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Induced Convolution Operator Norms for Discrete-Time Linear Systems

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Abstract

In this paper we develop explicit formulas for induced convolution operator norms and their bounds. These results generalize established induced operator norms for discrete-time linear systems with various classes of input-output signal pairs.

1. Introduction

In this paper we consider the dynamical system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = 0, \quad k \in \mathcal{N}, \quad (1)$$
$$y(k) = Cx(k), \quad (2)$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^l$, $k \in \mathcal{N}$, $A \in \mathbb{R}^{n\times n}$ is discrete-time asymptotically stable, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, and where $u(\cdot)$ is an input signal belonging to the class $\ell_{p,q}$ of input signals and $y(\cdot)$ is an output signal belonging to the class $\ell_{r,s}$ of output signals, where $\ell_{p,q}$ denotes the set of sequences in ℓ_p with q spatial norm. In applications, (1) and (2) may denote a control system in closed-loop configuration where the objective is to determine the "size" of the output $y(\cdot)$ for a disturbance $u(\cdot)$.

Operator norms induced by classes of input-output signal pairs can be used to capture disturbance rejection performance objectives for controlled dynamical systems [1]. In particular, discrete-time H_∞ control theory [2] has been developed to address the problem of disturbance rejection for systems with bounded energy $\ell_{2,2}$ signal norms on the disturbance and performance variables. Since the induced H_∞ transfer function norm corresponds to the worst-case disturbance attenuation, for systems with $\ell_{2,2}$ disturbances which possess significant power within arbitrarily small bandwidths, H_∞ theory is clearly appropriate. Alternatively, to address pointwise in time worst-case peak amplitude response due to bounded amplitude persistent $\ell_{\infty,\infty}$ disturbances, ℓ_1 theory is appropriate [3]. The problem of finding a stabilizing controller such that the closed-loop system gain from $\ell_{2,2}$ to $\ell_{\infty,q}$, where $q = 2$ or ∞, is below a specified level is given in [4].

In addition to the disturbance rejection problem, another application of induced operator norms is the problem of actuator amplitude and rate saturation [5]. In particular, since the convolution operator norm induced from $\ell_{2,2}$ to $\ell_{\infty,\infty}$ captures the worst-case peak amplitude response due to finite energy disturbances, defining the output (performance) variables y to correspond to the actuator amplitude and actuator rate signals, it follows that the induced $\ell_{2,2}$ to $\ell_{\infty,\infty}$ convolution operator norm bounds actuator amplitude and actuator rate excursion. Furthermore, since uncertain signals can also be used to model uncertainty in a system, the treatment of certain classes of uncertain disturbances also enable the development of controllers that are robust with respect to input-output uncertainty blocks [3, 6].

In a recent paper [7], explicit formulas for convolution operator norms of continuous-time dynamical systems induced by several classes of input-output signal pairs were developed. In this paper we present analogous results for discrete-time dynamical systems. Specifically, for a large class of input-output signal pairs we provide explicit formulas for induced convolution operator norms and operator norm bounds for linear dynamical systems. These results generalize several well known induced convolution operator norm results in the literature including results on $\ell_{\infty,\infty}$ equi-induced norms (ℓ_1 operator norms) and $\ell_{1,1}$ equi-induced norms (resource norms).

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Notation

$\mathbb{Z}, \mathcal{N}$	integers, nonnegative integers
$\mathbb{R}, \mathbb{R}^{m\times n}$	real numbers, $m \times n$ real matrices
x_i	ith entry of vector x
$\lvert x\rvert$	vector whose ith element is $\lvert x_i\rvert$
e_i	vector with 1 in ith position and 0's elsewhere
$A_{(i,j)}$	(i,j)th element of matrix A
$\mathrm{row}_i(A)$	ith row of A
$\mathrm{col}_i(A)$	ith column of A
A^{T}	transpose of A
$\sigma_{\max}(A)$	maximum singular value of A
$\mathrm{d}_{\max}(A)$	$\max_{i=1,\ldots,n} A_{(i,i)}$
$\lVert A\rVert_{\mathrm{F}}$	Frobenius norm of A $\left(= (\mathrm{tr}\, AA^{\mathrm{T}})^{1/2}\right.$
$\lVert A\rVert_p$	$\left[\sum_{i=1}^m \sum_{j=1}^n \lvert A_{(i,j)}\rvert^p\right]^{1/p}$, $1 \le p < \infty$
$\lVert A\rVert_\infty$	$\max_{i=1,\ldots,n;\ j=1,\ldots,m} \lvert A_{(i,j)}\rvert$
$\lvert\lvert\lvert f\rvert\rvert\rvert_{p,q}$	$\{\sum_{k=0}^\infty \lVert f(k)\rVert_q^p\}^{\frac{1}{p}}$, $1 \le p < \infty$
$\lvert\lvert\lvert f\rvert\rvert\rvert_{\infty,q}$	$\sup_{k\in\mathcal{N}} \lVert f(k)\rVert_q$
$\langle f, g\rangle$	$\sum_{k=0}^\infty f^{\mathrm{T}}(k)g(k)$
$\ell_{p,q}$	$\{f : \mathcal{N} \to \mathbb{R}^n : \lvert\lvert\lvert f\rvert\rvert\rvert_{p,q} < \infty\}$
$\bar{p}$	$p/(p-1)$, $p \in [1, \infty]$

