qquad \tilde{G}_c \triangleq \tilde{B}_c\tilde{G};$$

$$\tilde{V}_r \triangleq \tilde{G}_r\tilde{G}_r^T, \qquad \tilde{V}_m \triangleq \tilde{B}_mV\tilde{B}_m^T, \qquad \tilde{V}_e \triangleq \tilde{B}_e\tilde{V}\tilde{B}_e^T, \qquad \tilde{V}_c \triangleq \tilde{B}_c\tilde{V}\tilde{B}_c^T;$$

$$\tilde{C}_m \triangleq [C \quad -C_m], \qquad \tilde{C}_e \triangleq [L \quad -C_e], \qquad \tilde{C}_c \triangleq \begin{bmatrix} I_n & 0 \\ 0 & C_c \end{bmatrix};$$

$$\tilde{R}_m \triangleq \tilde{C}_m^TR\tilde{C}_m, \qquad \tilde{R}_e \triangleq \tilde{C}_e^TN\tilde{C}_e, \qquad \tilde{R}_c \triangleq \tilde{C}_c^T\tilde{R}\tilde{C}_c;$$

$$\mathscr{A} \triangleq A \oplus A + \sum_{i=1}^{p} A_i \otimes A_i, \qquad \tilde{x}_r = \begin{bmatrix} x \\ x_r \end{bmatrix};$$

$$\tilde{A}_m \triangleq \begin{bmatrix} A & 0 \\ 0 & A_m \end{bmatrix}, \qquad \tilde{A}_e \triangleq \begin{bmatrix} A & 0 \\ B_eC & A_e \end{bmatrix}, \qquad \tilde{A}_c \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix};$$

$$\tilde{A}_{mi} \triangleq \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{A}_{ei} \triangleq \begin{bmatrix} A_i & 0 \\ B_eC_i & 0 \end{bmatrix}, \qquad \tilde{A}_{ci} \triangleq \begin{bmatrix} A_i & B_iC_c \\ B_cC_i & 0 \end{bmatrix};$$

$$\tilde{\mathscr{A}}_r = \tilde{A}_r \oplus \tilde{A}_r + \sum_{i=1}^{p} \tilde{A}_{ri} \otimes \tilde{A}_{ri}.$$

For the following definitions, let $Q, P, \hat{Q}, \hat{P} \in \mathbb{R}^{n\times n}$:

$$\hat{R}_2 \triangleq R_2 + \sum_{i=1}^{p} B_i^T(P + \hat{P})B_i,$$

$$\hat{V}_2 \triangleq V_2 + \sum_{i=1}^{p} C_i(Q + \hat{Q})C_i^T,$$

$$\mathscr{Q} \triangleq QC^T + V_{12} + \sum_{i=1}^{p} A_i(Q + \hat{Q})C_i^T,$$

$$\mathscr{P} \triangleq B^TP + R_{12}^T + \sum_{i=1}^{p} B_i^T(P + \hat{P})A_i,$$

$$A_Q \triangleq A - \mathcal{Q}\hat{V}_2^{-1}C, \qquad A_P \triangleq A - B\hat{R}_2^{-1}\mathcal{P}.$$

Using the above notation, we can state the reduced-order modelling, estimation, and control problems.

Reduced-Order Modelling Problem. Given the nth-order model

$$dx_t = Ax_t\,dt + \sum_{i=1}^{p} A_i x_t\,dv_{it} + G_1\,dw_t, \tag{1}$$

$$y_t = Cx_t, \tag{2}$$

where $t \in [0, \infty)$, determine an n_mth-order model

$$dx_{mt} = A_m x_{mt}\,dt + B_m G\,dw_t, \tag{3}$$

$$y_{mt} = C_m x_{mt}, \tag{4}$$

which minimizes the model-reduction criterion

$$J_m(A_m, B_m, C_m) \triangleq \limsup_{t\to\infty} \mathbb{E}[(y_t - y_{mt})^T R (y_t - y_{mt})]. \tag{5}$$

Reduced-Order State Estimation Problem. Given the nth-order observed system

$$dx_t = Ax_t\,dt + \sum_{i=1}^{p} A_i x_t\,dv_{it} + BG\,dw_t, \tag{6}$$

$$dy_t = Cx_t\,dt + \sum_{i=1}^{p} C_i x_t\,dv_{it} + G_2\,dw_t, \tag{7}$$

where $t \in [0, \infty)$, design an n_eth-order state estimator

$$dx_{et} = A_e x_{et}\,dt + B_e\,dy_t, \tag{8}$$

$$y_{et} = C_e x_{et}, \tag{9}$$

which minimizes the state estimation criterion

$$J_e(A_r, B_e, C_e) \triangleq \limsup_{t\to\infty} \mathbb{E}[(Lx_t - y_{et})^T N (Lx_t - y_{et})]. \tag{10}$$

Reduced-Order Dynamic Compensation Problem. Given the nth-order observed and controlled system

$$dx_t = Ax_t\,dt + \sum_{i=1}^{p} A_i x_t\,dv_{it} + Bu_t\,dt + \sum_{i=1}^{p} B_i u_t\,dv_{it} + G_1\,dw_t, \tag{11}$$

$$dy_t = Cx_t\,dt + \sum_{i=1}^{p} C_i x_t\,dv_{it} + G_2\,dw_t, \tag{12}$$

where $t \in [0, \infty)$, design an n_cth-order dynamic compensator

$$dx_{ct} = A_c x_{ct}\,dt + B_c\,dy_t, \tag{13}$$

$$u_t = C_c x_{ct}, \tag{14}$$

which minimizes the dynamic compensation criterion

$$J_c(A_c, B_c, C_c) \triangleq \limsup_{t\to\infty} \mathbb{E}[x_t^T R_1 x_t + 2x_t^T R_{12} u_t + u_t^T R_2 u_t]. \tag{15}$$

Clearly, J_m, J_e, and J_c are nonnegative, extended, real-valued functionals defined on appropriate Euclidean spaces. Explicit expressions for these functionals are now given. Henceforth, we assume that $\mathbb{E}\|\tilde{x}_{r0}\|^2 < \infty$ and that $\tilde{x}_{r0}$ and $v_{1t}, \ldots, v_{pt}, w_t$ are uncorrelated, $t \geq 0$.

