Proceedings of the 36th
Conference on Decision & Control
San Diego, California USA • December 1997

Adaptive Disturbance Rejection Using ARMARKOV System Representations

Ravinder Venugopal and Dennis S. Bernstein [1]
Department of Aerospace Engineering
The University of Michigan
Ann Arbor, MI 48109-2118
{ravinder, dsbaero}@umich.edu

Abstract

An adaptive disturbance rejection algorithm is developed for the standard control problem. The MIMO system and controller are represented as ARMARKOV/Toeplitz models, and the parameter matrix of the compensator is updated on-line by means of a gradient algorithm. The algorithm requires minimal knowledge of the plant, specifically, the numerator of the ARMARKOV transfer function from control to performance is required. No knowledge about the spectrum of the disturbance is needed. Experimental results demonstrating tonal and broadband disturbance rejection in an acoustic duct are presented.

1 Introduction

An important objective of control system design is to minimize the effects of external disturbance signals. For applications such as active noise and vibration control, it is the primary focus. In cases where the system is time varying or difficult to identify, adaptive methods such as the feedforward LMS and RLMS algorithms are useful [1] - [5]. However, feedforward-type algorithms neglect the effect of the feedback path from control to measurement thus leading to poor performance and instability [6]. To remedy this problem, robust variations of the classical LMS algorithm have been proposed; see, for example [7].

Predictive models, which involve the Markov parameters of the system, are used in predictive control of systems with time delays [8] pp. 169-179, [9] pp. 331-365, [10], [11]. Markov-parameter-based representations of systems also provide a framework for direct controller synthesis based on input-output data [12]. In addition, predictive control algorithms such as the long range generalized predictive algorithm [9] pp. 353-362, [10] use windows of data. Predictive models are also used in [13, 14, 15, 16] for model identification within recursive and batch least squares techniques. In these works predictive models are termed ARMARKOV models to emphasize the presence of Markov parameters in ARMA-type models. In [15] it is shown that ARMARKOV models can be used to estimate Markov parameters in the presence of persistent, but not necessarily white, input signals.

In the present paper we provide a mathematical analysis of the adaptive disturbance rejection controller of [17]. This approach uses ARMARKOV plant and controller models, and is distinct from predictive control techniques due to the fact that the adaptation mechanism is based upon past data rather than future predicted error. A gradient algorithm that minimizes a performance cost function is used to update the entries of the controller parameter matrix. The update law uses an adaptive step-size, involving the past data and plant Markov parameters.

2 Standard Problem Representation of Disturbance Rejection

Consider the linear discrete-time two vector-input, two vector-output (TITO) system. The *disturbance* $w(k)$, the *control* $u(k)$, the *measurement* $y(k)$ and the *performance* $z(k)$ are in $\mathcal{R}^{m_w}, \mathcal{R}^{m_u}, \mathcal{R}^{l_y}$ and $\mathcal{R}^{l_z}$, respectively. The system can be written in state space form as

$$x(k+1) = Ax(k) + Bu(k) + D_1 w(k), \quad (1)$$
$$z(k) = E_1 x(k) + E_2 u(k) + E_0 w(k), \quad (2)$$
$$y(k) = Cx(k) + Du(k) + D_2 w(k), \quad (3)$$

or equivalently in terms of transfer matrices

$$z = G_{zw} w + G_{zu} u, \quad (4)$$
$$y = G_{yw} w + G_{yu} u. \quad (5)$$

The controller G_c generates the control signal $u(k)$ based on the measurement $y(k)$, that is,

$$u = G_c y. \quad (6)$$

The objective of the standard problem [18] is to determine a controller G_c that produces a control signal

[1]This research was supported in part by the Air Force Office of Scientific Research under grant F49620-95-1-0019 and the University of Michigan Office of the Vice President for Research.

0-7803-3970-8/97 $10.00 © 1997 IEEE

$u(k)$ based on the measurement $y(k)$ such that a performance measure involving $z(k)$ is minimized. In classical fixed-gain H_2 and H_∞ optimal control theory, the performance $z(k)$ is not required to be measured, but rather G_{zw} and G_{yw} are used analytically for off-line controller design. Fixed-gain controller design methods for disturbance rejection also require knowledge of all four transfer matrices, namely, the *primary path* G_{zw}, the *secondary path* G_{zu}, the *reference path* G_{yw} and the *feedback path* G_{yu}, as well as the spectrum of the disturbance $w(k)$. This terminology is standard in the noise control literature [2].