2. Mathematical Preliminaries

Let $\lVert\cdot\rVert'$ and $\lVert\cdot\rVert''$ denote vector norms on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, where $m, n \ge 1$. Then $\lVert\cdot\rVert : \mathbb{R}^{m\times n} \to \mathbb{R}$

defined by

$$\|A\| \triangleq \max_{\|x\|'=1} \|Ax\|''$$

is the *matrix norm induced by* $\|\cdot\|'$ and $\|\cdot\|''$. If $\|\cdot\|' = \|\cdot\|_p$ and $\|\cdot\|'' = \|\cdot\|_q$, where $p,q \in [1,\infty]$, then the matrix norm on $\mathbb{R}^{m\times n}$ induced by $\|\cdot\|_p$ and $\|\cdot\|_q$ is denoted by $\|\cdot\|_{q,p}$. Let $\|\cdot\|$ denote a vector norm on $\mathbb{R}^m$. Then the *dual norm* $\|\cdot\|_{\mathrm{D}}$ of $\|\cdot\|$ is defined by

$$\|y\|_{\mathrm{D}} \triangleq \max_{\|x\|=1} |y^{\mathrm{T}}x|,$$

where $y \in \mathbb{R}^m$ [8]. Note that $\|\cdot\|_{\mathrm{DD}} = \|\cdot\|$ [8]. Furthermore, if $p,q \in [1,\infty]$ satisfy $1/p + 1/q = 1$, then $\|\cdot\|_{p\mathrm{D}} = \|\cdot\|_q$ [8]. For $p \in [1,\infty]$ we denote the conjugate variable $q \in [1,\infty]$ satisfying $1/p + 1/q = 1$ by $\bar{p} = p/(p-1)$.

Lemma 2.1 [7]. Let $p \in [1,\infty]$ and let $A \in \mathbb{R}^{m\times n}$. Then

$$\|A\|_{2,2} = \sigma_{\max}(A), \tag{3}$$

$$\|A\|_{p,1} = \max_{i=1,\dots,n} \|\mathrm{col}_i(A)\|_p, \tag{4}$$

and

$$\|A\|_{\infty,p} = \max_{i=1,\dots,m} \|\mathrm{row}_i(A)\|_{\bar{p}}. \tag{5}$$

Remark 2.1. Note that (4) and (5) generalize the well-known expressions $\|A\|_{1,1} = \max_{i=1,\dots,n} \|\mathrm{col}_i(A)\|_1$ [9], $\|A\|_{\infty,\infty} = \max_{i=1,\dots,m} \|\mathrm{row}_i(A)\|_1$ [9], and $\|A\|_{\infty,1} = \|A\|_\infty$ [10]. Furthermore, since $\max_{i=1,\dots,n} \|\mathrm{col}_i(A)\|_2 = \mathrm{d}_{\max}^{1/2}(A^{\mathrm{T}}A)$ and $\max_{i=1,\dots,m} \|\mathrm{row}_i(A)\|_2 = \mathrm{d}_{\max}^{1/2}(AA^{\mathrm{T}})$, it follows from (4) with $p=2$ that $\|A\|_{2,1} = \mathrm{d}_{\max}^{1/2}(A^{\mathrm{T}}A)$ and from (5) with $p=2$ that $\|A\|_{\infty,2} = \mathrm{d}_{\max}^{1/2}(AA^{\mathrm{T}})$.

The following result generalizes Hölder's inequality to mixed-signal norms.

Lemma 2.2. Let $p,r \in [1,\infty]$, and let $f \in \ell_{p,r}$ and $g \in \ell_{\bar{p},\bar{r}}$. Then

$$\langle f,g\rangle \le |\!|\!| f |\!|\!|_{p,r} |\!|\!| g |\!|\!|_{\bar{p},\bar{r}}. \tag{6}$$

Finally, the following two results are needed for the results given in Section 3. The proofs of these results are similar to the results in [11] involving continuous-time signals and hence are omitted.