Proposition 2.1. The nonnegative-definite covariance

$$\tilde{Q}_r(t) \triangleq \mathbb{E}[\tilde{x}_{rt}\tilde{x}_{rt}^T], \qquad t \geq 0,$$

is given by

$$\dot{\tilde{Q}}_r(t) = \tilde{A}_r\tilde{Q}_r(t) + \tilde{Q}_r(t)\tilde{A}_r^T + \sum_{i=1}^{p} \tilde{A}_{ri}\tilde{Q}_r(t)\tilde{A}_{ri}^T + \tilde{V}_r, \qquad t \geq 0, \tag{16}$$

or explicitly by

$$\tilde{Q}_r(t) = \mathrm{vec}^{-1}_{(\tilde{n}_r,\tilde{n}_r)}\left(\exp(\tilde{\mathcal{A}}_r t)\,\mathrm{vec}\,\tilde{Q}_r(0) + \int_0^t \exp(\tilde{\mathcal{A}}_r \sigma)\,d\sigma\,\mathrm{vec}\,\tilde{V}_r\right). \tag{17}$$

The cost criteria J_m, J_e, J_c are given by

$$J_r(A_r, B_r, C_r) = \limsup_{t\to\infty} \mathrm{tr}\,\tilde{Q}_r(t)\tilde{R}_r, \tag{18}$$

or equivalently by

$$J_r(A_r, B_r, C_r) = \limsup_{t\to\infty} (\mathrm{vec}\,\tilde{R}_r)^T\left(\exp(\tilde{\mathcal{A}}_r t)\,\mathrm{vec}\,\tilde{Q}_r(0) + \int_0^t \exp(\tilde{\mathcal{A}}_r\sigma)\,d\sigma\,\mathrm{vec}\,\tilde{V}_r\right). \tag{19}$$

For the proof, see Appendix A.

The finiteness and smoothness of J_m, J_e, and J_c clearly depend upon the interrelationships among $\tilde{Q}_r(0)$, $\tilde{\mathcal{A}}_r$, $\tilde{R}_r$, and $\tilde{V}_r$. To avoid a detailed analysis and to guarantee that J_m, J_e, and J_c are finite and independent of initial data, we restrict our consideration to second-moment stable or second-moment stabilizing design triples. Furthermore, to avoid degeneracy in later developments (and without loss of generality), only minimal (i.e., controllable and observable realizations are admitted. Hence, for the modelling, estimation, and control problems, define the open sets

$$\mathcal{S}_r = \{(A_r, B_r, C_r): \tilde{\mathcal{A}}_r \text{ is stable and } (A_r, B_r, C_r) \text{ is minimal}\}.$$

In the following result, we abuse notation slightly and let

$$\tilde{Q}_r = \lim_{t\to\infty} \tilde{Q}_r(t).$$

Proposition 2.2. Suppose that $\mathcal{S}_r$ is nonempty. If $(A_r, B_r, C_r) \in \mathcal{S}_r$, then

$$\tilde{Q}_r \triangleq \lim_{t\to\infty} \tilde{Q}_r(t)$$

exists and is given by the unique, nonnegative-definite solution to

$$0 = \tilde{A}_r\tilde{Q}_r + \tilde{Q}_r\tilde{A}_r^T + \sum_{i=1}^{p} \tilde{A}_{ri}\tilde{Q}_r\tilde{A}_{ri}^T + \tilde{V}_r, \tag{20}$$

or explicitly by

$$\tilde{Q}_r = \text{vec}^{-1}_{(\tilde{n}_r,\tilde{n}_r)}(-\tilde{\mathcal{A}}_r^{-1} \text{vec } \tilde{V}_r). \tag{21}$$

Hence,

$$J_r(A_r, B_r, C_r) = \text{tr } \tilde{Q}_r\tilde{R}_r, \tag{22}$$

or equivalently

$$J_r(A_r, B_r, C_r) = -(\text{vec } \tilde{R}_r)^T\tilde{\mathcal{A}}_r^{-1} \text{vec } \tilde{V}_r. \tag{23}$$

For the proof, see Appendix A.

As a side note, we examine the evolution of the mean value of $\tilde{x}_r$.

Proposition 2.3. The mean

$$\tilde{m}(t) \triangleq \mathbb{E}\tilde{x}_{rt}, \qquad t \geq 0, \tag{24}$$

satisfies

$$\dot{\tilde{m}}(t) = \tilde{A}_r\tilde{m}(t), \qquad t \geq 0. \tag{25}$$

Furthermore, if $(A_r, B_r, C_r) \in \mathcal{S}_r$, then $\tilde{A}_r$ is stable and thus

$$\lim_{t\to\infty} \tilde{m}(t) = 0. \tag{26}$$

For the proof, see Appendix A.

Of course, it is useful to know when the sets $\mathcal{S}_m$, $\mathcal{S}_e$, and $\mathcal{S}_c$ are nonempty. Although for the closed-loop control problem the question is complex because of stabilizability concerns, the modelling and estimation problems permit considerable simplification.

Proposition 2.4. $\mathcal{S}_m$ (alternatively, $\mathcal{S}_e$) is nonempty if and only if $\mathcal{A}$ is stable. In this case, $\mathcal{S}_m$ and $\mathcal{S}_e$ are given by

$$\mathcal{S}_m = \{(A_m, B_m, C_m): A_m \text{ is stable and } (A_m, B_m, C_m) \text{ is minimal}\},$$

$$\mathcal{S}_e = \{(A_e, B_e, C_e): A_e \text{ is stable and } (A_e, B_e, C_e) \text{ is minimal}\}.$$

For the proof, see Appendix B.

The following observation concerns the smoothness of the cost functionals.

Proposition 2.5. The functionals J_r are infinitely Fréchet differentiable on $\mathcal{S}_r$.

Proof. From Lemma 3.7.2, p. 203 of ref. 38, it follows that the map $W \to W^{-1}$ defined on the set of invertible matrices is C^∞. The result follows from the chain rule and (23). □

It is now possible to proceed with the principal aim of the paper, which is to characterize solutions of the reduced-order modelling, estimation, and control problems by means of a first-order variational analysis. To this end, one additional assumption is required. In order to obtain closed-form expressions for extremal values of the closed-loop control gains, the dynamic compensation problem requires the technical assumption

$$[B_i \neq 0 \Rightarrow C_i = 0], \qquad i = 1, \ldots, p, \tag{27}$$

or equivalently,

$$[C_i \neq 0 \Rightarrow B_i = 0], \qquad i = 1, \ldots, p, \tag{28}$$

i.e., for each $i \in \{1, \ldots, p\}$, B_i and C_i are not both nonzero. Essentially, (27) expresses the condition that the control dependent and measurement dependent noises are independent. There are no constraints, however, on correlations with the state dependent noise.