Unlike fixed-gain controller design methods, adaptive control techniques require on-line measurement of $z(k)$ for use in adaptation. If $z(k)$ is measured and used for control, we say that the *performance assumption* is satisfied. However, in contrast to fixed-gain methods, adaptive methods [1]-[4] often require that only the secondary path transfer matrix G_{zu} be known. Other adaptive methods [7] identify G_{zu} on-line but require additional actuators and sensors.

3 ARMARKOV/Toeplitz Model of TITO Systems

We now derive the ARMARKOV representation of the TITO system described in Section 2. First, the ARMARKOV form of (1) - (3) is

$$z(k) = \sum_{j=1}^{n} -\alpha_j z(k-\mu-j+1) + \sum_{j=1}^{\mu} H_{zw,j-2} w(k-j+1) + \sum_{j=1}^{n} \mathcal{B}_{zw,j} w(k-\mu-j+1) + \sum_{j=1}^{\mu} H_{zu,j-2} u(k-j+1) + \sum_{j=1}^{n} \mathcal{B}_{zu,j} u(k-\mu-j+1), \quad (7)$$

$$y(k) = \sum_{j=1}^{n} -\alpha_j y(k-\mu-j+1) + \sum_{j=1}^{\mu} H_{yw,j-2} w(k-j+1) + \sum_{j=1}^{n} \mathcal{B}_{yw,j} w(k-\mu-j+1) + \sum_{j=1}^{\mu} H_{yu,j-2} u(k-j+1) + \sum_{j=1}^{n} \mathcal{B}_{yu,j} u(k-\mu-j+1), \quad (8)$$

where $\alpha_j \in \mathcal{R}$, $\mathcal{B}_{zw,j}, H_{zw,j} \in \mathcal{R}^{l_z \times m_w}$, $\mathcal{B}_{zu,j}, H_{zu,j} \in \mathcal{R}^{l_z \times m_u}$, $\mathcal{B}_{yw,j}, H_{yw,j} \in \mathcal{R}^{l_y \times m_w}$ and $\mathcal{B}_{yu,j}, H_{yu,j} \in \mathcal{R}^{l_y \times m_u}$. Note that the system order n is the same in (7) and (8). Next, define the *extended performance vector* $Z(k)$ and the *extended control vector* $U(k)$ by

$$Z(k) \triangleq [z(k) \ \cdots \ z(k-p+1)]^{\mathrm{T}}, \quad (9)$$

$$U(k) \triangleq [u(k) \ \cdots \ u(k-\mu-p-n+2)]^{\mathrm{T}} \quad (10)$$

the ARMARKOV regressor vector $\Phi_{zw}(k)$ by

$$\Phi_{zw}(k) \triangleq [z(k-\mu) \ \cdots \ z(k-\mu-p-n+2) \ \ w(k) \ \cdots \ w(k-\mu-p-n+2)]^{\mathrm{T}}, \quad (11)$$

the block-Toeplitz ARMARKOV weight matrix W_{zw} by

$$W_{zw} \triangleq \left[\begin{array}{cccccc} -\alpha_1 I_{l_z} & \cdots & -\alpha_n I_{l_z} & 0_{l_z} & \cdots & 0_{l_z} \\ 0_{l_z} & \ddots & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & 0_{l_z} \\ 0_{l_z} & \cdots & 0_{l_z} & -\alpha_1 I_{l_z} & \cdots & -\alpha_n I_{l_z} \end{array}\right. \quad (12)$$

$$\left.\begin{array}{ccccccccc} H_{zw,-1} & \cdots & H_{zw,\mu-2} & \mathcal{B}_{zw,1} & \cdots & \mathcal{B}_{zw,n} & 0_{l_z \times m_w} & \cdots & 0_{l_z \times m_w} \\ 0_{l_z \times m_w} & \ddots & & \ddots & \ddots & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \ddots & & \ddots & 0_{l_z \times m_w} \\ 0_{l_z \times m_w} & \cdots & 0_{l_z \times m_w} & H_{zw,-1} & \cdots & H_{zw,\mu-2} & \mathcal{B}_{zw,1} & \cdots & \mathcal{B}_{zw,n} \end{array}\right],$$

and the ARMARKOV control matrix $B_{zu} \triangleq$

$$\left[\begin{array}{ccccccccc} H_{zu,-1} & \cdots & H_{zu,\mu-2} & \mathcal{B}_{zu,1} & \cdots & \mathcal{B}_{zu,1} & 0_{l_z \times m_u} & \cdots & 0_{l_z \times m_u} \\ 0_{l_z \times m_u} & \ddots & & \ddots & \ddots & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \ddots & & \ddots & 0_{l_z \times m_u} \\ 0_{l_z \times m_u} & \cdots & 0_{l_z \times m_u} & H_{zu,-1} & \cdots & H_{zu,\mu-2} & \mathcal{B}_{zu,1} & \cdots & \mathcal{B}_{zu,n} \end{array}\right], \quad (13)$$

