[14] D. E. Dudgeon, "Two-dimensional recursive filtering," Sc.D. thesis, Dep. Elec. Eng. Comput. Sci., M.I.T., Cambridge, MA, May 1974.
[15] M. D. Ni and J. K. Aggarwal, "Two-dimensional digital filtering and its error analysis," *IEEE Trans. Comput.*, vol. C-23, pp. 942–954, Sept. 1974.
[16] S. K. Mitra, A. D. Sagar, and N. A. Pendergrass, "Realizations of two-dimensional recursive digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-22, pp. 177–184, Mar. 1975.
[17] S. Y. Kung, B. C. Levy, M. Morf, and T. Kailath, "New results in 2-D systems theory, Part II: 2-D state-space models," *Proc. IEEE*, vol. 65, pp. 945–961, June 1977.
[18] D. S. K. Chan, "A simple derivation of minimal and near-minimal realizations of 2-D transfer functions," *Proc. IEEE*, vol. 66, pp. 515–516, Apr. 1978.
[19] R. E. Mullans and D. L. Elliott, "Linear systems on partially ordered time sets," in *Proc. IEEE Conf. Decision Contr.*, San Diego, CA, pp. 334–337, Dec. 1973.
[20] R. E. Kalman, P. Falb, and M. A. Arbib, *Topics in Mathematical System Theory*. New York: McGraw-Hill, 1969.

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Optimal Periodic Control: The π Test Revisited

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***Abstract*—The paper examines second-order conditions for both steady-state and dynamic optimality in a periodic control problem. It centers on the π condition of Bittanti, Fronza, and Guardabassi [2] and has three main objectives: 1) to form a π "test" for a somewhat more general problem than considered in [2]; 2) to point out that certain auxiliary conditions must be added if the results of [2] are to be valid; and 3) to explore more fully the relationships between second-order conditions for steady-state optimality and second-order conditions for optimality in the dynamic problem.**

I. Introduction

FOR SOME dynamic processes which are normally operated in a steady-state mode, it may be possible to improve performance by time-dependent periodic control. This possibility has received much attention in recent years and there is a well-developed theory [1], [7], [10], [13]. The application of second-order conditions for optimality as a test for the possibility of improved performance was pioneered by Bittanti, Fronza, and Guardabassi [2], [9]. Their conditions involve a frequency domain criterion, called the π test, which has proved valuable in cases where first-order conditions for optimality [1], [2], [7], [13] have failed to give information. In this paper we have three main objectives: 1) to form a π "test" for a somewhat more general problem than considered in [2]; 2) to point out that certain auxiliary conditions must be added if the results of [2] are to be valid; and 3) to explore more fully the relationships between second-order conditions for steady-state optimality and optimality in the dynamic problem. The auxiliary conditions in (2) are normality conditions which are similar to those that appear in the classical calculus of variations. They rule out pathological cases in which the system is, in a certain sense, uncontrollable in the neighborhood of an optimal solution. The approach to (3) is similar in spirit to the treatment of first-order conditions presented in [7], although for reasons of brevity, there is no attempt to rival the completeness and generality of the discussion there.

The organization and content of this paper may be summarized as follows. In Section II we state an optimal periodic control (OPC) problem and its corresponding steady-state version (OSS), introduce notation and basic definitions, and comment on the relationship between conditions for proper (periodic control is better than optimal steady-state control) and conditions for optimality in

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OPC. In Section III well-known results from the mathematical programming literature are applied to obtain both second-order necessary conditions and second-order sufficient conditions for local optimality in OSS. To obtain the test for proper, the above-mentioned normality condition is needed. This is discussed in Section IV along with some properties of a related rank condition. Section V contains the test for proper (Theorem 5.1). Theorem 5.1 is similar to [2, Theorem 1] in that it concerns the sign definiteness of a π matrix for positive frequencies. Alternatively, as indicated in Theorem 5.2, Theorem 5.1 may be interpreted as providing necessary conditions for steady-state optimality in OPC. This clarifies the connections with the results of Section III and with second-order conditions in the classical calculus of variations. Sufficient conditions for steady-state optimality of OPC, which are a strengthening of the necessary conditions in Theorem 5.2, are given in Section VI. These conditions do not include normality and may be used to show that OPC is not proper. All of these results are limited to optimality in the neighborhood of steady-state solutions of the constraint equations. Section VII gives several simple examples which illustrate the need for some of the conditions stated in Sections V and VI. Since the proofs of the main theorems are quite lengthy, they are relegated to Appendices A and B.

II. Problem Formulation and Basic Definitions

In this section we formulate the periodic control problem and its steady-state specialization. For motivation and additional detail, see [2], [7], [9], [10], [13].

The optimal periodic control problem OPC is: minimize

$$J(x,u,\tau)=g_0(y) \tag{2.1}$$

subject to

$$\begin{aligned}
&g_i(y)\leq 0, \qquad i=-j,\cdots,-1,\\
&g_i(y)=0, \qquad i=1,\cdots,k,\\
&\quad y=\frac{1}{\tau}\int_0^{\tau}\tilde{f}(x(\sigma),u(\sigma))\,d\sigma,\\
&\dot{x}(t)=f(x(t),u(t)), \qquad \text{a.a. } t\in[0,T],\ x(0)=x(\tau),\\
&\qquad\qquad \tau\in(0,T]
\end{aligned} \tag{2.2}$$

where $0<T<+\infty$, $y\in R^l$, $x(t)\in R^n$, $u(t)\in R^m$. The convention $j=0$ means there are no inequality constraints on y; the convention $k=0$ means there are no equality constraints on y; the functions g_i: $R^l\to R$ for $i=-j,\cdots,k$, f: $R^n\times R^m\to R^n$, and $\tilde{f}$: $R^n\times R^m\to R^l$ are twice continuously differentiable. This formulation of OPC is a specialization of the one considered in [7] and includes the formulation considered in [2] as a special case. In [2], $\tau\in(0,+\infty)$ and $g_i(y)=$component of y; also the functional notation for $\tilde{f}$ is different and a maximum is sought (this changes the definition of π and the sign convention in the π test). For simplicity, it is not assumed, as in [7], that $u(t)\in U$, where U is a proper subset of R^m. If $u(t)$, $t\in(0,T]$, belongs to the interior of U, the subsequent analysis is applicable because it focuses on local, "weak" variations in $u(t)$.

By assuming x and u are constant (then τ has no effect), the optimal steady-state problem OSS is obtained: minimize

$$J_{ss}(x,u)=g_0(y) \tag{2.3}$$

subject to

$$\begin{aligned}
&g_i(y)\leq 0, \qquad i=-j,\cdots,-1,\\
&g_i(y)=0, \qquad i=1,\cdots,k,\\
&\quad y=\tilde{f}(x,u),\\
&\quad 0=f(x,u),
\end{aligned} \tag{2.4}$$

where $y\in R^l$, $x\in R^n$, $u\in R^m$. There is a chance that the cost in OPC may be made strictly less than the minimal cost in OSS. To make the discussion of this situation precise, we introduce some additional notation and terminology.

For $z\in R^p$ define $|z|=(\sum_{i=1}^{p}z_i^2)^{1/2}$, where z_i is the ith component of z. Let L_∞^p be the Banach space of all functions from $[0,T]$ into R^p which are measurable and essentially bounded with norm

$$\|z\|_\infty=\operatorname*{ess\,sup}_{t\in[0,T]}|z(t)|.$$

Define $\mathfrak{U}=L_\infty^m$ and $\mathfrak{X}=\{x\colon x\in L_\infty^n,\ x \text{ is absolutely continuous on } [0,T]\}$. By using $x(t)\equiv x$ and $u(t)\equiv u$, it is possible to interpret $(x,u)\in R^n\times R^m$ as $(x,u)\in\mathfrak{X}\times\mathfrak{U}$. This notational convenience will be exploited without further comment in what follows.

Next, consider some terminology concerning OPC and OSS. The triple (x,u,τ) is *admissible* if $(x,u,\tau)\in\mathfrak{X}\times\mathfrak{U}\times(0,T]$ and $x(t),u(t)$, τ satisfy (2.2). The triple (x,u,τ) *solves* OPC if (x,u,τ) minimizes $J(x,u,\tau)$ over the class of admissible triples. The pair (x,u) is *steady-state admissible* if $(x,u)\in R^n\times R^m$ and x,u satisfy (2.4). The pair (x,u) *solves* OSS if (x,u) minimizes $J_{ss}(x,u)$ over the class of steady-state admissible pairs. Assume OSS has a solution, say $(\hat{x},\hat{u})$. OPC is *proper* if for some admissible triple (x,u,τ), $J(x,u,\tau)<J_{ss}(\hat{x},\hat{u})$. Because $(\hat{x},\hat{u},\hat{\tau})$ is an admissible triple for all $\hat{\tau}\in(0,T]$ it is known that there exists an admissible triple $(x,u,\tau)=(\hat{x},\hat{u},\hat{\tau})$ such that $J(x,u,\tau)=J_{ss}(\hat{x},\hat{u})$. Thus, if OPC is not proper $(\hat{x},\hat{u},\hat{\tau})$ solves OPC. Our subsequent theory centers on "local optimality" in OPC and OSS. This prompts the following definitions.

Definition 2.1: The pair $(\bar{x},\bar{u})$ is a *local minimum* of OSS if: i) $(\bar{x},\bar{u})$ is steady-state admissible and ii) there exists an $\epsilon>0$ such that for all steady-state admissible pairs (x,u) satisfying $|x-\bar{x}|+|u-\bar{u}|<\epsilon$, it follows that $J_{ss}(x,u)\geq J_{ss}(\bar{x},\bar{u})$. If in ii), $J_{ss}(x,u)=J_{ss}(\bar{x},\bar{u})$ only for $(x,u)=(\bar{x},\bar{u})$, $(\bar{x},\bar{u})$ is an *isolated* local minimum of OSS.