In order to state the main results, we require some additional notation and a lemma concerning a pair of nonnegative-definite matrices. For a real, semisimple matrix $X \in \mathbb{R}^{n\times n}$, define the set of diagonalizing matrices

$$\mathcal{D}(X) \triangleq \{\Psi \in \mathbb{R}^{n\times n}: \Psi^{-1}X\Psi \text{ is diagonal}\},$$

and, for a pair of nonnegative-definite matrices X, $Y \in \mathbb{R}^{n\times n}$, define the set of contragradiently diagonalizing matrices

$$\mathcal{C}(X, Y) \triangleq \{\Psi \in \mathbb{R}^{n\times n}: \Psi^{-1}X\Psi^{-T} \text{ and } \Psi^TY\Psi \text{ are diagonal}\}$$

and the subset of balancing transformations

$$\mathcal{B}(X, Y) \triangleq \{\Psi \in \mathcal{C}(X, Y): \Psi^{-1}X\Psi^{-T} = \Psi^TY\Psi\}.$$

The following result unifies and extends similar results found in Refs. 32, 33, and 12.

Lemma 2.1. Suppose that $\hat{Q}, \hat{P} \in \mathbb{R}^{n \times n}$ are nonnegative definite and $\rho(\hat{Q}\hat{P}) = n_r$. Then, the following statements hold:

(i) $\varnothing \neq \mathscr{C}(\hat{Q}, \hat{P}) \subset \mathscr{D}(\hat{Q}\hat{P})$;

(ii) $\hat{Q}\hat{P}$ is nonnegative semisimple;

(iii) the $n \times n$ matrix

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} \tag{29}$$

is idempotent, i.e., τ is an oblique projection;

(iv) there exists $\Psi \in \mathscr{C}(\hat{Q}, \hat{P})$, with $(\Psi^{-1}\hat{Q}\hat{P}\Psi)_{(i,i)} \neq 0$, $i = 1, \ldots, n_r$, such that τ is given by

$$\tau = \sum_{i=1}^{n_r} \Pi_i(\Psi); \tag{30}$$

(v) if $\rho(\hat{Q}) = \rho(\hat{P}) = n_r$, then $\mathscr{B}(\hat{Q}, \hat{P}) \neq \varnothing$;

(vi) if $\rho(\hat{Q}) = \rho(\hat{P}) = n_r$, then there exists $\Psi \in \mathscr{B}(\hat{Q}, \hat{P})$, with $(\Psi^{-1}\hat{Q}\hat{P}\Psi)_{(i,i)} \neq 0$, $i = 1, \ldots, n_r$, such that τ is given by (30);

(vii) if $\rho(\hat{Q}) = \rho(\hat{P}) = n_r$, then

$$\hat{Q} = \tau\hat{Q} = \hat{Q}\tau^T = \tau\hat{Q}\tau^T, \tag{31}$$

$$\hat{P} = \tau^T\hat{P} = \hat{P}\tau = \tau^T\hat{P}\tau; \tag{32}$$

(viii) there exist $G, \Gamma \in \mathbb{R}^{n_r \times n}$ and positive-semisimple $M \in \mathbb{R}^{n_r \times n_r}$ such that

$$\hat{Q}\hat{P} = G^T M \Gamma, \tag{33}$$

$$\Gamma G^T = I_{n_r}; \tag{34}$$

(ix) if $\bar{G}, \bar{\Gamma} \in \mathbb{R}^{n_r \times n}$ and $\bar{M} \in \mathbb{R}^{n_r \times n_r}$ satisfy

$$\hat{Q}\hat{P} = \bar{G}^T \bar{M} \bar{\Gamma}, \tag{35}$$

$$\bar{\Gamma}\bar{G}^T = I_{n_r}, \tag{36}$$

then there exists invertible $S \in \mathbb{R}^{n_r \times n_r}$ such that

$$\bar{G} = S^{-T}G, \qquad \bar{\Gamma} = S\Gamma, \qquad \bar{M} = SMS^{-1};$$

(x) if $G, \Gamma \in \mathbb{R}^{n_r \times n}$, and $M \in \mathbb{R}^{n_r \times n_r}$ satisfy (33) and (34), then M is invertible, $(\hat{Q}\hat{P})^{\#} = G^T M^{-1} \Gamma$, and

$$\tau = G^T \Gamma. \tag{37}$$

For the proof, see Appendix C.

For convenience, we shall call $G, \Gamma \in \mathbb{R}^{n_r \times n}$, and $M \in \mathbb{R}^{n_r \times n_r}$ satisfying (33) and (34) a projective factorization of $\hat{Q}\hat{P}$. Furthermore, define the complementary oblique projection

$$\tau_\perp \triangleq I_n - \tau, \tag{38}$$

and let $J_r'(A_r, B_r, C_r)$ denote the Fréchet derivative of J_r evaluated at (A_r, B_r, C_r).

It is now possible to state the main results, which provide a parametrization of triples $(A_r, B_r, C_r) \in \mathscr{S}_r$ for which the first Fréchet derivative of J_r vanishes.

Theorem 2.1. Assume that $\mathscr{A}$ is stable. Then, for $(A_m, B_m, C_m) \in \mathscr{S}_m$,

$$J_m'(A_m, B_m, C_n) = 0 \tag{39}$$

if and only if there exist $n \times n$ nonnegative-definite matrices $\hat{Q}$ and $\hat{P}$ such that, for some projective factorization G, M, Γ of $\hat{Q}\hat{P}$, A_m, B_m, and C_m are given by

$$A_m = \Gamma A G^T, \tag{40}$$

$$B_m = \Gamma B, \tag{41}$$

$$C_m = C G^T, \tag{42}$$

and are such that, with

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} = G^T\Gamma \quad \text{and} \quad \tau_\perp \triangleq I_n - \tau,$$

the following conditions are satisfied:

$$0 = A\hat{Q} + \hat{Q}A^T + BVB^T - \tau_\perp BVB^T\tau_\perp^T, \tag{43}$$

$$0 = A^T\hat{P} + \hat{P}A + C^TRC - \tau_\perp^T C^TRC\tau_\perp, \tag{44}$$

$$\rho(\hat{Q}) = \rho(\hat{P}) = \rho(\hat{Q}\hat{P}) = n_m. \tag{45}$$

Furthermore, if $(A_m, B_m, C_m) \in \mathscr{S}_m$ satisfies (39), then the extremal cost is given by

$$J_m(A_m, B_m, C_m) = \mathrm{tr}[(W_c - \hat{Q})C^TRC] = \mathrm{tr}[(W_0 - \hat{P})BVB^T]$$
$$= 2\,\mathrm{tr}[(\hat{Q}\hat{P} - W_cW_0)A] - 2\sum_{i=1}^{p} \mathrm{tr}\; W_cA_i^TW_0A_i, \tag{46}$$

where W_c, $W_0 \in \mathbb{R}^{n \times n}$ are the unique, nonnegative-definite solutions to