Then (7) can be written in the form

$$Z(k) = W_{zw}\Phi_{zw}(k) + B_{zu}U(k). \quad (14)$$

Similarly, (8) can be written as

$$Y(k) = W_{yw}\Phi_{yw}(k) + B_{yu}U(k), \quad (15)$$

where $Y(k)$, Φ_{yw}, W_{yw} and B_{yw} are defined analogous to (10), (11), (12) and (13) respectively to yield the ARMARKOV weight matrix representation of (1) - (3). The length of the vector $U(k)$, $p_c = m_u(\mu+n+p-1)$.

4 Adaptive Disturbance Rejection Algorithm

In this section we formulate an adaptive disturbance rejection feedback algorithm for the TITO system represented by (14) and (15). We use a strictly proper controller in ARMARKOV form of order n_c with μ_c Markov parameters, so that, the control $u(k)$ is given by

$$u(k) = \sum_{j=1}^{n_c} -\alpha_{c,j}(k) u(k-\mu-j+1) + \sum_{j=1}^{\mu_c-1} H_{c,j-1}(k) y(k-j+1) + \sum_{j=1}^{n_c} \mathcal{B}_{c,j}(k) y(k-\mu-j+1), \quad (16)$$

where $H_{c,j} \in \mathcal{R}^{m_u \times l_y}$ are the Markov parameters of the controller. Next, define the *controller parameter block vector*

$$\theta(k) \triangleq [-\alpha_{c,1}(k)I_{m_u} \quad \cdots \quad -\alpha_{c,n_c}(k)I_{m_u} \quad H_{c,0}(k) \quad \cdots \quad H_{c,\mu_c-2}(k) \quad \mathcal{B}_{c,1}(k) \quad \cdots \quad \mathcal{B}_{c,n}(k)]. \quad (17)$$

Now from (10) and (16) it follows that $U(k)$ is given by

$$U(k) = \sum_{i=1}^{p_c} L_i\theta(k-i+1)R_i\Phi_{uy}(k), \quad (18)$$

where

$$\Phi_{uy}(k) \triangleq [u(k-\mu_c) \quad \cdots \quad u(k-\mu_c-n_c-p_c+2) \quad y(k-1) \quad \cdots \quad y(k-\mu_c-n_c-p_c+2)]^{\mathrm{T}}, \quad (19)$$

and

$$L_i \triangleq \begin{bmatrix} 0_{(i-1)m_u \times m_u} \\ I_{m_u} \\ 0_{(p_c-i)m_u \times m_u} \end{bmatrix}, \quad (20)$$

$$R_i \triangleq \begin{bmatrix} 0_{q_1\times(i-1)m_u} & I_{q_1\times q_1} & 0_{q_1\times(p_c-i)m_u} & 0_{q_1\times(i-1)l_y} & 0_{q_1\times q_2} & 0_{q_1\times(p_c-i)l_y} \\ 0_{q_2\times(i-1)m_u} & 0_{q_2\times q_1} & 0_{q_2\times(p_c-i)m_u} & 0_{q_1\times(i-1)l_y} & I_{q_2\times q_2} & 0_{q_2\times(p_c-i)l_y} \end{bmatrix}, \quad (21)$$

with $q_1 \triangleq n_c m_u$ and $q_2 \triangleq (n_c+\mu_c-1)l_y$. Thus, from (14) and (18) we obtain

$$Z(k) = W_{zw}\Phi_{zw}(k) + B_{zu}\sum_{i=1}^{p_c} L_i\theta(k-i+1)R_i\Phi_{uy}(k). \quad (22)$$