Definition 2.2: The pair $(\bar{x},\bar{u})$ is a *local steady-state minimum* of OPC if: i) $(\bar{x},\bar{u})$ is steady-state admissible and ii) there exists an $\epsilon>0$ such that for all admissible triples

(x,u,τ) satisfying $\|x-\bar{x}\|_\infty+\|u-\bar{u}\|_\infty<\epsilon$, it follows that $J(x,u,\tau)\geqslant J_{ss}(\bar{x},\bar{u})$. If in ii), $J(x,u,\tau)=J_{ss}(\bar{x},\bar{u})$ only for $(x,u)=(\bar{x},\bar{u})$, $(\bar{x},\bar{u})$ is an *isolated* local steady-state minimum of OPC.

Definition 2.3: OPC is locally proper at $(\bar{x},\bar{u})$ if: i) $(\bar{x},\bar{u})$ is a local minimum of OSS and ii) for all $\epsilon>0$ there exists an admissible triple (x,u,τ) such that $\|x-\bar{x}\|_\infty+\|u-\bar{u}\|_\infty<\epsilon$ and $J(x,u,\tau)<J_{ss}(\bar{x},\bar{u})$.

Remark 2.4: If $(\bar{x},\bar{u})$ is a local steady-state minimum of OPC, then $(\bar{x},\bar{u})$ is a local minimum of OSS.

Remark 2.5: If $(\bar{x},\bar{u})$ solves OSS and OPC is locally proper at $(\bar{x},\bar{u})$, then OPC is proper. Thus, if OSS can be solved, a test for locally proper becomes a test for proper.

Remark 2.6: Definition 2.3 differs from the one given in [2] in that $(\bar{x},\bar{u})$ does not necessarily solve OSS. Definition 2.3 is consistent with the emphasis on local optimality and, because of the preceding remark, represents no loss in the practical application of the theory.

Remark 2.7: The pair $(\bar{x},\bar{u})$ is a local steady-state minimum of OPC if and only if: i) $(\bar{x},\bar{u})$ is a local minimum of OSS and ii) OPC is *not* locally proper at $(\bar{x},\bar{u})$. This equivalence is important because it shows that necessary (sufficient) conditions for OPC to be locally proper at $(\bar{x},\bar{u})$ are related to sufficient (necessary) conditions for $(\bar{x},\bar{u})$ being a local steady-state minimum of OPC.

III. Conditions for Optimality in OSS

Since OSS is a finite dimensional minimization problem with equality and inequality constraints, necessary and sufficient conditions for (local) minimality can be obtained by applying known results such as those found in [5], [14]. The details involve notational issues only and are therefore omitted from what follows.

The conditions are stated most conveniently in terms of a function H, which also appears in the theorems of the following sections. For $\lambda\in R^n$, $\mu\in R^l$, $\alpha\in R^{k+j+1}$ define $H: R^n\times R^m\times R^l\times R^n\times R^l\times R^{k+j+1}\to R$ by

$$H(x,u,y,\lambda,\mu,\alpha)=\alpha' g(y)+\lambda' f(x,u)+\mu'(y-\tilde{f}(x,u)) \tag{3.1}$$

where the prime denotes transpose and $g(y)=(g_{-j}(y),\cdots,g_k(y))'$. Partial derivatives are indicated by subscripts, e.g.,

$$H_x(x,u,y,\lambda,\mu,\alpha)=\lambda' f_x(x,u)-\mu'\tilde{f}_x(x,u) \tag{3.2}$$

where f_x and $\tilde{f}_x$ are the Jacobian matrices of f and $\tilde{f}$ with respect to x. Similarly, H_{xx} is the Hessian matrix of H with respect to x. When the various functions and their partial derivatives are evaluated at $\bar{x},\bar{u},\bar{y}=\tilde{f}(\bar{x},\bar{u}),\bar{\lambda},\bar{\mu},\bar{\alpha}$ they will be denoted by an overbar. For instance, $\bar{H}=H(\bar{x},\bar{u},\bar{y},\bar{\lambda},\bar{\mu},\bar{\alpha})$.

Theorem 3.1: Let $(\bar{x},\bar{u})$ be a local minimum of OSS. Then there exist $\bar{\lambda}\in R^n$, $\bar{\mu}\in R^l$, $\bar{\alpha}=(\bar{\alpha}_{-j},\cdots,\bar{\alpha}_k)\in R^{j+k+1}$ such that the following conditions hold:

$$\begin{aligned}
&\bar{H}_x,\bar{H}_u,\bar{H}_y=0\\
&\bar{\alpha}_i\geqslant 0, \qquad i=-j,\cdots,0\\
&\bar{\alpha}_i\bar{g}_i=0, \qquad i=-j,\cdots,-1\\
&(\bar{\lambda},\bar{\mu},\bar{\alpha})\neq 0.
\end{aligned} \tag{3.3}$$

From $\bar{H}_y=0$ it is seen that $\bar{\mu}=-\bar{g}_y'\bar{\alpha}$. Thus, $\bar{\mu}$ can be eliminated from the statement of the theorem (and many of the following theorems). To circumvent notational complexity, it is convenient to avoid the elimination. Usually, but not always, $\bar{\alpha}_0>0$. This can be assured by introducing a constraint qualification [5], [14]. If $\bar{\alpha}_0>0$, it can be assumed without loss of generality that $\bar{\alpha}_0=1$.

Theorem 3.1 gives the "first-order" necessary conditions [14, Theorem 1]. To introduce the "second-order" necessary conditions [14, Theorem 4], some additional notation is required:

$$A=f_x,\ B=f_u,\ C=\tilde{f}_x,\ D=\tilde{f}_u,$$

$$M(I)=\begin{bmatrix} g_{i_1y}\\ \vdots\\ g_{i_sy}\end{bmatrix}. \tag{3.4}$$

Here, I is an index set $\{i_1,i_2,\cdots,i_s\}$, where for definiteness, $i_1<i_2<\cdots<i_s$.

Theorem 3.2: Let $(\bar{x},\bar{u})$ be a local minimum of OSS and suppose OSS satisfies first- and second-order constraint qualifications [14] at $(\bar{x},\bar{u})$. Then (3.3) holds for some $\bar{\lambda},\bar{\mu},\bar{\alpha}$ with $\bar{\alpha}_0=1$. Moreover, for all $x\in R^n$, $u\in R^m$, $y\in R^l$ which satisfy

$$\begin{aligned}
\bar{A}x+\bar{B}u&=0,\\
\bar{C}x+\bar{D}u&=y,\\
\bar{M}(\hat{I})y&=0,\\
\bar{M}(\bar{I}-\hat{I})y&\leqslant 0
\end{aligned} \tag{3.5}$$

where

$$\bar{I}=\{i: i<0, \bar{g}_i=0\}\cup\{1,\cdots,k\} \tag{3.6}$$

and

$$\hat{I}=\{i: i<0, \bar{\alpha}_i>0\}\cup\{1,\cdots,k\}, \tag{3.7}$$

it follows that

$$x'\bar{H}_{xx}x+2x'\bar{H}_{xu}u+u'\bar{H}_{uu}u+y'\bar{H}_{yy}y\geqslant 0. \tag{3.8}$$

Finally, sufficient conditions for a local minimum of OSS are obtained from [14, Theorem 6].

Theorem 3.3: Let $(\bar{x},\bar{u})$ be a steady-state admissible pair and suppose that (3.3) holds for $(\bar{\lambda},\bar{\mu},\bar{\alpha})\in R^n\times R^m\times R^{j+k+1}$ with $\bar{\alpha}_0=1$. If

$$x'\bar{H}_{xx}x+2x'\bar{H}_{xu}u+u'\bar{H}_{uu}u+y'\bar{H}_{yy}y>0 \tag{3.9}$$

for all nonzero (x,u,y) which satisfy (3.5), then $(\bar{x},\bar{u})$ is an isolated local minimum of OSS.

The conditions in this theorem are a strengthening of the conditions in Theorem 3.2 in that (3.9) is a strict inequality.

Remark 3.4: There are a number of ways in which the constraint qualifications mentioned in Theorem 3.2 can be assured. The most common condition involves linear independence of the gradients of the "active" constraint functions. See [5, Corollary 3, Section 2.1 and Theorem 3, Section 2.2]. In the present context this condition is satisfied if and only if

$$\operatorname{rank}\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{M}\bar{C} & \bar{M}\bar{D} \end{bmatrix} = n+s \tag{3.10}$$

where $\bar{M}=\bar{M}(\bar{I})$ and s = number of elements in $\bar{I}$ = number of "active" constraints on $\bar{y}$.

IV. The Normality Condition

Before stating the theorems which have to do with OPC being locally proper at $(\bar{x},\bar{u})$, we need to discuss a normality condition. This is not surprising in view of Remark 2.7. Normality conditions appear in the development of second-order necessary conditions in the classical calculus of variations.

Let $(\bar{x},\bar{u})$ be a steady-state admissible pair. We say OPC satisfies a full rank condition at $(\bar{x},\bar{u})$ and τ if

$$\operatorname{rank}\begin{bmatrix} \left(e^{\bar{A}\tau}-I_n\right) & \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B} \\ \left(\bar{M}\bar{C}\int_0^\tau e^{\bar{A}\sigma}d\sigma\right) & \bar{M}\bar{D} & \bar{M}\bar{C}\bar{B} & \cdots & \bar{M}\bar{C}\bar{A}^{n-2}\bar{B} \end{bmatrix} = n+s \tag{4.1}$$

where $I_n = n\times n$ identity matrix, and the remaining notation is given in Remark 3.4. It is clear that (4.1) is a controllability condition. Appendix A shows that it guarantees the existence of a one-parameter family of solutions of (2.2): $x(t,\epsilon), u(t,\epsilon), y(\epsilon)$. This family can be chosen so that $x(t,0)\equiv\bar{x}$, $u(t,0)\equiv\bar{u}$, $y(0)=\bar{f}(\bar{x},\bar{u})$ and $x_\epsilon(t,0)=\hat{x}(t)$, $u_\epsilon(t,0)=\hat{u}(t)$, $y_\epsilon(0)=\hat{y}$ where $\hat{x},\hat{u},\hat{y}$ are a solution of a linearized version of (2.2). Specifically,

$$\bar{g}_{iy}\hat{y}\leqslant 0, \qquad i\in\bar{I},\ i<0,$$

$$\bar{g}_{iy}\hat{y}=0, \qquad i\in\bar{I},\ i>0,$$

$$\hat{y}=\frac{1}{\tau}\int_0^\tau\left(\bar{C}\hat{x}(\sigma)+\bar{D}\hat{u}(\sigma)\right)d\sigma,$$

$$\dot{\hat{x}}(t)=\bar{A}\hat{x}(t)+\bar{B}\hat{u}(t), \qquad \hat{x}(0)=\hat{x}(\tau). \tag{4.2}$$

Let the $(n+s)\times(n+nm)$ matrix in (4.1) be denoted by $\Psi(\tau)$. The rank condition is satisfied if and only if the rows of $\Psi(\tau)$ are linearly independent. This holds if and only if the Gramian matrix $\Psi(\tau)\Psi'(\tau)$ is nonsingular, i.e., $\psi(\tau)=\det\Psi(\tau)\Psi'(\tau)\neq 0$. Because $\psi(\tau)$ is analytic there are two possibilities: i) $\psi(\tau)\equiv 0$ on $[0,T]$ and ii) $\psi(\tau)=0$ for a finite number of values of τ in $[0,T]$. Thus, if the rank condition is satisfied for some τ in $[0,T]$, it is satisfied for all but a finite number of values of τ in $[0,T]$. This motivates the following.