$$0 = AW_c + W_cA^T + \sum_{i=1}^{p} A_iW_cA_i^T + BVB^T, \tag{47}$$

$$0 = A^TW_0 + W_0A + \sum_{i=1}^{p} A_i^TW_0A_i + C^TRC. \tag{48}$$

Theorem 2.2. Assume that $\mathscr{A}$ is stable. Then, for $(A_e, B_e, C_e) \in \mathscr{S}_e$,

$$J_e'(A_e, B_e, C_e) = 0 \tag{49}$$

if and only if there exist $n \times n$ nonnegative-definite matrices Q, $\hat{Q}$, and $\hat{P}$ such that, for some projective factorization G, M, Γ of $\hat{Q}\hat{P}$, A_e, B_e, and C_e are given by

$$A_e = \Gamma(A - \mathscr{Q}\hat{V}_2^{-1}C)G^T, \tag{50}$$

$$B_e = \Gamma\mathscr{Q}\hat{V}_2^{-1}, \tag{51}$$

$$C_e = LG^T, \tag{52}$$

and are such that, with

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} = G^T\Gamma \quad \text{and} \quad \tau_\perp \triangleq I_n - \tau,$$

the following conditions are satisfied:

$$0 = AQ + QA^T + V_1 + \sum_{i=1}^{p} A_i(Q+\hat{Q})A_i^T$$
$$+ \mathscr{Q}\hat{V}_2^{-1}\mathscr{Q}^T + \tau_\perp\mathscr{Q}\hat{V}_2^{-1}\mathscr{Q}^T\tau_\perp^T, \tag{53}$$

$$0 = A\hat{Q} + \hat{Q}A^T + \mathscr{Q}\hat{V}_2^{-1}\mathscr{Q}^T - \tau_\perp\mathscr{Q}\hat{V}_2^{-1}\mathscr{Q}^T\tau_\perp^T, \tag{54}$$

$$0 = A_Q^T\hat{P} + \hat{P}A_Q + L^TNL - \tau_\perp^T L^TNL\tau_\perp, \tag{55}$$

$$\rho(\hat{Q}) = \rho(\hat{P}) = \rho(\hat{Q}\hat{P}) = n_e. \tag{56}$$

Furthermore, if $(A_e, B_e, C_e) \in \mathscr{S}_e$ satisfies (49), then the extremal cost is given by

$$J_e(A_e, B_e, C_e) = \operatorname{tr} QL^TNL. \tag{57}$$

Theorem 2.3. Assume that (27) holds and $\mathscr{S}_c$ is nonempty. Then, for $(A_c, B_c, C_c) \in \mathscr{S}_c$,

$$J_c'(A_c, B_c, C_c) = 0 \tag{58}$$

if and only if there exist $n \times n$ nonnegative-definite matrices Q, P, $\hat{Q}$, and $\hat{P}$ such that, for some projective factorization G, M, Γ of $\hat{Q}\hat{P}$, A_c, B_c, and C_c are given by

$$A_c = \Gamma(A - B\hat{R}_2^{-1}\mathscr{P} - \mathscr{Q}\hat{V}_2^{-1}C)G^T, \tag{59}$$

$$B_c = \Gamma\mathscr{Q}\hat{V}_2^{-1}, \tag{60}$$

$$C_c = -\hat{R}_2^{-1}\mathscr{P}G^T, \tag{61}$$

and are such that, with

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} = G^T\Gamma \quad \text{and} \quad \tau_\perp \triangleq I_n - \tau,$$

the following conditions are satisfied:

$$0 = AQ + QA^T + V_1$$
$$+ \sum_{i=1}^{p} [A_iQA_i^T + (A_i - B_i\hat{R}_2^{-1}\mathscr{P})\hat{Q}(A_i - B_i\hat{R}_2^{-1}\mathscr{P})^T]$$
$$- \mathscr{Q}\hat{V}_2^{-1}\mathscr{Q}^T + \tau_\perp\mathscr{Q}\hat{V}_2^{-1}\mathscr{Q}^T\tau_\perp^T, \tag{62}$$

$$0 = A^TP + PA + R_1$$
$$+ \sum_{i=1}^{p} [A_i^TPA_i + (A_i - \mathscr{Q}\hat{V}_2^{-1}C_i)^T\hat{P}(A_i - \mathscr{Q}\hat{V}_2^{-1}C_i)]$$
$$- \mathscr{P}^T\hat{R}_2^{-1}\mathscr{P} + \tau_\perp^T\mathscr{P}^T\hat{R}_2^{-1}\mathscr{P}\tau_\perp, \tag{63}$$

$$0 = A_P\hat{Q} + \hat{Q}A_P^T + \mathscr{Q}\hat{V}_2^{-1}\mathscr{Q}^T - \tau_\perp\mathscr{Q}\hat{V}_2^{-1}\mathscr{Q}^T\tau_\perp^T, \tag{64}$$

$$0 = A_Q^T\hat{P} + \hat{P}A_Q + \mathscr{P}^T\hat{R}_2^{-1}\mathscr{P} - \tau_\perp^T\mathscr{P}^T\hat{R}_2^{-1}\mathscr{P}\tau_\perp, \tag{65}$$

$$\rho(\hat{Q}) = \rho(\hat{P}) = \rho(\hat{Q}\hat{P}) = n_c. \tag{66}$$

Furthermore, if $(A_c, B_c, C_c) \in \mathscr{S}_c$ satisfies (58), then the extremal cost is given by

$$J_c(A_c, B_c, C_c)$$
$$= \operatorname{tr}[(Q+\hat{Q})R_1 - 2R_{12}\hat{R}_2^{-1}\mathscr{P}\hat{Q} + \mathscr{P}^T\hat{R}_2^{-1}R_2\hat{R}_2^{-1}\mathscr{P}\hat{Q}]. \tag{67}$$

3. Appendix A: Proof of Propositions 2.1, 2.2, and 2.3

To prove Proposition 2.1, note that (1)-(4), (6)-(9), and (11)-(14) can be written as

$$d\tilde{x}_{rt} = \tilde{A}_r\tilde{x}_{rt}\,dt + \sum_{i=1}^{p} \tilde{A}_{ri}\tilde{x}_{rt}\,dv_{it} + \tilde{G}_r\,dw_t. \tag{68}$$