Next, we derive an update law for the parameter block vector $\theta(k)$. To do this, we consider a cost function that evaluates the performance of the current value of $\theta(k)$ based upon the behavior of the system during the previous p steps. Therefore, we define the *estimated performance* $\hat{Z}(k)$ by

$$\hat{Z}(k) \triangleq W_{zw}\Phi_{zw}(k) + B_{zu}\sum_{i=1}^{p_c} L_i\theta(k)R_i\Phi_{uy}(k), \quad (23)$$

which has the same form as (22) but with $\theta(k-i+1)$ replaced by the current parameter block vector $\theta(k)$. Using (23) we define the *estimated performance cost function*

$$J(k) = \frac{1}{2}\hat{Z}^{\mathrm{T}}(k)\hat{Z}(k). \quad (24)$$

Lemma 1 The gradient of $J(k)$ with respect to $\theta(k)$ is given by

$$\frac{\partial J(k)}{\partial \theta(k)} = \sum_{i=1}^{p_c} L_i^{\mathrm{T}} B_{zu}^{\mathrm{T}} \hat{Z}(k)\Phi_{uy}^{\mathrm{T}}(k)R_i^{\mathrm{T}}. \quad (25)$$

Proof: Substituting (23) in (24) and using matrix derivative formulae, we obtain (25). ■

Note that $\hat{Z}(k)$ cannot be evaluated using (23) since $w(k)$ is not available which implies that $\Phi_{zw}(k)$ is unknown. However, it follows from (14) and (23) that

$$\hat{Z}(k) = Z(k) - B_{zu}\left(U(k) - \sum_{i=1}^{p_c} L_i\theta(k)R_i\Phi_{uy}(k)\right), \quad (26)$$

which can be used to evaluate (25).

The gradient (25) is used in the update law

$$\theta(k+1) = \theta(k) - \eta(k)\frac{\partial J(k)}{\partial \theta(k)}, \quad (27)$$

where $\eta(k)$ is the *adaptive step size*. To determine the adaptive step size $\eta(k)$, we make the following assumption which is analogous to the assumption given in [8], pp. 281-282.

Assumption 1
There exists a matrix $\theta^* \in \mathcal{R}^{m_u \times n_c m_u + (n_c+\mu_c)l_y}$ that minimizes $J(k)$ for all k.

Under Assumption 1, we define the *desired performance*

$$Z^*(k) \triangleq W_{zw}\Phi_{zw}(k) + B_{zu}\sum_{i=1}^{p_c} L_i\theta^* R_i\Phi_{uy}(k), \quad (28)$$

the *error matrix*

$$E(k) \triangleq \theta^* - \theta(k), \quad (29)$$

the *performance error*

$$\varepsilon(k) \triangleq Z^*(k) - \hat{Z}(k), \quad (30)$$

and the *error matrix cost function*

$$\mathcal{J}(k,\eta(k)) \triangleq \|E(k+1)\|_{\mathrm{F}}^2 - \|E(k)\|_{\mathrm{F}}^2. \quad (31)$$

Our goal is to determine $\eta(k)$ such that $\|E(k)\|_{\mathrm{F}}^2$ is decreasing, that is, $\mathcal{J}(k,\eta(k))$ is negative. For convenience in stating the following result, we define the *optimal adaptive step size*

$$\eta_{\mathrm{opt}}(k) \triangleq \frac{\|\varepsilon(k)\|_2^2}{\left\|\frac{\partial J(k)}{\partial \theta(k)}\right\|_{\mathrm{F}}^2}. \quad (32)$$

Theorem 1 Consider the update law (27) and suppose Assumption 1 is satisfied. Furthermore, let $k \geq 0$ and assume that $\frac{\partial J(k)}{\partial \theta(k)} \neq 0$. Then

$$\mathcal{J}(k,\eta(k)) < 0 \quad (33)$$

if and only if

$$0 < \eta(k) < 2\eta_{\rm opt}(k). \tag{34}$$

In particular, $\eta(k) = \eta_{\rm opt}(k)$ minimizes $\mathcal{J}(k, \eta(k))$. Finally,

$$\mathcal{J}(k, \eta_{\rm opt}(k)) = -\frac{\|\varepsilon(k)\|_2^4}{\left\|\frac{\partial J(k)}{\partial L(k)}\right\|_{\rm F}^2}. \tag{35}$$

Proof: See Appendix A. ∎

In practice $\eta_{\rm opt}(k)$ is not computable since $\varepsilon(k)$ is not available. Hence, we define the ***implementable adaptive step size*** $\eta_{\rm imp}(k)$ by

$$\eta_{\rm imp}(k) \triangleq \frac{1}{p_{\rm c}\|B_{zu}\|_{\rm F}^2 \|\Phi_{uy}(k)\|_2^2}. \tag{36}$$

Note that if B_{zu} is known, then $\eta_{\rm imp}(k)$ can be calculated and used to implement (27). The following result shows that $\eta_{\rm imp}(k)$ satisfies the requirements of Theorem 1.