Definition 4.1: Let $(\bar{x},\bar{u})$ be a steady-state admissible pair. OPC is *normal* at $(\bar{x},\bar{u})$ if the rank condition (4.1) is satisfied for some $\tau\in[0,T]$.

Remark 4.2: Suppose OPC is normal at $(\bar{x},\bar{u})$. Then the rank condition is satisfied for all but a finite number of values of τ in $[0,T]$.

It turns out that normality is equivalent to the constraint qualification condition mentioned in Remark 3.4. Thus it is easily verified by a simple algebraic test.

Theorem 4.3: OPC is normal at $(\bar{x},\bar{u})$ if and only if (3.10) is satisfied.

Proof: In (4.1) replace $e^{\bar{A}\tau}-I_n$ with $\bar{A}\int_0^\tau e^{\bar{A}\sigma}d\sigma$ and note that $\det\int_0^\tau e^{\bar{A}\sigma}d\sigma\neq 0$ for $\tau>0$, τ sufficiently small. For this value of τ the span of the first n columns of the matrix in (4.1) contains the span of the last $(n-1)m$ columns. Thus, (4.1) reduces to

$$\operatorname{rank}\begin{bmatrix} \bar{A}\int_0^\tau e^{\bar{A}\sigma}d\sigma & \bar{B} \\ \bar{M}\bar{C}\int_0^\tau e^{\bar{A}\sigma}d\sigma & \bar{M}\bar{D} \end{bmatrix} = n+s \tag{4.3}$$

which is equivalent to (3.10).

V. The Condition for Locally Proper

Before stating the main theorem we need some further notation. Let $\Omega(A)$ be the set of nonnegative real numbers such that $\omega\in\Omega(A)$ if and only if $j\omega$ is an eigenvalue of A. Define

$$G(s)=\left(sI_n-\bar{A}\right)^{-1}\bar{B} \tag{5.1}$$

and for $\omega\geqslant 0$, $\omega\notin\Omega(\bar{A})$ let

$$\pi(\omega)=G'(-j\omega)\bar{H}_{xx}G(j\omega)+\bar{H}_{ux}G(j\omega) + G'(-j\omega)\bar{H}_{xu}+\bar{H}_{uu}. \tag{5.2}$$

Clearly, $\pi(\omega)$ is a complex-valued $m\times m$ matrix which is Hermitian and depends on $\bar{x},\bar{u},\bar{y},\bar{\lambda},\bar{\mu},\bar{\alpha}$. Let $\mathcal{C}$ be the set of complex numbers and $*$ denote complex conjugate transpose.

Theorem 5.1: Assume $(\bar{x},\bar{u})$ is a local minimum of OSS and OPC is normal at $(\bar{x},\bar{u})$. Then (3.3) is satisfied with $\bar{\alpha}_0=1$ and for $\bar{\alpha}_0=1$, $\bar{\lambda},\bar{\mu},\bar{\alpha}$ and $\pi(\omega)$ are unique. Further, suppose there exists $\eta \in \mathcal{C}^m$ and $\omega \geq 2\pi/T$, $\omega \notin \Omega(\bar{A})$ such that

$$\eta^* \pi(\omega)\eta < 0. \tag{5.3}$$

Then OPC is locally proper at $(\bar{x},\bar{u})$.

When Theorem 5.1 is specialized to the problem considered in [2], it corresponds to the first part of [2, Theorem 1]. Apart from notation, it differs in three fundamental respects: it guarantees the existence of $\bar{\alpha}_0 = 1$ and the uniqueness of $\pi(\omega)$, it places no restriction on $\Omega(\bar{A})$ (in [2], $\Omega(\bar{A})$ is empty), and it requires the normality condition. It will be seen in Section VII that [2, Theorem 1] may fail because it does not include a normality condition. Fortunately, the normality condition is satisfied in many problems, including most which have been treated in the literature. One case of practical interest is $k=j=0$ and $\bar{A}$ nonsingular. Then OPC automatically satisfies the normality condition. A special version of this case was treated correctly in [9].

As indicated in Remark 2.7 the conditions in Theorems 3.2 and 5.1 can be viewed as necessary conditions for optimality. In particular, they yield the following.

Theorem 5.2: Assume $(\bar{x},\bar{u})$ is a local steady-state minimum of OPC and that OPC is normal at $(\bar{x},\bar{u})$. Then there exist $\bar{\lambda} \in R^n$, $\bar{\mu} \in R^l$, $\bar{\alpha} \in R^{j+k+1}$ with $\bar{\alpha}_0 = 1$ such that: i) (3.3) is satisfied, ii) (3.8) holds for all x,u,y which satisfy (3.5), and iii)

$$\eta^* \pi(\omega)\eta \geq 0 \qquad \text{for all } \eta \in \mathcal{C}^m,\ \omega \geq \frac{2\pi}{T},\ \omega \notin \Omega(\bar{A}). \tag{5.4}$$

Conditions i) and ii) follow from Theorems 3.2 and 4.3 and Remark 3.4. Condition iii), which follows from Theorem 5.1, is analogous to the Jacobi condition in calculus of variations. Because of $\lim_{\omega\to\infty}\pi(\omega) = \bar{H}_{uu}$ and condition iii), $\eta^* \bar{H}_{uu}\eta \geq 0$. This is the "Legendre condition."

The idea behind the proof of Theorem 5.1 is simple. Using the normality condition, a one-parameter family of the type described in Section IV is generated, where in (4.2) it is assumed that $\hat{u}(t) = \mathrm{Re}\,\eta e^{j\omega t}$. Then $J(x,u,\tau) = \hat{J}(\epsilon)$ is evaluated in terms of H and $\hat{J}(\epsilon) < \bar{J}$ ($\epsilon > 0$, small) if the conditions of the theorems are satisfied. Because of Remark 4.2 and the continuity of $\pi(\omega)$, $\omega \notin \Omega(\bar{A})$, it is possible to avoid special conditions having to do with τ values for which (4.1) fails. The first variation ($J_\epsilon(0)$) is zero and $\hat{J}(\epsilon) < \bar{J}$ is obtained from the second variation ($J_{\epsilon\epsilon}(0)$). For the details see Appendix A.

Since Theorem 3.2 involves steady-state variations, it might be conjectured that it could be stated in terms of $\pi(0)$. This is true if $\bar{A}$ is nonsingular. Then the condition corresponding to (3.8) can be restated as follows:

$$u'\left[\pi(0) + \left(\bar{C}G(0) + \bar{D}\right)'\bar{H}_{yy}\left(\bar{C}G(0)+\bar{D}\right)\right]u \geq 0 \tag{3.8$'$}$$

holds for all $u \in R^m$ such that

$$\bar{M}(\hat{I})\left(\bar{C}G(0) + \bar{D}\right)u = 0,$$
$$\bar{M}(\bar{I} - \hat{I})\left(\bar{C}G(0) + \bar{D}\right)u \leq 0. \tag{3.5$'$}$$

Even for $\bar{H}_{yy} = 0$ (this happens for the problem statement in [2] where $g(y)$ is affine) the presence of (3.5)$'$ complicates the issue of relating (3.8)$'$ to (5.4). For example, suppose (3.8)$'$ is satisfied for all $u \in R^m$ satisfying (3.5)$'$ but $u'\pi(0)u < 0$ for some $u \in R^m$ which does not satisfy (3.5)$'$. Then (5.3) will hold if T is sufficiently large ($\bar{A}$ nonsingular implies $\pi(\omega)$ is continuous at $\omega = 0$). Thus, very low frequency forcing can improve performance. This is the content of [2, Remark 1].

VI. Local Steady-State Optimality in OPC

If the conditions in Theorem 5.2 are suitably strengthened, they become sufficient for $(\bar{x},\bar{u})$ to be a local steady-state minimum of OPC. This requires much weaker hypotheses than a strengthening based on first-order necessary conditions. See, for example, [8].

Theorem 6.1: Let $(\bar{x},\bar{u})$ be a steady-state admissible pair and suppose the following conditions are satisfied: i) $\bar{A}$ is nonsingular ($0 \notin \Omega(\bar{A})$), ii) $\omega \notin \Omega(\bar{A})$ for all $\omega \geq 2\pi/T$, iii) there exist $\bar{\lambda}, \bar{\mu}, \bar{\alpha}$ with $\alpha_0 = 1$ such that (3.3) holds, iv) for $\bar{I}$ and $\hat{I}$ defined by (3.6) and (3.7),

$$u'\left[\pi(0) + \left(\bar{C}G(0) + \bar{D}\right)'\bar{H}_{yy}\left(\bar{C}G(0)+\bar{D}\right)\right]u > 0 \tag{6.1}$$

is satisfied for all $u \in R^m$ such that $u \neq 0$ and

$$\bar{M}(\hat{I})\left(\bar{C}G(0) + \bar{D}\right)u = 0,$$
$$\bar{M}(\bar{I} - \hat{I})\left(\bar{C}G(0) + \bar{D}\right)u \leq 0, \tag{6.2}$$

and v) there exists $\gamma > 0$ such that

$$\eta^* \pi(\omega)\eta \geq \gamma\eta^*\eta \qquad \text{for all } \eta \in \mathcal{C}^m,\ \omega \geq \frac{2\pi}{T}. \tag{6.3}$$

Then $(\bar{x},\bar{u})$ is an isolated local steady-state minimum of OPC.