From Theorem 8.5.5, p. 142 of Ref. 17 (or from the Ito differential rule), it follows that the nonnegative-definite covariance $\tilde{Q}_r(t)$ is given by (16). Furthermore, (5), (10), and (15) are equivalent to (18). Rewriting $\tilde{Q}_r(t)$ in the form (see Refs. 35 and 36)

$$\operatorname{vec}\dot{\tilde{Q}}_r(t) = \tilde{\mathscr{A}}_r \operatorname{vec}\tilde{Q}_r(t) + \operatorname{vec}\tilde{V}_r \tag{69}$$

leads to (19). □

To prove Proposition 2.2, note that the stability of $\tilde{\mathscr{A}}_r$ implies, by (69), that

$$\tilde{Q}_r \triangleq \lim_{t\to\infty} \tilde{Q}_r(t)$$

exists and is given by (21), which satisfies (20). Clearly, $\tilde{Q}_r \geq 0$, since $\tilde{Q}_r(t) \geq 0$, $t \geq 0$. Now, (22) and (23) follow from (18) and (19). □

To prove Proposition 2.3, note that the differential equation for $\tilde{m}(t)$ is an immediate consequence of (68). To show that $\tilde{A}_r$ is stable, we proceed as in Lemma 2.2 of Ref. 4. Repeating the steps leading to (22), with $\tilde{V}_r$ replaced by $I_{\tilde{n}_r}$, it follows (see Ref. 4) that $(\tilde{A}_r, I_{\tilde{n}_r})$ is stabilizable. Hence,

$$\int_0^t \exp(\tilde{A}_r\sigma)\exp(\tilde{A}_r^T\sigma)\,d\sigma$$

is bounded as $t \to \infty$, which implies that $\tilde{A}_r$ is stable. □

4. Appendix B: Proof of Proposition 2.4

We require some elementary properties of the Kronecker product (Refs. 35 and 36) applied to partitioned matrices. For $X \in \mathbb{R}^{s\times t}$ and $Y \in \mathbb{R}^{p\times q}$ partitioned by

$$X = \begin{matrix} & t_1 & t_2 \\ s_1 & X_1 & X_{12} \\ s_2 & X_{21} & X_2 \end{matrix}, \qquad y = \begin{matrix} & q_1 & q_2 \\ p_1 & Y_1 & Y_{12} \\ p_2 & Y_{21} & Y_2 \end{matrix},$$

it follows that

$$X\otimes Y = \begin{bmatrix} X_1\otimes Y & X_{12}\otimes Y \\ X_{21}\otimes Y & X_2\otimes Y \end{bmatrix}$$

$$= \begin{bmatrix} U_{s_1\times p} & 0 \\ 0 & U_{s_2\times p} \end{bmatrix} \begin{bmatrix} Y\otimes X_1 & Y\otimes X_{12} \\ Y\otimes X_{21} & Y\otimes X_2 \end{bmatrix} \begin{bmatrix} U_{q\times t_1} & 0 \\ 0 & U_{q\times t_2} \end{bmatrix}$$

$$= \begin{bmatrix} U_{s_1\times p} & 0 \\ 0 & U_{s_2\times p} \end{bmatrix}$$

$$\times \begin{bmatrix} Y_1\otimes X_1 & Y_{12}\otimes X_1 & Y_1\otimes X_{12} & Y_{12}\otimes X_{12} \\ Y_{21}\otimes X_1 & Y_2\otimes X_1 & Y_{21}\otimes X_{12} & Y_2\otimes X_{12} \\ Y_1\otimes X_{21} & Y_{12}\otimes X_{21} & Y_1\otimes X_2 & Y_{12}\otimes X_2 \\ Y_{21}\otimes X_{21} & Y_2\otimes X_{21} & Y_{21}\otimes X_2 & Y_2\otimes X_2 \end{bmatrix}$$

$$\times \begin{bmatrix} U_{q\times t_1} & 0 \\ 0 & U_{q\times t_2} \end{bmatrix},$$

where $U_{i\times j}$ is the permutation matrix defined in Refs. 35 and 36. Since $U_{i\times j} = U_{i\times j}^{-1}$ (i.e., $U_{i\times j}$ is involutory), the stability of (square) $X\otimes Y$ is equivalent to the stability of the above block 4×4 matrix. Hence, note that

$$\tilde{\mathscr{A}}_m = \text{block-diagonal}(\mathscr{A}, A\oplus A_m, A_m\oplus A, A_m\oplus A_m).$$

If (A_m, B_m, C_m) is such that $\tilde{\mathscr{A}}_m$ is stable, then clearly $\mathscr{A}$ is also stable and, by the elementary properties of the Kronecker sum, A_m is stable. Conversely, if $\mathscr{A}$ and A_m are stable, then $\tilde{\mathscr{A}}_m$ is stable. The result for $\tilde{\mathscr{A}}_e$ is obtained analogously noting only that $\tilde{\mathscr{A}}_e$ is lower block triangular. □

5. Appendix C: Proof of Lemma 2.1

(i) From Theorem 6.2.5, p. 123 of Ref. 39, it follows that there exists an $n\times n$ invertible matrix Ψ such that the nonnegative-definite matrices

$$D_{\hat{Q}} \triangleq \Psi^{-1}\hat{Q}\Psi^{-T} \quad \text{and} \quad D_{\hat{P}} \triangleq \Psi^T\hat{P}\Psi$$

are both diagonal. Hence, $\mathscr{C}(\hat{Q}, \hat{P}) \neq \varnothing$. Since $D_{\hat{Q}}D_{\hat{P}} = \Psi^{-1}\hat{Q}\hat{P}\Psi$ is also diagonal, $\mathscr{C}(\hat{Q}, \hat{P}) \subset \mathscr{D}(\hat{Q}\hat{P})$.

(ii) Since $\hat{Q}\hat{P} = \Psi\Lambda\Psi^{-1}$, where $\Lambda \triangleq D_{\hat{Q}}D_{\hat{P}}$ is nonnegative diagonal, $\hat{Q}\hat{P}$ is nonnegative semisimple.

(iii) Since $\hat{Q}\hat{P}$ is semisimple, it is group invertible. By properties of the group inverse (Ref. 37, p. 124), $\tau^2 = \tau$.