Proposition 1 The step size $\eta_{\rm imp}(k)$ satisfies

$$0 < \eta_{\rm imp}(k) \le \eta_{\rm opt}(k). \tag{37}$$

Proof: See Appendix B. ∎

Thus, under the assumptions of Theorem 1 it follows from (37) that $\mathcal{J}(k, \lambda\eta_{\rm imp}(k)) < 0$ for all $\lambda \in (0, 2)$.

Finally, we show that the update law (27) with step size $\eta_{\rm imp}(k)$ drives $\hat{Z}(k)$ to $Z^*(k)$ as k tends to infinity if $\{\Phi_{uy}(k)\}_{k=0}^{\infty}$ is bounded.

Proposition 2 Suppose Assumption 1 is satisfied, let $\frac{\partial J(k)}{\partial \theta(k)} \ne 0$ for all $k \ge 0$ and let $\eta(k) = \eta_{\rm imp}(k)$. If $\{\Phi_{uy}(k)\}_{k=0}^{\infty}$ is bounded, then

$$\lim_{k\to\infty} \varepsilon(k) = 0. \tag{38}$$

Proof: See Appendix C. ∎

We observe that of the four transfer matrices G_{zw}, G_{zu}, G_{yw} and G_{yu} in the MIMO standard problem, the algorithm described above requires that we identify only one transfer matrix, namely, G_{zu}. The signals that we require to be measured are $y(k)$ and $z(k)$.

5 Experimental Results

Experimental demonstration of the ARMARKOV adaptive disturbance algorithm is performed on an acoustic duct of circular cross-section. The duct is 80 inches long and has a diameter of 4 inches. The disturbance speaker (w) is located at one end of the duct and the measurement microphone (y) is located 4 inches in from the same end of the duct. The performance microphone (z) is positioned 6 inches in from the other end while the control speaker (u) is placed 16 inches in from that end of the duct. The signals from the two microphones are amplified by a dbx 760x microphone preamplifier while the control signal is amplified by an Alesis RA-100 amplifier. Both speakers are Radio Shack 6 inch woofers.

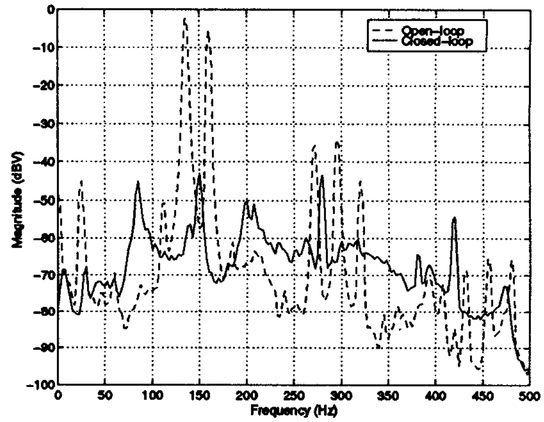

Figure 1: Open-loop and closed-loop frequency domain performance with a two-tone disturbance at 135.74 Hz and 160.4 Hz

The algorithm is tested on a two-tone disturbance (135.74 Hz and 160.4 Hz) and band-limited white noise (up to 390 Hz). The algorithm uses $n = 4$ and $\mu = 12$ for the matrix B_{zu}, and $n_{\rm c} = 2$, $\mu_{\rm c} = 10$ and $p = 2$ for control. The controller is implemented on a dSPACE ds1102 real time controller running a TMS320C30 DSP processor at a sampling frequency of 800 Hz. The microphone signals are processed through an Ithaco low pass filter that rolls off at 315 Hz. The tonal and band-limited white noise disturbances are generated by a Stanford Research Systems 770 FFT network analyzer and amplified by an Optimus STA-825 stereo receiver.

Figure 1 represents the open-loop and closed-loop performance with a two-tone disturbance. In this case, disturbance attenuation of over 35 dB is observed. Figure 2 shows the open-loop and closed-loop magnitude plots of the transfer function from disturbance to performance with a white noise disturbance, and noise suppression of up to 15 dB is observed over a frequency range from 0 - 300 Hz. Further experimental results which compare the performance of the proposed algorithm with LMS type algorithms are presented in [19].