The proof of this theorem appears in Appendix B. The general idea is to assume that there exists a sequence of admissible triples $\{(x_i,u_i,\tau_i)\}$ with $(x_i,u_i) \neq (\bar{x},\bar{u})$ and $\|x_i - \bar{x}\|_\infty + \|u_i - \bar{u}\|_\infty \to 0$ such that $J(x_i,u_i,\tau_i) \leq J_{ss}(\bar{x},\bar{u})$ and show that this leads to a contradiction. It is necessary to consider a subsequence which is directionally convergent in a certain sense. The arguments are similar to those used in [5], but more complex because of the function space setting (see [12] indirect sufficiency proofs). It is also possible to derive a sufficiency theorem without i) and ii) but then the conditions are not so simply stated and the proof is more difficult. An entirely different approach to the proof of the theorem exploits a Riccati equation. This path has been followed in [3] for the case of no constraints on y ($j=k=0$). Its use here seems to offer no advantages.

Since by Theorem 3.3, conditions i), iii), and iv) imply $(\bar{x},\bar{u})$ is a local steady-state minimum of OSS, it is of interest to ask if iv) can be replaced by the hypothesis that $(\bar{x},\bar{u})$ is a local minimum of OSS. It seems that the answer is no, although an example which illustrates the need for iv) has not been found.

Remark 6.2: Because $G(j\omega)\to 0$ as $\omega\to\infty$, condition v) implies $\bar{H}_{uu}>0$. This corresponds to the strengthened Legendre condition in the calculus of variations. Using the properties of $G(j\omega)$, it is easy to show that the uniform bound (6.3) may be replaced by $\bar{H}_{uu}>0$ and

$$\eta^*\pi(\omega)\eta>0, \qquad \text{for all } \eta\in\mathcal{C}^m,\ \eta\neq 0,\ \omega\geq\frac{2\pi}{T}. \quad (6.4)$$

Remark 6.3: For $\bar{H}_{uu}>0$, $\bar{H}_{yy}=0$ (true if $g(y)$ is affine), and $\Omega(\bar{A})$ empty, hypotheses iv) and v) may be replaced by the (stronger) hypothesis:

$$\eta^*\pi(\omega)\eta>0, \qquad \text{for all } \eta\in\mathcal{C}^m,\ \eta\neq 0,\ \omega>0. \quad (6.5)$$

This follows from (5.2), the continuity of $G(s)$ at $s=0$ when $\bar{A}$ is nonsingular, and Remark 6.2.

Using this Remark, together with Remark 2.7, proves the following.

Corollary 6.4: Assume OPC is locally proper at $(\bar{x},\bar{u})$, $\bar{H}_{uu}>0$, $\bar{H}_{yy}=0$, $\Omega(\bar{A})$ is empty, and (3.3) is satisfied for $\bar{\lambda},\bar{\mu},\bar{\alpha}$ with $\bar{\alpha}_0=1$. Then there exist $\eta\in\mathcal{C}^m$, $\eta\neq 0$, and $\omega>0$ such that

$$\eta^*\pi(\omega)\eta\leq 0. \quad (6.6)$$

This corollary is equivalent to [2, Theorem 1, part 2] except there the condition $\bar{H}_{uu}>0$ is omitted. An example (see the next section) shows that the conclusion (6.6) may be false if $\bar{H}_{uu}>0$ does not hold.

VII. Examples

The first example illustrates that Theorem 5.1 fails if the normality assumption is omitted. The problem data are: $j=0$ (no inequality constraints on y), $k=1$, $n=1$, $m=1$, $l=2$, $g_0(y)=y_1$, $g_1(y)=y_2$, $f(x,u)=u^2+xu-4$, $\tilde{f}_1(x,u)=2x+4u$, $\tilde{f}_2(x,u)=(u-2)^2$, $T=$any positive number. The only steady-state admissible pair is $\bar{x}=0$, $\bar{u}=2$. Thus, $(\bar{x},\bar{u})$ is both a minimum and local minimum of OSS. By Theorem 3.1 conditions (3.3) must be satisfied. In fact, $\bar{\alpha}_0=1$, $\bar{\mu}_1=-1$, $\bar{\alpha}_1=-\bar{\mu}_2$, $\bar{\lambda}=-1$, where $\bar{\alpha}_1$ is arbitrary, is a solution. This gives

$$\pi(\omega)=16(\omega^2+4)^{-1}-2+2\bar{\alpha}_1. \quad (7.1)$$

Since $\overline{MC}=\overline{MD}=0$ the rank condition (3.10) is not satisfied. Thus, OPC is not normal at $(\bar{x},\bar{u})$. This accounts for the fact that $\pi(\omega)$ is not uniquely determined. For $\bar{\alpha}_1=0$ it follows that $\eta^*\pi(\omega)\eta<0$ when $\eta\neq 0$, $\omega>2$, $\omega\geq 2\pi/T$. However, OPC is *not* locally proper at $(\bar{x},\bar{u})$. This is clear because (2.2) implies $u(t)\equiv 2$ and $x(t)\equiv 0$.

The second example shows that (6.3) cannot be replaced by (6.4) unless $\bar{H}_{uu}>0$ (see Remark 6.2). Let $j=k=0$ (no constraints on y), $n=1$, $m=1$, $l=1$, $T=1$, $g_0(y)=y$, $f(x,u)=-x+u$, $\tilde{f}(x,u)=2ux-x^2-G(u-x)$ where

$$G(\eta)=0, \qquad \eta<0$$
$$=\rho\eta^3, \qquad \eta\geq 0. \quad (7.2)$$

It is easy to see the pair $(\bar{x},\bar{u})$ is steady-state admissible if and only if $\bar{x}=\bar{u}$. Substituting this into $J_{ss}=y=\tilde{f}$ shows OSS has a unique local minimum at $\bar{x}=\bar{u}=0$ and $J_{ss}(0,0)=0$. Conditions (3.3) hold at $\bar{\alpha}=1$, $\bar{\mu}=-1$, $\bar{\lambda}=0$. Moreover, $\bar{H}_{uu}=0$ and

$$\pi(\omega)=2(1+\omega^2)^{-1}>0, \qquad \omega>0. \quad (7.3)$$

Now let

$$u(t)=q\sin\omega t, \qquad \tau=\frac{2\pi}{\omega}. \quad (7.4)$$

A simple calculation shows (2.2) has a unique solution and

$$u(t)-x(t)=w(t)$$
$$=q\omega(1+\omega^2)^{-1}(\omega\sin\omega t+\cos\omega t). \quad (7.5)$$

This gives

$$J(x,u,\tau)=q^2(1+\omega^2)^{-1}-\frac{1}{\tau}\int_0^\tau G(w(t))\,dt$$
$$=q^2(1+\omega^2)^{-1}-\nu\rho q^3\omega^3\sqrt{1+\omega^2}^{\,-3} \quad (7.6)$$

where ν is a positive constant. Setting

$$\tau=\frac{2\pi}{\omega}=2\pi q^{1/2}(1-q)^{-1/2}, \quad \rho=\frac{2}{\nu}, \quad (7.7)$$

gives (for all $q>0$)

$$J(x,u,\tau)=-q^3\left(2\sqrt{1-q}^{\,3}-1\right). \quad (7.8)$$

Since $J(x,u,\tau)<0$ and $\|x-\bar{x}\|_\infty+\|u-\bar{u}\|_\infty=q(1+q^{1/2})$, it is clear that $(\bar{x},\bar{u})$ is not a local steady-state minimum of OPC.

VIII. Conclusions

A variety of second-order conditions for optimality in OSS and OPC have been presented and their interrelationship has been examined. The π test of Bittanti, Fronza, and Guardabassi [2] has been extended to a more general class of periodic control problems and the importance of normality in tests for proper and the strengthened Legendre condition in tests for not proper has been stressed. Although the applicability of the π test is more restricted than indicated in [2], the additional requirements (3.10) or $\bar{H}_{uu}>0$) are easily evaluated.

Just before going to press the authors became aware of [15]. This reference treats the original problem of [3] and under a different normality condition, which is both stronger and more difficult to verify than Definition 4.1, proves that the π test is a condition for proper. Under the normality condition it also gives a sequence condition which implies $\pi(\omega)$ is not positive definite for all $\omega>0$.

From the last example in Section VII it can be seen that the sequence condition is stronger than "proper."

APPENDIX A
PROOFS OF NECESSARY CONDITIONS

We begin by verifying some simple lemmas.