(iv) Note that, by means of a basis rearrangement, it can be assumed that Ψ in (i) is such that

$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{n_r}, 0, \ldots, 0),$$

where

$$\lambda_i \triangleq (D_{\hat{Q}}D_{\hat{P}})_{(i,i)} \neq 0, \qquad i = 1, \ldots, n_r.$$

Hence, since

$$\Lambda^{\#} = \text{diag}(\lambda_1^{-1}, \ldots, \lambda_{n_r}^{-1}, 0, \ldots, 0),$$

we have

$$\tau = \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} = \Psi\Lambda\Lambda^{\#}\Psi^{-1} = \sum_{i=1}^{n_r} \Psi E_i \Psi^{-1}.$$

(v) Since

$$\rho(\hat{Q}) = \rho(\hat{P}) = \rho(\hat{Q}\hat{P}),$$

it follows that

$$(D_{\hat{Q}})_{(i,i)} \neq 0, \quad (D_{\hat{P}})_{(i,i)} \neq 0, \qquad i = 1, \ldots, n_r.$$

Hence, let $\Lambda_{\hat{Q}}$ and $\Lambda_{\hat{P}}$ denote the upper left positive-diagonal blocks of $D_{\hat{Q}}$ and $D_{\hat{P}}$, respectively, and define

$$\hat{\Psi} \triangleq \Psi \begin{bmatrix} (\Lambda_{\hat{Q}}\Lambda_{\hat{P}}^{-1})^{1/4} & 0 \\ 0 & I_{n-n_r} \end{bmatrix}.$$

It now follows that

$$\hat{\Psi}^{-1}\hat{Q}\hat{\Psi}^{-T} = \hat{\Psi}^T\hat{P}\hat{\Psi}^{-1} = \begin{bmatrix} (\Lambda_{\hat{Q}}\Lambda_{\hat{P}})^{1/2} & 0 \\ 0 & 0 \end{bmatrix}, \tag{70}$$

as desired.

(vi) This is an immediate consequence of (70).

(vii) This is an immediate consequence of (70) and (30).

(viii) With Ψ as in (iv) and

$$\Lambda_0 \triangleq \operatorname{diag}(\tau_1, \ldots, \tau_{n_r}),$$

it follows that, for arbitrary invertible $S \in \mathbb{R}^{n \times n}$,

$$\hat{Q}\hat{P} = \Psi \begin{bmatrix} S \\ 0 \end{bmatrix} (S^{-1}\Lambda_0 S)[S^{-1} \quad 0]\Psi^{-1};$$

thus, (33) and (34) hold with

$$G = [S^T \quad 0], \qquad M = S^{-1}\Lambda_0 S, \qquad \Gamma = [S^{-1} \quad 0]\Psi^{-1}.$$

(ix) The result follows from

$$S = \bar{M}^{-1}\bar{\Gamma}G^T M^{-1}, \qquad \text{with } S^{-1} = M\bar{\Gamma}\bar{G}^T\bar{M}^{-1}.$$

(x) The result is a consequence of (viii) and (ix). □

6. Appendix D: Proof of Theorem 2.3

First note that, by arguments similar to those used in Appendix A, the dual of (22), given by

$$0 = \tilde{A}_r^T\tilde{P}_r + \tilde{P}_r\tilde{A}_r + \sum_{i=1}^{p} \tilde{A}_{ri}^T\tilde{P}_r\tilde{A}_{ri} + \tilde{R}_r, \tag{71}$$

has a unique, nonnegative-definite solution given explicitly by

$$\tilde{P}_r = \operatorname{vec}^{-1}_{(\tilde{n}_r,\tilde{n}_r)}(-\tilde{\mathscr{A}}_r^{-T} \operatorname{vec} \tilde{R}_r). \tag{72}$$

Define the partitionings

$$\tilde{Q}_r = \begin{matrix} & \begin{matrix} n & n_r \end{matrix} \\ \begin{matrix} n \\ n_r \end{matrix} & \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \end{matrix}, \qquad \tilde{P}_r = \begin{matrix} & \begin{matrix} n & n_r \end{matrix} \\ \begin{matrix} n \\ n_r \end{matrix} & \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix} \end{matrix},$$

where, for notational convenience, we suppress the subscript r. Also, define the notation

$$\tilde{P}\tilde{Q} = \begin{bmatrix} Z_1 & Z_{12} \\ Z_{21} & Z_2 \end{bmatrix},$$

where

$$Z_1 \triangleq P_1Q_1 + P_{12}Q_{12}^T, \qquad Z_{12} \triangleq P_1Q_{12} + P_{12}Q_2,$$
$$Z_{21} \triangleq P_{12}^TQ_1 + P_2Q_{12}^T, \qquad Z_2 \triangleq P_{12}^Tq_{12} + P_2Q_2,$$

and let

$$(\delta_{A_c}, \delta_{B_c}, \delta_{C_c}) \in \mathbb{R}^{n_c \times n_c} \times \mathbb{R}^{n_c \times l} \times \mathbb{R}^{m \times n_c}.$$

We now specialize to the control problem.

Lemma 6.1. Under the assumptions of Theorem 2.3,

$$\begin{aligned} &J_c'(A_c, B_c, C_c)(\delta_{A_c}, \delta_{B_c}, \delta_{C_c}) \\ &= 2\operatorname{tr}[Z_2^T\delta_{A_c}] \\ &+ 2\operatorname{tr}\left[\left(\hat{V}_2B_c^TP_2 + CZ_{21}^T + \left[V_{12}^T + \sum_{i=1}^{p} C_iQ_1A_i^T\right]P_{12}\right)\delta_{B_c}\right] \\ &+ 2\operatorname{tr}\left[\left(Q_2C_c^T\hat{R}_2 + Z_{12}^TB + Q_{12}^T\left[R_{12} + \sum_{i=1}^{p} A_i^TP_1B_i\right]\right)\delta_{C_c}\right]. \end{aligned} \tag{73}$$

Proof. From Lemma 3.7.2, p. 203 of Ref. 38, it follows that the Fréchet derivative of the map $W \to W^{-1}$ is given by

$$\delta_W \to -W^{-1}\delta_W W^{-1}.$$

Also, recall from Refs. 35 and 36 the identities

$$(\operatorname{vec} X)^T \operatorname{vec} Y = \operatorname{tr} X^TY,$$
$$(X \otimes Y)\operatorname{vec} Z = \operatorname{vec} YZX^T.$$

Hence, using (23) and noting that $\tilde{V}_c$ and $\tilde{R}_c$ are independent of A_c, we compute

$$\begin{aligned} &[\partial J_c(A_c, B_c, C_c)/\partial A_c]\delta_{A_c} \\ &= (\operatorname{vec} \tilde{R}_c)^T\tilde{\mathscr{A}}_c^{-1}\left(\left(\frac{\partial\tilde{\mathscr{A}}_c}{\partial A_c}\right)\delta_{A_c}\right)\tilde{\mathscr{A}}_c^{-1}\operatorname{vec}\tilde{V}_c \\ &= (\tilde{\mathscr{A}}_c^{-T}\operatorname{vec}\tilde{R}_c)^T\left(\begin{bmatrix} 0 & 0 \\ 0 & \delta_{A_c} \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & \delta_{A_c} \end{bmatrix}\right)(\tilde{\mathscr{A}}_c^{-1}\operatorname{vec}\tilde{V}_c) \\ &= (\operatorname{vec}\tilde{P}_c)^T\left(\begin{bmatrix} 0 & 0 \\ 0 & \delta_{A_c} \end{bmatrix} \otimes I_{\tilde{n}_c} + I_{\tilde{n}_c} \otimes \begin{bmatrix} 0 & 0 \\ 0 & \delta_{A_c} \end{bmatrix}\right)\operatorname{vec}\tilde{Q}_c \\ &= 2\operatorname{tr}\tilde{Q}_c\tilde{P}_c\begin{bmatrix} 0 & 0 \\ 0 & \delta_{A_c} \end{bmatrix} \\ &= 2\operatorname{tr} Z_2^T\delta_{A_c}. \end{aligned}$$