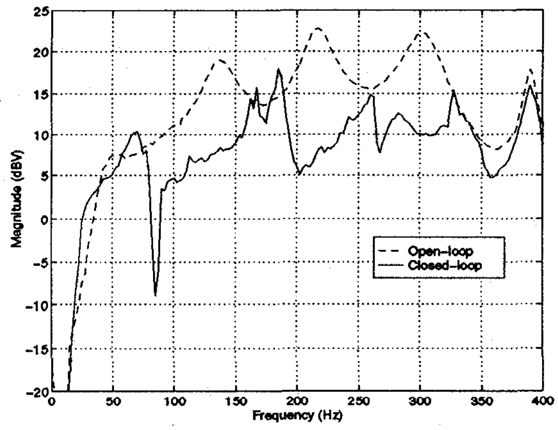

Figure 2: Open-loop and closed-loop performance with band-limited white noise

References

[1] P. A. Nelson and S. J. Elliot, *Active Control of Sound.* New York: Academic Press, 1992.

[2] S. M. Kuo and D. R. Morgan, *Active Noise Control Systems.* New York: Wiley, 1996.

[3] S. J. Elliot, I. M. Stothers and P. A. Nelson, "A Multiple Error LMS Algorithm and its Applications to the Active Control of Sound and Vibration," *IEEE Transactions on Acoustics, Speech and Signal Processing,* Vol. ASSP-35, pp. 1423 - 1434, 1987.

[4] B. Widrow and E. Walach, *Adaptive Inverse Control.* New Jersey: Prentice Hall, 1996.

[5] C. R. Fuller, C. A. Rogers and H. H. Robertshaw, "Control of Sound Radiation with Active/Adaptive Structures," *Journal of Sound and Vibration,* Vol. 157, pp. 19 - 39, 1992.

[6] P. L. Feintuch, N. J. Bershad and A. K. Lo, "A Frequency Domain Model for 'filtered' LMS Algorithm - Stability Analysis, Design, and Elimination of the Training Mode," *IEEE Transactions on Acoustics, Speech, and Signal Processing,* Vol. ASSP-41, pp. 1518 - 1531, 1993.

[7] F. Jiang, N. Ojiro, H. Ohmori and A. Sano, "Adaptive Active Noise Control Schemes in Time Domain and Transform-Domains," *Proceedings of the 34th Conference on Decision and Control, New Orleans,* pp. 2165 - 2172, 1995.

[8] K. J. Astrom and B. Wittenmark, *Adaptive Control,* second edition, Reading, MA: Addison-Wesley, 1995.

[9] P. P. Kanjilal, *Adaptive Prediction and Predictive Control,* Herts, U. K.:Peter Peregrinus, 1995.

[10] D. W. Clarke, C. H. Mohtadi and P. S. Tuffs, "Generalized Predictive Algorithm - Part I, The Basic Algorithm," *Automatica,* Vol. 23, pp. 137-148, 1987.

[11] J. Richalet, "Industrial Applications of Model Based Predictive Control," *Automatica,* Vol. 29, No. 5, pp. 1251-1274, 1993.

[12] K. Furuta and M. Wongsaisuwan, "Discrete-time LQG Dynamic Controller Design Using Plant Markov Parameters," *Automatica,* Vol. 31, pp. 1317-1324, 1995.

[13] D. C. Hyland, E. G. Collins, Jr., W. M. Haddad and D. L. Hunter, "Neural Network System Identification for Improved Noise Rejection," *Proceedings of the American Control Conference, Seattle,* pp. 345 - 349, June 1995.

[14] C. Ahn,"A Recursive Algorithm for Identification of IIR Systems," *Proc. IEEE COnf. Dec. Contr.,* pp. 2595-2600, San Antonio, TX, December 1993.

[15] J. C. Akers and D. S. Bernstein, "Time-Domain Identification Using ARMARKOV/Toeplitz Models," *Amer. Contr. Conf.,* pp. 191-195, Albuquerque, NM, June 1997.

[16] J. C. Akers and D. S. Bernstein, "ARMARKOV Least-Squares Identification," *Amer. Contr. Conf.,* pp. 186-190, Albuquerque, NM, June 1997.

[17] R. Venugopal and D. S. Bernstein, "Adaptive Disturbance Rejection Using ARMARKOV/Toeplitz Models," *Amer. Contr. Conf.,* pp. 1657-1661, Albuquerque, NM, June 1997.