Lemma A.1: Let V be a Banach space and consider Q: $\mathfrak{N} \to R^q$ where $\mathfrak{N}$ is a neighborhood of $\bar{v} \in V$. Assume i) $Q(\bar{v}) = 0$, ii) Q has a Frechet derivative $Q^1(\bar{v})$: $V \to R^q$ at $\bar{v}$, iii) for $v_1, \cdots, v_{q+1} \in V$ the function g determined by

$$g(\beta_1, \cdots, \beta_{q+1}) = Q\left(\sum_{i=1}^{q+1} \beta_i v_i + \bar{v}\right) \tag{A.1}$$

is C^2 in a neighborhood of $(\beta_1, \cdots, \beta_{q+1}) = 0$, and iv) Range $Q^1(\bar{v}) = R^q$. Let $h \in R^q$. Then there exists $\epsilon_0 > 0$ and v: $(-\epsilon_0, \epsilon_0) \to V$ such that $v(0) = \bar{v}$, v is C^2 and

$$Q(v(\epsilon)) + \epsilon h = 0, \qquad |\epsilon| < \epsilon_0. \tag{A.2}$$

Furthermore, if $\hat{v} \in V$ satisfies

$$Q^1(\bar{v})\hat{v} + h = 0, \tag{A.3}$$

then $v(\epsilon)$ can be chosen so that

$$\frac{dv}{d\epsilon}(0) = \hat{v}. \tag{A.4}$$

Proof: By assumption there exist $v_1, \cdots, v_q \in V$ such that $\{Q^1(\bar{v})v_i\}_{i=1}^q$ is a linearly independent set. Consider $f(\beta_1, \cdots, \beta_q, \epsilon) = g(\beta_1, \cdots, \beta_q, \epsilon) + \epsilon h$ for arbitrary $v_{q+1} \in V$. Clearly, f is C^2 and the Jacobian of f with respect to $\beta_1, \cdots, \beta_q$ at $(\beta_1, \cdots, \beta_q, \epsilon) = 0$, which has rows $(Q^1(\bar{v})v_i)'$, is nonsingular. Thus, the implicit function theorem, [4, p. 202], guarantees the existence of $\epsilon_0 > 0$ and C^2 functions β_i: $(-\epsilon_0, \epsilon_0) \to R$ such that $\beta_i(0) = 0$ and $f(\beta_1(\epsilon), \cdots, \beta_q(\epsilon), \epsilon) \equiv 0$. Setting

$$v(\epsilon) = \bar{v} + \sum_{i=1}^{q} \beta_i(\epsilon) v_i + \epsilon v_{q+1} \tag{A.5}$$

gives (A.2). Next, take the derivative of (A.2) with respect to ϵ at $\epsilon = 0$ and substitute $dy/d\epsilon(0)$ as obtained from (A.5). Choosing $v_{q+1} = \hat{v}$ and using (A.3) shows

$$\frac{d\beta_i}{d\epsilon}(0) = 0, \qquad i = 1, \cdots, q.$$

This proves (A.4).

Define Q: $R^n \times L_\infty^m \to R^{s+n}$ in the following way: $Q(v) = \nu$ where $v = (\xi, u)$ and the components of ν are given by

$$\begin{aligned} \nu_j &= x_j(\tau) - x_j(0), \qquad j = 1, \cdots, n \\ \nu_{j+n} &= g_{i_j}(y), \qquad j = 1, \cdots, s. \end{aligned} \tag{A.6}$$

Here $x(\tau)$ is determined by the solution of

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = \xi, \tag{A.7}$$

$\{i_1, \cdots, i_s\} = \bar{I}$, and

$$y = \frac{1}{\tau}\int_0^\tau \tilde{f}(x(t), u(t))\, dt. \tag{A.8}$$

Lemma A.2: Let $\bar{x}(t) \equiv \bar{x}$, $\bar{u}(t) \equiv \bar{u}$ where $\bar{x} \in R^n$ and $\bar{u} \in R^m$ satisfy $f(\bar{x}, \bar{u}) = 0$. Then for $\bar{v} = (\bar{x}, \bar{u})$, Q satisfies conditions (2) and (3) of Lemma A.1. Moreover, $Q^1(\bar{v})$ is characterized by $Q^1(\bar{v})(\hat{\xi}, \hat{u}) = \hat{\nu}$ where

$$\hat{\nu} = \begin{bmatrix} \hat{x}(\tau) - \hat{x}(0) \\ \overline{M}\hat{y} \end{bmatrix} \tag{A.9}$$

and $\hat{x}(t)$, $\hat{y}$ are defined by

$$\dot{\hat{x}}(t) = \bar{A}\hat{x}(t) + \bar{B}\hat{u}(t), \qquad \hat{x}(0) = \hat{\xi}, \tag{A.10}$$

$$\hat{y} = \frac{1}{\tau}\int_0^\tau \left(\bar{C}\hat{x}(t) + \bar{D}\hat{u}(t)\right) dt. \tag{A.11}$$

Proof: Following arguments almost identical to those in [6], it can be seen that Q and its first variation are defined in a neighborhood of $\bar{v}$. Also, the first variation at $\bar{v}$ is characterized by (A.9)–(A.11). Since it is linear and bounded, it is the Frechet derivative of Q. Condition (3) follows from the assumptions of f and $\tilde{f}$ (they are C^2) and the resulting differentiability of solutions of differential equations with respect to parameters.

Lemma A.3: If (4.1) is satisfied, Range $Q^1(\bar{v}) = R^{n+s}$.

Proof: Using the variation of parameters formula, it follows from (A.9)–(A.11) that

$$\hat{\nu} = \begin{bmatrix} \left(e^{\bar{A}\tau} - I_n\right) \\ \overline{M}\bar{C}\int_0^\tau e^{\bar{A}t}\, dt \end{bmatrix} \xi + y^*(\tau) \tag{A.12}$$

where $y^*(\tau)$ is given by

$$\begin{aligned} \dot{x}^*(t) &= A^* x^*(t) + B^* \hat{u}(t), \qquad x^*(0) = 0 \\ y^*(t) &= C^* x^*(t) \end{aligned} \tag{A.13}$$

where $x^*(t) \in R^{n+l}$ and

$$A^* = \begin{bmatrix} \bar{A} & 0 \\ \tau^{-1}\bar{C} & 0 \end{bmatrix}, \quad B^* = \begin{bmatrix} \bar{B} \\ \tau^{-1}\bar{D} \end{bmatrix}, \quad C^* = \begin{bmatrix} I_n & 0 \\ 0 & \overline{M} \end{bmatrix}. \tag{A.14}$$

From the theory of linear systems, $y^*(\tau)$ can be generated by $\hat{u}$ if and only if

$$\begin{aligned} y^*(\tau) &\in \text{Range } C^*\left[B^* \; A^*B^* \; \cdots \; (A^*)^{n+l-1} \; B^*\right] \\ &= \text{Range } C^*\left[B^* \; A^*B^* \; \cdots \; (A^*)^{n-1} \; B^*\right]. \end{aligned} \tag{A.15}$$

The equality of the ranges follows because rank $A^* \leqslant n$. Using (A.14) in (A.15) it is seen that (A.12) has a solution for all $\hat{\nu} \in R^{n+s}$ if (4.1) holds.

From the lemmas it is clear that (4.1) implies the existence of $\epsilon_0 > 0$ and $u(t,\epsilon)$, $x(t,\epsilon)$, $y(\epsilon)$ such that

$$\dot{x}(t,\epsilon) = f(x(t,\epsilon), u(t,\epsilon)), \qquad x(\tau,\epsilon) = x(0,\epsilon), \tag{A.16}$$

$$y(\epsilon)=\frac{1}{\tau}\int_0^\tau \tilde{f}(x(t,\epsilon),u(t,\epsilon))\,dt, \tag{A.17}$$

$$g_i(y(\epsilon))=0 \qquad i\in\bar{I},\ i>0, \tag{A.18}$$

$$g_i(y(\epsilon))+\epsilon h_i=0, \qquad i\in\bar{I},\ i<0 \tag{A.19}$$

are satisfied for all ϵ, $|\epsilon|<\epsilon_0$, where the $h_i\in R$ may be chosen arbitrarily. Hereafter, assume $h_i\geqslant 0$ for $i\in\bar{I}$, $i<0$, and $\epsilon>0$. Then (A.19) can be replaced by

$$g_i(y(\epsilon))\leqslant 0, \qquad i\in\bar{I},\ i<0. \tag{A.20}$$

Finally, from Lemmas A.1 and A.2 it follows that $u(t,\epsilon)$, $x(t,\epsilon)$, $y(\epsilon)$ may be chosen so that $u_\epsilon(t,0)=\hat{u}(t)$, $x_\epsilon(t,0)=\hat{x}(t)$, $y_\epsilon(0)=\hat{y}$ where $\hat{u}(t)$, $\hat{x}(t)$, $\hat{y}$ satisfy (4.2).

Now consider the proof of Theorem 5.1. Because of Theorem 4.3, OSS satisfies the linear independence condition of Remark 3.4. Thus, by Theorem 3.2, $\bar{\alpha}_0=1$ is possible. For this choice of $\bar{\alpha}_0$, it is easily verified from $\overline{H}_x, \overline{H}_u, \overline{H}_y=0$ and (3.10) that $\bar{\lambda},\bar{\mu},\bar{\alpha}$ are unique. Thus, $\overline{H}_{xx}$, $\overline{H}_{ux}$, $\overline{H}_{xu}$, $\overline{H}_{uu}$, and $\pi(\omega)$ are unique.

Now let $u(t,\epsilon)$, $x(t,\epsilon)$, $y(\epsilon)$ satisfy (A.16)–(A.19) where in (A.19) $h_i=0$, $i<0$, $i\in\hat{I}$. Thus, equality holds in (A.20) for $i<0$, $i\in\hat{I}$. Since OPC is normal at $(\bar{x},\bar{u})$, it can be assumed without loss of generality that (4.1) holds at $\tau=2\pi/\omega$. If (4.1) does not hold at $\tau=2\pi/\omega$, the continuity of π at ω and Remark 4.2 guarantee the existence of $\hat{\omega}$ close to ω such that (5.3) is satisfied with ω replaced by $\hat{\omega}$, (4.1) is satisfied at $\tau=2\pi/\hat{\omega}$, and $\hat{\omega}\notin\Omega(\bar{A})$. Now, choose $u_\epsilon(t,0)=\hat{u}(t)$, $x_\epsilon(t,0)=\hat{x}(t)$, and $y_\epsilon(0)=\hat{y}$ where

$$\begin{aligned}\hat{u}(t)&=2\,\mathrm{Re}\,\eta e^{j\omega t},\\ \hat{x}(t)&=2\,\mathrm{Re}\big(j\omega I-\bar{A}\big)^{-1}\bar{B}\eta e^{j\omega t},\\ \hat{y}&=0\end{aligned} \tag{A.21}$$

is a solution of (4.2) with the added proviso that $\bar{g}_{iy}\hat{y}=0$ for all $i\in\hat{I}$.