Furthermore, noting that

$$\text{vec } \tilde{V}_c = \text{vec } \tilde{B}_c \tilde{V} \tilde{B}_c^T = (\tilde{B}_c \otimes \tilde{B}_c) \text{ vec } \tilde{V},$$

we obtain

$$[\partial J_c(A_c, B_c, C_c)/\partial B_c]\delta_{B_c}$$

$$= (\text{vec } \tilde{P}_c)^T \left(\left(\frac{\partial \tilde{\mathscr{A}}_c}{\partial B_c} \right) \delta_{B_c} \right) \text{vec } \tilde{Q}_c$$

$$- (\text{vec } \tilde{P}_c)^T \left(\left(\frac{\partial (\tilde{B}_c \otimes \tilde{B}_c)}{\partial B_c} \right) \delta_{B_c} \right) \text{vec } \tilde{V}$$

$$= (\text{vec } \tilde{P}_c)^T \left(\begin{bmatrix} 0 & 0 \\ \delta_{B_c} C & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ \delta_{B_c} C & 0 \end{bmatrix} + \sum_{i=1}^{p} \tilde{A}_{ci} \otimes \begin{bmatrix} 0 & 0 \\ \delta_{B_c} C_i & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \delta_{B_c} C_i & 0 \end{bmatrix} \otimes \tilde{A}_{ci} \right) \text{vec } \tilde{Q}_c$$

$$- (\text{vec } \tilde{P}_c)^T \left(\tilde{B}_c \otimes \begin{bmatrix} 0 & 0 \\ 0 & \delta_{B_c} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \delta_{B_c} \end{bmatrix} \otimes \tilde{B}_c \right) \text{vec } \tilde{V}$$

$$= 2 \text{ tr } \tilde{Q}_c \tilde{P}_c \begin{bmatrix} 0 & 0 \\ \delta_{B_c} C & 0 \end{bmatrix} + 2 \text{ tr} \sum_{i=1}^{p} \tilde{Q}_c \tilde{A}_{ci}^T \tilde{P}_c \begin{bmatrix} 0 & 0 \\ \delta_{B_c} C_i & 0 \end{bmatrix}$$

$$+ 2 \text{ tr } \tilde{V} \tilde{B}_c^T \tilde{P}_c \begin{bmatrix} 0 & 0 \\ 0 & \delta_{B_c} \end{bmatrix}$$

$$= 2 \text{ tr } C Z_{21}^T \delta_{B_c} + 2 \text{ tr} \sum_{i=1}^{p} (C_i Q_1 A_i^T P_{12} + C_i Q_1 C_i^T)$$

$$+ 2 \text{ tr}(V_{12}^T P_{12} + V_2 B_c^T P_2) \delta_{B_c}$$

$$= 2 \text{ tr} \left(\hat{V}_2 B_c^T P_2 C Z_{21}^T + \left[V_{12}^T + \sum_{i=1}^{p} C_i Q_1 A_i^T \right] P_{12} \right) \delta_{B_c}.$$

A similar computation for $(\partial J_c(A_c, B_c, C_c)/\partial C_c)\delta_{C_c}$ yields (73). □

We can now proceed with the proof of Theorem 2.3. Obviously, (58) is equivalent to

$$0 = Z_2, \tag{74}$$

$$0 = P_2 B_c \hat{V}_2 + Z_{21} C^T + P_{12}^T \left(V_{12} + \sum_{i=1}^{p} A_i Q_1 C_i^T \right), \tag{75}$$

$$0 = \hat{R}_2 C_c Q_2 + B^T Z_{12} + \left(R_{12}^T + \sum_{i=1}^{p} B_i^T P_1 A_i \right) Q_{12}. \tag{76}$$

Expanding the $n \times n$, $n \times n_c$, and $n_c \times n_c$ blocks of (20) and (71) yields

$$0 = AQ_1 + Q_1 A^T + V_1 + BC_c Q_{12}^T + Q_{12}(BC_c)^T + \sum_{i=1}^{p} [A_i Q_1 A_i^T + B_i C_c Q_{12}^T A_i^T + A_i Q_{12}(B_i C_c)^T + B_i C_c Q_2 (B_i C_c)^T], \tag{77}$$

$$0 = AQ_{12} + Q_{12} A_c^T + BC_c Q_2 + Q_1 (B_c C)^T + \sum_{i=1}^{p} A_i Q_1 (B_c C_i)^T + V_{12} B_c^T, \tag{78}$$

$$0 = A_c Q_2 + Q_2 A_c^T + B_c C Q_{12} + Q_{12}^T (B_c C)^T + B_c \hat{V}_2 B_c^T, \tag{79}$$

$$0 = A^T P_1 + P_1 A + R_1 + (B_c C)^T P_{12}^T + P_{12} B_c C + \sum_{i=1}^{p} [A_i^T P_1 A_i + (B_c C_i)^T P_{12}^T A_i + A_i^T P_{12} B_c C_i + (B_c C_i)^T P_2 B_c C_i, \tag{80}$$

$$0 = A^T P_{12} + P_{12} A_c + (B_c C)^T P_2 + P_1 B C_c + \sum_{i=1}^{p} A_i^T P_1 B_i C_c + R_{12} C_c, \tag{81}$$

$$0 = A_c^T P_2 + P_2 A_c + (BC_c)^T P_{12} + P_{12}^T B C_c + C_c^T \hat{R}_2 C_c. \tag{82}$$

Obviously, $V_2 > 0$ and $R_2 > 0$ imply $\hat{V}_2 > 0$ and $\hat{R}_2 > 0$. Next, note that, since (A_c, B_c) is controllable and $V_2 > 0$, it follows that $(A_c + B_c C Q_{12} Q_2^{\#}, B_c \hat{V}_2^{1/2})$ is controllable. Using $Q_{12} = Q_{12} Q_2^{\#} Q_2$ (Refs. 39 and 40), (79) can be rewritten as

$$0 = (A_c + B_c C Q_{12} Q_2^{\#}) Q_2 + Q_2 (A_c + B_c C Q_{12} Q_2^{\#})^T + B_c \hat{V}_2 B_c^T. \tag{83}$$

Now, using Lemma 12.2 of Ref. 41, it follows from (83) that Q_2 is positive definite. Similarly, P_2 is positive definite.