[18] B. Francis, *A Course in H_∞ Control Theory,* New York: Springer-Verlag, 1987.

[19] T. H. Van Pelt, R. Venugopal and D. S. Bernstein, "Experimental Comparison of Adaptive Noise Control Algorithms," *Conf. Contr. Appl.,* October 1997.

Appendix A

From (23), (28) and (30) it follows that

$$\varepsilon(k) = B_{zu} \sum_{i=1}^{p_c} L_i[\theta^* - \theta(k)] R_i \Phi_{uy}(k). \tag{39}$$

Using (29), (39) can be written as

$$\varepsilon(k) = B_{zu} \sum_{i=1}^{p_c} L_i E(k) R_i \Phi_{uy}(k). \tag{40}$$

By Assumption 1, θ^* minimizes $J(k)$, and thus it follows from (25) that

$$\left.\frac{\partial J(k)}{\partial \theta(k)}\right|_{\theta(k)=\theta^*} = \sum_{i=1}^{p_c} L_i^{\mathrm{T}} B_{zu}^{\mathrm{T}} \hat{Z}^*(k) \Phi_{uy}^{\mathrm{T}}(k) R_i^{\mathrm{T}} = 0. \tag{41}$$

Subtracting (41) from (25) and substituting $\varepsilon(k)$ from (30) into the resulting equation yields

$$\frac{\partial J(k)}{\partial \theta(k)} = -\sum_{i=1}^{p_c} L_i^{\mathrm{T}} [B_{zu}^{\mathrm{T}} \varepsilon(k) \Phi_{uy}^{\mathrm{T}}(k)] R_i^{\mathrm{T}}. \tag{42}$$

Next, using (27) and (29) we obtain

$$E(k+1) = E(k) + \eta(k)\frac{\partial J(k)}{\partial \theta(k)}, \qquad (43)$$

and thus

$$\|E(k+1)\|_{\rm F}^2 - \|E(k)\|_{\rm F}^2 = 2\eta(k){\rm tr}\left(E(k)\frac{\partial J(k)}{\partial \theta(k)}^{\rm T}\right) + \eta^2(k)\left\|\frac{\partial J(k)}{\partial \theta(k)}\right\|_{\rm F}^2. \qquad (44)$$

Using (40) and (42) we obtain

$$\begin{aligned} {\rm tr}\left(E(k)\frac{\partial J(k)}{\partial \theta(k)}^{\rm T}\right) &= -{\rm tr}\left(E(k)\sum_{i=1}^{p_c} R_i\Phi_{uy}(k)\varepsilon^{\rm T}(k)B_{zu}L_i\right) \\ &= -\sum_{i=1}^{p_c}\left(\varepsilon^{\rm T}(k)B_{zu}L_iE(k)R_i\Phi_{uy}(k)\right) \\ &= -\varepsilon^{\rm T}(k)\left(B_{zu}\sum_{i=1}^{p_c} L_iE(k)R_i\Phi_{uy}(k)\right) \\ &= -\|\varepsilon(k)\|_2^2. \end{aligned} \qquad (45)$$

Thus, (31), (44) and (45) imply that

$$\mathcal{J}(k,\eta(k)) = -2\eta(k)\|\varepsilon(k)\|_2^2 + \eta^2(k)\left\|\frac{\partial J(k)}{\partial \theta(k)}\right\|_{\rm F}^2. \qquad (46)$$

Now, from (46) it follows that $\mathcal{J}(k,\eta(k)) < 0$ if and only if

$$\eta(k) < \frac{2\|\varepsilon(k)\|_2^2}{\|\frac{\partial J(k)}{\partial \theta(k)}\|_{\rm F}^2} = 2\eta_{\rm opt}(k), \qquad (47)$$

which proves the first statement of the theorem.