Since $\bar{\alpha}_0=1$, $g_i(y(\epsilon))=g_i(\bar{y})=0$, $i\in\hat{I}$ and $\bar{\alpha}_i=0$, $i\in\bar{I}-\hat{I}$, it follows that

$$\begin{aligned}g_0(y(\epsilon))-g_0(\bar{y})&=H\big(x(t,\epsilon),u(t,\epsilon),y(\epsilon),\bar{\lambda},\bar{\mu},\bar{\alpha}\big)\\&\quad-H\big(\bar{x},\bar{u},\bar{y},\bar{\lambda},\bar{\mu},\bar{\alpha}\big)\\&\quad-\bar{\lambda}'f(x(t,\epsilon),u(t,\epsilon))\\&\quad-\bar{\mu}'\big(y(\epsilon)-\tilde{f}(x(t,\epsilon),u(t,\epsilon))\big).\end{aligned} \tag{A.22}$$

Integrating both sides of (A.22) from 0 to τ, dividing by τ and using (A.16) and (A.17) gives

$$g_0(y(\epsilon))-g_0(\bar{y})=\frac{1}{\tau}\int_0^\tau H\big(x(t,\epsilon),u(t,\epsilon),y(\epsilon),\bar{\lambda},\bar{\mu},\bar{\alpha}\big)-H\big(\bar{x},\bar{u},\bar{y},\bar{\lambda},\bar{\mu},\bar{\alpha}\big)\,dt \tag{A.23}$$

which because of (3.3) and the C^2 differentiability implies

$$\begin{aligned}g_0(y(\epsilon))-g_0(\bar{y})&=\frac{\epsilon^2}{2\tau}\int_0^\tau \hat{x}(t)'\overline{H}_{xx}\hat{x}(t)+2\hat{x}(t)'\overline{H}_{xu}\hat{u}(t)\\&\quad+\hat{u}(t)'\overline{H}_{uu}\hat{u}(t)\,dt+o(\epsilon^2)\\&=\epsilon^2\eta^*\pi(\omega)\eta+o(\epsilon^2).\end{aligned} \tag{A.24}$$

By taking $\epsilon>0$ sufficiently small, it follows from (5.3) that OPC is locally proper at $(\bar{x},\bar{u})$.

Appendix B
Proof of Theorem 6.1

We begin by stating and proving a series of lemmas. Consider

$$\dot{x}(t)=\bar{A}x(t)+f(t), \qquad \text{a.a. } t\in[0,\tau],\ x(0)=x(\tau) \tag{B.1}$$

where $x(t), f(t)\in L_\infty^n[0,\tau]$, $\tau>0$ and $\bar{A}$ is a constant $n\times n$ matrix.

Lemma B.1: If the matrix $I_n-e^{\bar{A}\tau}$ is nonsingular, (B.1) has a unique solution which is given by

$$x(t)=\int_0^\tau G(t,\sigma)f(\sigma)\,d\sigma \tag{B.2}$$

where

$$\begin{aligned}G(t,\sigma)&=e^{\bar{A}t}\big(I_n-e^{\bar{A}\tau}\big)^{-1}e^{-\bar{A}\sigma}, &&0\leqslant\sigma<t,\\&=e^{\bar{A}t}\big(I_n-e^{\bar{A}\tau}\big)^{-1}e^{\bar{A}\tau}e^{-\bar{A}\sigma}, &&t\leqslant\sigma\leqslant\tau.\end{aligned}$$

Proof: Express the solution of (B.1) in terms of $x(0)$ and $f(t)$ by means of the variation of parameters formula. Set $x(\tau)=x(0)$ and solve for $x(0)$.

Our principle concern is the system

$$\dot{x}(t)=f(x(t),u(t)), \qquad \text{a.a. } t\in[0,\tau],\ x(0)=x(\tau) \tag{B.3}$$

where the assumptions and notations of Sections II and III apply. Here, $\bar{x}\in R^n$ and $\bar{u}\in R^m$ are assumed only to satisfy $f(\bar{x},\bar{u})=0$. Let $z(t)=x(t)-\bar{x}$, $\delta u(t)=u(t)-\bar{u}$. The solution of (B.3) is equivalent to the solution of

$$\dot{z}(t)=f(z(t)+\bar{x},\delta u(t)+\bar{u}), \qquad \text{a.a. } t\in[0,\tau],\quad z(0)=z(\tau). \tag{B.4}$$

Because $t\in[0,\tau]$ instead of $[0,T]$, we work with $L_\infty^m[0,\tau]$ and $L_\infty^n[0,\tau]$. Thus, $\|\nu\|_\infty=\mathrm{ess\,sup}_{t\in[0,\tau]}|\nu(t)|$. The $L_1^m[0,\tau]$, $L_1^n[0,\tau]$, $L_2^m[0,\tau]$, $L_2^n[0,\tau]$ norms are defined in the usual way: $\|\nu\|_1=\int_0^\tau|\nu(t)|\,dt$, $\|\nu\|_2=(\int_0^\tau|\nu(t)|^2dt)^{1/2}$. For brevity the dependence of the norms on m, n and τ is not explicitly designated. Since $0<\tau\leqslant T$ it follows from the familiar relationships of norms that

$$\|\nu\|_1\leqslant\sqrt{T}\,\|\nu\|_2, \tag{B.5}$$

$$\|\nu\|_2 \leqslant \sqrt{T}\,\|\nu\|_\infty. \tag{B.6}$$

We now consider the existence of z and bounds for $\|z\|_\infty$.

Lemma B.2: Assume $I_n - e^{\bar{A}\tau}$ is nonsingular. Then there exist $K_1>0$ and $\epsilon>0$ such that for all $\delta u \in L^m_\infty$, $\|\delta u\|_\infty \leqslant \epsilon$, the system (B.4) has a unique solution $z \in \mathfrak{X} \subset L^n_\infty$ which satisfies

$$\|z\|_\infty < K_1 \|\delta u\|_\infty. \tag{B.7}$$

Proof: Equation (B.4) can be written

$$\dot{z} = \bar{A}z + \bar{B}\delta u + \Gamma(z,\delta u), \qquad z(0)=z(\tau) \tag{B.8}$$

where Γ is C^2 and $\Gamma(0,0)=0$, $\Gamma_x(0,0)=0$, $\Gamma_u(0,0)=0$. By applying Lemma B.1 it follows that z is characterized by

$$z(t) = \int_0^\tau G(t,\sigma)\big[\bar{B}\delta u(\sigma) + \Gamma(z(\sigma),u(\sigma))\big]\,d\sigma \tag{B.9}$$

which has the form

$$z = \mathfrak{L}(\delta u) + \mathfrak{F}(z,\delta u) \tag{B.10}$$

where $\mathfrak{L}: L^m_\infty \to L^n_\infty$ is bounded and linear and $\mathfrak{F}: L^n_\infty \times L^m_\infty \to L^n_\infty$. Using the properties of Γ it is easy to see that there exists an $\epsilon_1>0$ such that $\|\delta u\|_\infty < \epsilon_1$, $\|z_i\|_\infty < \epsilon_1$ imply

$$\|\mathfrak{F}(z_1,\delta u)\| < \frac{1}{2}\|z_1\| + \frac{1}{2}\|\delta u\| \tag{B.11}$$

$$\|\mathfrak{F}(z_1,\delta u) - \mathfrak{F}(z_2,\delta u)\| < \frac{1}{2}\|z_1 - z_2\|. \tag{B.12}$$

Choose $\bar{K}_2>1$ so that $\|\mathfrak{L}(\delta u)\|_\infty < \bar{K}_2\|\delta u\|_\infty$ and impose the further requirement that $\|\delta u\|_\infty \leqslant \epsilon = \frac{1}{4}\bar{K}_2^{-1}\epsilon_1$. Then for fixed δu, $\mathfrak{L}(\delta u) + \mathfrak{F}(z_1,\delta u)$ is a contraction in z for all $z \in \{z : \|z\|_\infty \leqslant \epsilon_1\}$. Moreover, for fixed z, $\mathfrak{L}(\delta u) + \mathfrak{F}(z,\delta u)$ is continuous in δu. Thus, by [11, Theorem 3.2, p. 7], (B.10) has a unique solution. From (B.9) it may be deduced that z is absolutely continuous. From (B.10) and the above bounds,

$$\|z\|_\infty < \bar{K}_2\|\delta u\|_\infty + \frac{1}{2}\|\delta u\|_\infty + \frac{1}{2}\|z\|_\infty \tag{B.13}$$

which proves (B.7).

Lemma B.3: If ϵ in Lemma B.2 is chosen sufficiently small, there exists $K_2>0$ such that

$$\|z\|_\infty \leqslant K_2 \|\delta u\|_1. \tag{B.14}$$

Proof: Let $\alpha'\bar{f}_{xx}\alpha$, $\alpha \in R^n$, denote the vector whose ith component is $\alpha' f_{ixx}(\bar{x},\bar{u})\alpha$. Similarly, define $\alpha'\bar{f}_{xu}\beta$, $\beta'\bar{f}_{uu}\beta$ where $\beta \in R^m$. Then, because of the properties of Γ,

$$z(t) = \int_0^\tau G(t,\sigma)\Big[\bar{B}\delta u(\sigma) + \frac{1}{2}z(\sigma)'\bar{f}_{xx}z(\sigma) + z(\sigma)'\bar{f}_{xu}\delta u(\sigma) + \frac{1}{2}\delta u(\sigma)'\bar{f}_{uu}\delta u(\sigma) + F(z(\sigma),\delta u(\sigma))\Big]\,d\sigma \tag{B.15}$$

where there exists $\epsilon_2>0$ such that $|z|+|u|<\epsilon_2$ implies

$$|F(z,u)| < |z|^2 + |u|^2. \tag{B.16}$$

Thus, there are $\hat{K}_1, \hat{K}_2$ such that

$$|z(t)| \leqslant \int_0^\tau \big(\hat{K}_1|\delta u(\sigma)| + \hat{K}_2|z(\sigma)|\big)\,d\sigma \tag{B.17}$$

where, because of (B.7), $\hat{K}_2$ can be made arbitrarily small by the choice of ϵ. For $\hat{K}_2\tau < 1/2$, it follows that

$$\|z\|_\infty \leqslant \hat{K}_1\|\delta u\|_1 + \frac{1}{2}\|z\|_\infty \tag{B.18}$$

which yields (B.14).