Since $\hat{R}_2$, $\hat{V}_2$, Q_2, and P_2 are invertible, (74)-(76) can be written as

$$-P_2^{-1} P_{12}^T Q_{12} Q_2^{-1} = I_{n_c}, \tag{84}$$

$$B_c = -P_2^{-1} \left[Z_{21} C^T + P_{12}^T \left(V_{12} + \sum_{i=1}^{p} A_i Q_1 C_i^T \right) \right] \hat{V}_2^{-1}, \tag{85}$$

$$C_c = -\hat{R}_2^{-1} \left[B^T Z_{12} + \left(R_{12}^T + \sum_{i=1}^{p} B_i^T P_1 A_i \right) Q_{12} \right] Q_2^{-1}. \tag{86}$$

Now, define new variables

$$Q \triangleq Q_1 - Q_{12} Q_2^{-1} Q_{12}^T, \qquad P \triangleq P_1 - P_{12} P_2^{-1} P_{12}^T, \tag{87}$$

$$\hat{Q} \triangleq Q_{12} Q_2^{-1} Q_{12}^T, \qquad \hat{P} \triangleq P_{12} P_2^{-1} P_{12}^T, \tag{88}$$

which are $n \times n$ nonnegative-definite matrices. Note that, because of (84),

(33) and (34) hold with

$$G \triangleq Q_2^{-1}Q_{12}^T, \qquad M \triangleq Q_2P_2, \qquad \Gamma \triangleq -P_2^{-1}P_{12}^T, \tag{89}$$

where M is positive semisimple, since

$$Q_2P_2 = Q_2^{1/2}(Q_2^{1/2}P_2Q_2^{1/2})Q_2^{-1/2}.$$

It is helpful to note the identities

$$\hat{Q} = Q_{12}G = G^TQ_{12}^T = G^TQ_2G,$$

$$\hat{P} = -P_{12}\Gamma = -\Gamma^TP_{12}^T = \Gamma^TP_2\Gamma, \tag{90}$$

$$\tau G^T = G^T, \qquad \Gamma\tau = \tau, \tag{91}$$

$$\hat{Q} = \tau\hat{Q}, \qquad \hat{P} = \hat{P}\tau, \tag{92}$$

$$\hat{Q}\hat{P} = -Q_{12}P_{12}^T. \tag{93}$$

Using (34) and Sylvester's inequality, it follows that

$$\rho(G) = \rho(\Gamma) = \rho(Q_{12}) = \rho(P_{12}) = n_c,$$

which in turn imply (66).

The components of $\tilde{Q}_c$ and $\tilde{P}_c$ can be written in terms of Q, P, $\hat{Q}$, $\hat{P}$, G, and Γ as

$$Q_1 = Q + \hat{Q}, \qquad P_1 = P + \hat{P}, \tag{94}$$

$$Q_{12} = \hat{Q}\Gamma^T, \qquad P_{12} = -\hat{P}G^T, \tag{95}$$

$$Q_2 = \Gamma\hat{Q}\Gamma^T, \qquad P_2 = G\hat{P}G^T. \tag{96}$$

The gain expressions (60) and (61) can now be seen to be equivalent to (85) and (86). Substituting (94)-(96) into (77)-(82) yields

$$0 = AQ + QA^T + V_1 + \sum_{i=1}^{p} [A_iQA_i^T + (A_i - B_i\hat{R}_2^{-1}\mathscr{P})\hat{Q}(A_i - B_i\hat{R}_2^{-1}\mathscr{P})^T] + A_P\hat{Q} + \hat{Q}A_P^T, \tag{97}$$

$$0 = [A_P\hat{Q} + \hat{Q}(\Gamma^TA_cG + C^T\hat{V}_2^{-1}\mathscr{Q}^T) + \mathscr{Q}\hat{V}_2^{-1}\mathscr{Q}^T]\Gamma^T, \tag{98}$$

$$0 = \Gamma[(G^TA_c\Gamma + \mathscr{Q}\hat{V}_2^{-1}C)\hat{Q} + \hat{Q}(G^TA_c\Gamma + \mathscr{Q}\hat{V}_2^{-1}C)^T + \mathscr{Q}\hat{V}_2^{-1}\mathscr{Q}^T]\Gamma^T, \tag{99}$$

$$0 = A^TP + PA + R_1 + \sum_{i=1}^{p} [A_i^TPA_i + (A_i - \mathscr{Q}V_2^{-1}C_i)^T \times \hat{P}(A_i - \mathscr{Q}\hat{V}_2^{-1}C_i)] + A_Q^T\hat{P} + \hat{P}A_Q, \tag{100}$$

$$0 = [A_Q^T\hat{P} + \hat{P}(G^TA_c\Gamma + B\hat{R}_2^{-1}\mathscr{P}) + \mathscr{P}^T\hat{R}_2^{-1}\mathscr{P}]G^T, \tag{101}$$

$$0 = G[(G^TA_c\Gamma + B\hat{R}_2^{-1}\mathscr{P})^T\hat{P} + \hat{P}(G^TA_c\Gamma + B\hat{R}_2^{-1}\mathscr{P}) + \mathscr{P}^T\hat{R}_2^{-1}\mathscr{P}]G^T. \tag{102}$$

The remaining calculations proceed as follows. Computing either (99)-Γ(98) or (102)-G(101) yields (59). Inserting this expression for A_c into (98), (99), (101), and (102) and computing the equivalent equations $G^T(98)^T$, $G^T(99)G$, (101)Γ, and $\Gamma^T(102)\Gamma$, it follows that $G^T(99)G = G^T(98)\tau^T$ and $\Gamma^T(102)\Gamma = \tau(101)\Gamma$. Hence, (99) and (102) are superfluous. Furthermore, $G^T(98)^T$ and (101)Γ are equivalent respectively to

$$0 = \tau[A_P\hat{Q} + \hat{Q}A_P^T + \mathscr{Q}\hat{V}_2^{-1}\mathscr{Q}^T], \tag{103}$$

$$0 = [A_Q^T\hat{P} + \hat{P}A_Q + \mathscr{P}^T\hat{R}_2^{-1}\mathscr{P}]\tau. \tag{104}$$

Finally, to obtain (62)-(65) note that

$$(64) = (103) + (103)^T - (103)\tau^T,$$

$$(62) = (97) - (64),$$

$$(65) = (104) + (104)^T - \tau^T(104),$$

$$(63) = (100) - (65).$$

□

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