To prove the second statement of the theorem we note from (46) that

$$\begin{aligned} \mathcal{J}(k,\eta(k)) &= \left[\eta^2(k)\left\|\frac{\partial J(k)}{\partial \theta(k)}\right\|_{\rm F}^2 - 2\|\varepsilon(k)\|_2^2\eta(k)\right] \qquad (48) \\ &= \left[\eta^2(k) - 2\eta(k)\eta_{\rm opt}(k)\right]\left\|\frac{\partial J(k)}{\partial \theta(k)}\right\|_{\rm F}^2, \qquad (49) \\ &= \left[(\eta(k)-\eta_{\rm opt}(k))^2 - \eta_{\rm opt}^2(k)\right]\left\|\frac{\partial J(k)}{\partial \theta(k)}\right\|_{\rm F}^2 \qquad (50) \end{aligned}$$

Since the quadratic function $(\eta(k)-\eta_{\rm opt}(k))^2 - \eta_{\rm opt}^2(k)$ achieves its minimum at $\eta(k) = \eta_{\rm opt}(k)$, it follows from (50) that $\mathcal{J}(p,\eta(k))$ is minimized by $\eta(k) = \eta_{\rm opt}(k)$. Substituting (32) into (46) yields (35). □

Appendix B

Note that

$$\begin{aligned} \eta_{\rm imp}(k) &= \frac{\|\varepsilon(k)\|_2^2}{p_{\rm c}\|B_{zu}\|_{\rm F}^2\|\varepsilon(k)\|_2^2\|\Phi_{uy}(k)\|_2} \\ &\le \frac{\|\varepsilon(k)\|_2^2}{p_{\rm c}\|B_{zu}^{\rm T}\varepsilon(k)\Phi_{uy}^{\rm T}(k)\|_{\rm F}^2}. \end{aligned} \qquad (51)$$

Using (42) it follows that

$$\begin{aligned} \left\|\frac{\partial J(k)}{\partial \theta(k)}\right\|_{\rm F}^2 &= \left\|\sum_{i=1}^{p_c} L_i^{\rm T}[B_{zu}^{\rm T}\varepsilon(k)\Phi_{uy}^{\rm T}(k)]R_i^{\rm T}\right\|_{\rm F}^2 \\ &\le p_{\rm c}\|B_{zu}^{\rm T}\varepsilon(k)\Phi_{uy}^{\rm T}(k)\|_{\rm F}^2, \end{aligned} \qquad (52)$$

and hence (51) implies

$$\eta_{\rm imp}(k) \le \frac{\|\varepsilon(k)\|_2^2}{\left\|\frac{\partial J(k)}{\partial \theta(k)}\right\|_{\rm F}^2} = \eta_{\rm opt}(k), \qquad (53)$$

which proves that $\eta_{\rm imp}(k)$ satisfies (37). □

Appendix C

From (46) and (52) it follows that

$$\begin{aligned} \mathcal{J}(k,\eta(k)) &\le -2\eta(k)\|\varepsilon(k)\|_2^2 \\ &+\eta^2(k)\|B_{zu}\|_{\rm F}^2\|\varepsilon(k)\|_2^2\|\Phi_{uy}(k)\|_2^2. \end{aligned} \qquad (54)$$

Setting $\eta(k) = \eta_{\rm imp}(k)$ in (54) and using (31) we obtain

$$\|E(k+1)\|_{\rm F}^2 - \|E(k)\|_{\rm F}^2 \le -\frac{\|\varepsilon(k)\|_2^2}{p_{\rm c}\|B_{zu}\|_{\rm F}^2\|\Phi_{uy}(k)\|_2^2}. \qquad (55)$$

Next,

$$\begin{aligned} \|E(0)\|_{\rm F}^2 &> \|E(0)\|_{\rm F}^2 - \|E(r+1)\|_{\rm F}^2 \\ &= \sum_{p=0}^{r}\left(\|E(k)\|_{\rm F}^2 - \|E(k+1)\|_{\rm F}^2\right). \end{aligned} \qquad (56)$$

Substituting (55) into (56) yields

$$\sum_{k=0}^{r}\frac{\|\varepsilon(k)\|_2^2}{p_{\rm c}\|B_{zu}\|_{\rm F}^2\|\Phi_{uy}(k)\|_2^2} < \|E(0)\|_{\rm F}^2. \qquad (57)$$

Since $\{\Phi_{uy}(k)\}_{k=0}^{\infty}$ is assumed to be bounded, there exists $\beta > 0$ such that $\|\Phi_{uy}(k)\|_2 < \beta$, $k \ge 0$, and thus it follows from (57) that

$$\sum_{k=0}^{r}\|\varepsilon(k)\|_2^2 < p_{\rm c}\left(\beta\|B_{zu}\|_{\rm F}\right)^2\|E(0)\|_{\rm F}^2. \qquad (58)$$

Letting $r \to \infty$, (58) implies that $\sum_{k=0}^{\infty}\|\varepsilon(k)\|_2^2 < \infty$, and thus we obtain (38). □