By Lemma B.1, the "linearized equation" corresponding to (B.3),

$$\delta\dot{x} = \bar{A}\delta x + \bar{B}\delta u, \quad \delta x(0) = \delta x(\tau), \tag{B.19}$$

has a unique solution $\delta x \in L^n_\infty$ for all $\delta u \in L^m_\infty$. Define

$$w = x - \bar{x} - \delta x = z - \delta x \tag{B.20}$$

where w is the "error" produced by using the linearized equation.

Lemma B.4: If $I_n - e^{\bar{A}\tau}$ is nonsingular there exist $K_3>0$ and $\epsilon>0$ such that for $\|\delta u\|_\infty \leqslant \epsilon$

$$\|w\|_\infty \leqslant K_3\|\delta u\|_2^2. \tag{B.21}$$

Proof: From (B.3) and (B.19) we have

$$\begin{aligned}\dot{w} &= f(\bar{x}+\delta x + w, \bar{u}+\delta u) - f(\bar{x},\bar{u}) - \bar{A}\delta x - \bar{B}\delta u\\ &= \bar{A}w + \frac{1}{2}(\delta x + w)'\bar{f}_{xx}(\delta x + w) + (\delta x + w)'\bar{f}_{xu}\delta u + \frac{1}{2}\delta u'\bar{f}_{uu}\delta u\\ &\quad + F(\delta x + w, \delta u), \qquad w(0)=w(\tau)\end{aligned} \tag{B.22}$$

where F has the property (B.16). Applying Lemma B.1 to (B.22) with $z = \delta x + w$ gives

$$w(t) = \int_0^\tau G(t,\sigma)\Big(\frac{1}{2}z(\sigma)'\bar{f}_{xx}z(\sigma) + z(\sigma)'\bar{f}_{xu}\delta u(\sigma) + \frac{1}{2}\delta u(\sigma)'\bar{f}_{uu}\delta u(\sigma) + F(z(\sigma),\delta u(\sigma))\Big)\,d\sigma. \tag{B.23}$$

By Lemma B.2, ϵ may be selected so that $|z(t)| + |\delta u(t)| < \epsilon_2$. Thus, Lemma B.3 shows there are $\hat{K}_3$ and $\hat{K}_4$ such that

$$\|w\|_\infty \leqslant \hat{K}_3\|\delta u\|_1^2 + \hat{K}_4\|\delta u\|_2^2. \tag{B.24}$$

Bound (B.21) follows from (B.5).

Clearly, it is possible to write

$$y=\frac{1}{\tau}\int_0^\tau \tilde{f}(x(\sigma),u(\sigma))\,d\sigma$$
$$=\frac{1}{\tau}\int_0^\tau \tilde{f}(\bar{x}+\delta x(\sigma)+w(\sigma),\bar{u}+\delta u(\sigma))\,d\sigma$$
$$=\bar{y}+\delta y+\tilde{w} \quad (B.25)$$

where

$$\delta y=\frac{1}{\tau}\int_0^\tau (\bar{C}\delta x+\bar{D}\delta u)\,d\sigma,$$
$$\tilde{w}=\frac{1}{\tau}\int_0^\tau \bar{C}w\,d\sigma+\frac{1}{\tau}\int_0^\tau\Big(\frac{1}{2}z'\bar{\tilde{f}}_{xx}z+z'\bar{\tilde{f}}_{xu}\delta u$$
$$+\frac{1}{2}\delta u'\bar{\tilde{f}}_{uu}\delta u+\tilde{F}(z,\delta u)\Big)\,d\sigma \quad (B.26)$$

and there exists an $\epsilon_3>0$ such that $|z|+|\delta u|<\epsilon_3$ implies $\tilde{F}$ satisfies a condition of the form (B.16). This is sufficient to show there exists $K_4>0$ such that

$$|\tilde{w}|\leq K_4\|\delta u\|_2^2,\ \|\delta u\|_\infty\leq\epsilon \quad (B.27)$$

where $\epsilon>0$ is sufficiently small.

Let ΔJ be defined by

$$\Delta J=\frac{1}{\tau}\int_0^\tau H\big(x(\sigma),u(\sigma),y,\bar{\lambda},\bar{\mu},\bar{\alpha}\big)$$
$$-H\big(\bar{x},\bar{u},\bar{y},\bar{\lambda},\bar{\mu},\bar{\alpha}\big)\,d\sigma \quad (B.28)$$

where (3.1) and (3.3) apply. Using $x(t)=\bar{x}+\delta x(t)+w(t)$ and $u(t)=\bar{u}+\delta u(t)$

$$\Delta J=\delta^2 J+\hat{R} \quad (B.29)$$

where

$$\delta^2 J=\frac{1}{2\tau}\int_0^\tau\begin{bmatrix}\delta x\\ \delta u\end{bmatrix}'\begin{bmatrix}\bar{H}_{xx} & \bar{H}_{xu}\\ \bar{H}_{ux} & \bar{H}_{uu}\end{bmatrix}\begin{bmatrix}\delta x\\ \delta u\end{bmatrix}d\sigma$$
$$+\frac{1}{2}\delta y'\bar{H}_{yy}\delta y \quad (B.30)$$

and $\hat{R}$ can be made small in the following sense.

Lemma B.5: Assume $I_n-e^{\bar{A}\tau}$ is nonsingular. Given any $\eta>0$ there exists $\epsilon>0$ such that

$$|\hat{R}|\leq\eta\|\delta u\|_2^2 \quad (B.31)$$

for all $\delta u\in L_\infty^m$ such that $\|\delta u\|_\infty\leq\epsilon$.

Proof: From (B.28) and (3.3)

$$\hat{R}=\frac{1}{\tau}\int_0^\tau\Big(\delta x'\bar{H}_{xx}w+\frac{1}{2}w'\bar{H}_{xx}w+\delta u'\bar{H}_{ux}w\Big)\,d\sigma$$
$$+\delta y'\bar{H}_{yy}\tilde{w}+\frac{1}{2}\tilde{w}'\bar{H}_{yy}\tilde{w}$$
$$+\int_0^\tau \bar{R}(z,\delta u,\delta y+\tilde{w})\,d\sigma \quad (B.33)$$

where for all $\bar{\eta}>0$ there exists an $\bar{\epsilon}>0$ such that $|z|+|u|+|y|\leq\bar{\epsilon}$ implies $|\bar{R}(z,u,y)|\leq\bar{\eta}(|z|^2+|u|^2+|y|^2)$. By (B.2), (B.19), (B.26), and (B.6) there exists $K_0>0$ such that

$$\|\delta x\|_\infty,|\delta y|\leq K_0\|\delta u\|_1\leq K_0\sqrt{T}\,\|\delta u\|_\infty. \quad (B.34)$$

From (B.5), (B.6), (B.27), and (B.34) there exists $\hat{K}_5$ such that

$$|\delta y+\tilde{w}|\leq\hat{K}_5\|\delta u\|_2\leq\hat{K}_5\sqrt{T}\,\|\delta u\|_\infty. \quad (B.35)$$

Thus, it is possible to pick ϵ so that $|z(t)|+|u(t)|+|\delta y+\tilde{w}|<\bar{\epsilon}$ for all $t\in[0,\tau]$. Using (B.27), (B.33), (B.34), and (B.35) it follows that there are $\hat{K}_6$, $\hat{K}_7$, $\hat{K}_8$, $\hat{K}_9$ such that

$$|\hat{R}|\leq\hat{K}_6\|\delta u\|_\infty\|\delta u\|_2^2+\hat{K}_7\|\delta u\|_2^4$$
$$+\bar{\eta}\big(\hat{K}_8\|\delta u\|_1^2+\hat{K}_9\|\delta u\|_2^2\big). \quad (B.36)$$

From (B.5) and (B.6)

$$|\hat{R}|\leq\big(\hat{K}_6\|\delta u\|_\infty+\hat{K}_7 T\|\delta u\|_\infty+\bar{\eta}\hat{K}_8 T+\bar{\eta}\hat{K}_9\big)\|\delta u\|_2^2 \quad (B.37)$$

and the method for choosing ϵ so that (B.31) holds is clear.

Note that i) and ii) of the theorem statement imply $I_n-e^{\bar{A}\tau}$ is nonsingular for all $\tau\in(0,T]$. From this and the proofs of the lemmas it can be seen that the determination of K_1, K_2, K_3 and the choice of ϵ in Lemma B.5 can be made independently of τ.

We introduce a special notation for the average values of δu and δx:

$$U=\frac{1}{\tau}\int_0^\tau \delta u(t)\,dt,\quad X=\frac{1}{\tau}\int_0^\tau \delta x(t)\,dt. \quad (B.38)$$

Define

$$\delta U(t)=\delta u(t)-U,\quad \delta X(t)=\delta x(t)-X. \quad (B.39)$$

Then

$$\int_0^\tau \delta U(t)\,dt=0,\quad \int_0^\tau \delta X(t)\,dt=0. \quad (B.40)$$

Because of this

$$\int_0^\tau|\delta u(t)|^2dt=\int_0^\tau|U|^2dt+\int_0^\tau|\delta U(t)|^2dt\geq\tau|U|^2 \quad (B.41)$$

and

$$|U|\leq(\tau)^{-1/2}\|\delta u\|_2. \quad (B.42)$$

Finally, from (B.19), (B.26) and the nonsingularity of $\bar{A}$

$$X=\bar{A}^{-1}\bar{B}U, \quad (B.43)$$
$$\delta y=(-\bar{C}\bar{A}^{-1}\bar{B}+\bar{D})U. \quad (B.44)$$

Suppose the conclusion of Theorem 6.1 does not follow from the hypotheses. Then there exists a sequence of triples, $\{(x_i,u_i,\tau_i)\}_{i=1}^\infty$, that satisfy (2.2),

$$\|x_i-\bar{x}\|_\infty+\|u_i-\bar{u}\|_\infty>0, \quad (B.45)$$
$$\lim_{i\to\infty}(\|x_i-\bar{x}\|_\infty+\|u_i-\bar{u}\|_\infty)=0, \quad (B.46)$$
$$\tau_i\in(0,T], \quad (B.47)$$

$$g_0(y_i)-g_0(\bar{y})\leq 0. \tag{B.48}$$

Here y_i is given by y in (2.2) with $x=x_i$ and $u=u_i$. We complete the proof of the theorem by showing this must lead to a contradiction. Subscripts i denote obvious changes of previous notation when $u=u_i$, $x=x_i$ and $\tau=\tau_i$.

Without loss of generality

$$\tau_i\in\left[\frac{1}{2}T,T\right]. \tag{B.49}$$

To see this let n_i be the integer satisfying $T(n_i+1)^{-1}<\tau_i\leq Tn_i^{-1}$ where $\tau_i\in(0,T]$. By replacing τ_i with $\hat{\tau}_i=n_i\tau_i$ and u_i by its periodic extension $\hat{u}_i, \hat{u}_i(\hat{n}\tau_i+t)=u_i(t)$, $t\in[0,\tau_i]$, $\hat{n}=0,\cdots,n_i-1$, we see y_i is unchanged. Also note that $\|\delta u_i\|_\infty>0$. Otherwise, Lemma B.1 shows (B.45) is not satisfied. This implies $\|\delta u_i\|_2>0$ and by (B.42)

$$\|\delta u_i\|_2^{-1}|U_i|\leq(\tau_i)^{-1/2}\leq(T)^{-1/2}\sqrt{2}\,. \tag{B.50}$$

Thus, there is a subsequence of $\{u_i\}_{i=1}^{\infty}$ such that

$$\|\delta u_i\|_2^{-1}U_i\rightarrow\hat{U}. \tag{B.51}$$

Hereafter, without introducing new notation, we limit our attention to this subsequence.

From hypothesis iii) and (B.28)

$$g_0(y_i)-g_0(\bar{y})=\Delta J_i-\sum_{j\in I^*}\bar{\alpha}_j g_j(y_i) \tag{B.52}$$

where $I^*=\{j:j<0,\ \bar{\alpha}_j>0\}$. By Lemma B.5, $\|\delta u_i\|_2^{-1}\Delta J_i\rightarrow 0$. Thus, (B.48), (B.44), and (B.51) imply

$$\lim_{i\to\infty}\|\delta u_i\|_2^{-1}\Big(-\sum_{j\in I^*}\bar{\alpha}_j g_j(y_i)\Big)=-\sum_{j\in I^*}\bar{\alpha}_j g_{jy}(\bar{y})\cdot(-\bar{C}\bar{A}^{-1}\bar{B}+\bar{D})\hat{U}\leq 0. \tag{B.53}$$

From $g_j(y_i)\leq 0$, $j\in\bar{I}$, $j<0$, it follows by an identical argument that

$$0\geq g_{jy}(\bar{y})(-\bar{C}\bar{A}^{-1}\bar{B}+\bar{D})\hat{U},\qquad j\in\bar{I},\ j<0. \tag{B.54}$$

Combining (B.53) and (B.54) and recalling that $\bar{\alpha}_j>0$, $j\in I^*$, we see that

$$g_{jy}(\bar{y})(-\bar{C}\bar{A}^{-1}\bar{B}+\bar{D})\hat{U}=0 \tag{B.55}$$

for $j\in I^*$. From $g_j(y_i)=0$, $j=1,\cdots,k$ (B.55) holds for $j\in\bar{I}$.

All of this may be summarized by saying $\hat{U}=u$ satisfies (6.2). Thus, by hypothesis iv), $\hat{U}=u$ satisfies (6.1).

Since $\bar{\alpha}_j g_j(y_i)\leq 0$ for $j\in I^*$, it follows from (B.52), (B.48), (B.29), and (B.31) that

$$\delta^2 J_i+\eta\|\delta u_i\|_2^2\leq 0. \tag{B.56}$$

Because $\|\delta u_i\|_\infty\rightarrow 0$ Lemma B.5 implies (B.56) holds for any $\eta>0$ provided i is sufficiently large. If (B.39) is substituted into (B.30), X and δy are given by (B.43) and (B.44), and (B.40) is observed,

$$\delta^2 J_i=\frac{1}{2\tau_i}\int_0^{\tau_i}\begin{bmatrix}\delta X_i\\ \delta U_i\end{bmatrix}'\begin{bmatrix}\bar{H}_{xx} & \bar{H}_{xu}\\ \bar{H}_{ux} & \bar{H}_{uu}\end{bmatrix}\begin{bmatrix}\delta X_i\\ \delta U_i\end{bmatrix}d\sigma+\frac{1}{2}U_i'WU_i \tag{B.57}$$

where

$$W=\pi(0)+(\bar{C}G(0)+\bar{D})'\bar{H}_{yy}(\bar{C}G(0)+\bar{D}). \tag{B.58}$$

Since $\delta U_i\in L_2^m[0,\tau_i]$

$$\left\|\delta U_i(t)-\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty}\xi_i^k e^{j(2\pi k/\tau_i)t}\right\|_2=0 \tag{B.59}$$

where $\xi_i^k\in\mathcal{C}^m$ and the $k=0$ term is missing because of (B.40). Furthermore, from (B.19) δX_i and δU_i satisfy

$$\delta\dot{X}_i=\bar{A}\delta X_i+\bar{B}\delta U_i,\quad \delta X_i(0)=\delta X_i(\tau_i). \tag{B.60}$$

Using (B.59) and (B.60), δX_i can be expressed in a Fourier series in terms of the ξ_i^k. By the L_2 convergence of these series it follows that

$$\frac{1}{2\tau_i}\int_0^{\tau_i}\begin{bmatrix}\delta X_i\\ \delta U_i\end{bmatrix}'\begin{bmatrix}\bar{H}_{xx} & \bar{H}_{xu}\\ \bar{H}_{ux} & \bar{H}_{uu}\end{bmatrix}\begin{bmatrix}\delta X_i\\ \delta U_i\end{bmatrix}d\sigma=\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty}\xi_i^{k*}\pi\left(\frac{2\pi k}{\tau_i}\right)\xi_i^k. \tag{B.61}$$

From (B.51) we can write

$$U_i=\|\delta u_i\|_2(\hat{U}+Q(i)) \tag{B.62}$$

where

$$\lim_{i\to\infty}Q(i)=0. \tag{B.63}$$

Using (B.62) and assumption v) of Theorem 6.1, (B.56) implies

$$0\geq\gamma\rho_i+\|\delta u_i\|_2^2(\hat{U}'W\hat{U}+\hat{U}'WQ(i)+Q(i)'WQ(i)+\eta) \tag{B.64}$$

where

$$\rho_i=\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty}|\xi_i^k|^2=\|\delta U_i\|_2^2. \tag{B.65}$$

First suppose $\hat{U}\neq 0$. Then it is possible to choose $\epsilon>0$ such that (B.64) gives

$$0\geq\hat{U}'W\hat{U}\|\delta u_i\|_2^2 \tag{B.66}$$

which by (6.1) is impossible.

Next consider the case $\hat{U}=0$. Then (B.64) becomes

$$0\geq\frac{\gamma\|\delta U_i\|_2^2}{\|\delta u_i\|_2^2}+Q(i)'WQ(i)+\eta. \tag{B.67}$$

From (B.41) and (B.62) we have

$$\|\delta u_i\|_2^2 = \|\delta U_i\|_2^2 + \tau_i |U_i|^2$$
$$= \|\delta U_i\|_2^2 + \tau_i \|\delta u_i\|_2^2 |Q(i)|^2, \quad \text{(B.68)}$$

which for i sufficiently large gives

$$\|\delta u_i\|_2^2 \leq 2\|\delta U_i\|_2^2. \quad \text{(B.69)}$$

Combining (B.67) and (B.69) yields

$$0 \geq \frac{\gamma}{2} + Q(i)' W Q(i) + \eta \quad \text{(B.70)}$$

which for i sufficiently large gives $\gamma/4 \leq 0$. Thus, (B.48) is false and the theorem is proved.

References

[1] J. E. Bailey, "Periodic operation of chemical reactors: A review," *Chem. Eng. Commun.*, vol. 1, pp. 111–124, 1973.
[2] S. Bittanti, G. Fronza, and G. Guardabassi, "Periodic control: A frequency domain approach," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 33–38, 1973.
[3] S. Bittanti, A. Locatelli, and C. Maffezzoni, "Second-variation methods in periodic optimization," *J. Opt. Theory Appl.*, vol. 14, pp. 31–49, 1974.
[4] C. H. Edwards, *Advanced Calculus of Several Variables*. New York: Academic, 1973.
[5] A. Fiacco and G. McCormick, *Nonlinear Programming*. New York: Wiley, 1968.
[6] E. G. Gilbert, "Functional expansions for the response of nonlinear differential systems," *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 909–921, 1977.
[7] ——, "Optimal periodic control: A general theory of necessary conditions," *SIAM J. Contr. Opt.*, vol. 15, pp. 717–746, 1977.
[8] ——, "Optimal periodic control: A solution set theory of necessary and sufficient conditions," *Preprint, IFAC, 7th Triennial World Congr.*, Helsinki, 1978.
[9] G. Guardabassi, "Optimal steady state versus periodic control," *Richerche Di Automatica*, vol. 2, pp. 240–252, 1971.
[10] G. Guardabassi, A. Locatelli, and S. Rinaldi, "Status of periodic optimization of dynamical systems," *J. Opt. Theory. Appl.*, vol. 12, pp. 1–20, 1974.
[11] J. K. Hale, *Ordinary Differential Equations*. New York: Wiley, 1969.
[12] M. R. Hestenes, *Calculus of Variations and Optimal Control Theory*. New York: Wiley, 1966.
[13] R. Marzollo, *Periodic Optimization*, vols. 1, 2. New York: Springer-Verlag, 1972.
[14] G. P. McCormick, "Optimality criteria in nonlinear programming," in *SIAM-AMS Proc.*, vol. 9, 1976, pp. 27–38.
[15] S. K. Ng and W. L. Chan, "Normality and proper performance improvement in periodic control," *J. Opt. Theory Appl.*, vol. 29, pp. 215–229, 1979.

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