Lemma 2.3. Let $p \in [1,\infty)$ and $r \in [1,\infty]$, and let $f \in \ell_{p,r}$. Then

$$|\!|\!| f |\!|\!|_{p,r} = \sup_{g\in\mathcal{G}} \langle f,g\rangle, \tag{7}$$

where $\mathcal{G} \triangleq \{g \in \ell_{\bar{p},\bar{r}} : |\!|\!| g |\!|\!|_{\bar{p},\bar{r}} \le 1\}$.

Lemma 2.4. Let $p \in [1,\infty)$, $r \in [1,\infty]$, and $f : \mathcal{N} \times \mathcal{N} \to \mathbb{R}^n$ be such that $f(k,\cdot)$ is summable for all $k \in \mathcal{N}$, $f(\cdot,\kappa) \in \ell_{p,r}$ for all $\kappa \in \mathcal{N}$, and $g \in \ell_{1,1}$, where $g(\kappa) \triangleq [\sum_{k=0}^{\infty} \|f(k,\kappa)\|_r^p]^{1/p}$. Then

$$|\!|\!| y |\!|\!|_{p,r} \le \sum_{k=0}^{\infty} g(k), \tag{8}$$

where

$$y(k) = \sum_{\kappa=0}^{\infty} f(k,\kappa), \qquad k \in \mathcal{N}. \tag{9}$$

3. Induced Convolution Operator Norms for Discrete-Time Linear Systems

In this section we develop induced convolution operator norms. For the system (1), (2), let $G : \mathbb{Z} \to \mathbb{R}^{l\times m}$ denote the impulse response function

$$G(k) \triangleq \begin{cases} 0, & k \le 0, \\ CA^{k-1}B, & k > 0. \end{cases} \tag{10}$$

Next, let $\mathcal{G} : \ell_{p,r} \to \ell_{q,s}$ denote the convolution operator

$$y(k) = (\mathcal{G} * u)(k) \triangleq \sum_{\kappa=0}^{\infty} G(k-\kappa)u(\kappa), \tag{11}$$

and define the induced norm $|\!|\!| \mathcal{G} |\!|\!|_{(q,s),(p,r)}$ as

$$|\!|\!| \mathcal{G} |\!|\!|_{(q,s),(p,r)} \triangleq \sup_{|\!|\!| u |\!|\!|_{p,r}=1} |\!|\!| \mathcal{G} * u |\!|\!|_{q,s}. \tag{12}$$

The following lemma provides an explicit expression for $|\!|\!| \mathcal{G} |\!|\!|_{(\infty,\infty),(p,p)}$ for the case in which $\mathcal{G}$ is a single-input/single-output operator.

Lemma 3.1. Let $r \in [1,\infty]$ and let $l = m = 1$. Then $\mathcal{G} : \ell_{r,r} \to \ell_{\infty,\infty}$, and

$$|\!|\!| \mathcal{G} |\!|\!|_{(\infty,\infty),(r,r)} = |\!|\!| G |\!|\!|_{\bar{r},\bar{r}}. \tag{13}$$

Proof. For $r = 1$ and $r = \infty$, (13) is standard; see [11] and [6, pp. 23-24] as applied to discrete-time signals, respectively. Next, let $r \in (1,\infty)$ and note for all $k \in \mathcal{N}$, it follows from Lemma 2.2 with $p = r$ that

$$\begin{aligned} |y(k)| &= \left|\sum_{\kappa=0}^{\infty} G(k-\kappa)u(\kappa)\right| \le \left[\sum_{\kappa=0}^{\infty} |G(k-\kappa)|^{\bar{r}}\right]^{1/\bar{r}} |\!|\!| u |\!|\!|_{r,r} \\ &= \left[\sum_{\kappa=0}^{k} |G(\kappa)|^{\bar{r}}\right]^{1/\bar{r}} |\!|\!| u |\!|\!|_{r,r} \le |\!|\!| G |\!|\!|_{\bar{r},\bar{r}} |\!|\!| u |\!|\!|_{r,r}, \end{aligned}$$

which implies

$$|\!|\!| \mathcal{G} |\!|\!|_{(\infty,\infty),(r,r)} \le |\!|\!| G |\!|\!|_{\bar{r},\bar{r}}. \tag{14}$$

Next, let $K > 0$ and let $u(\cdot)$ be such that $u(k) = \mathrm{sgn}(G(K-k))|G(K-k)|^{1/(r-1)}$, $k \in \mathcal{N}$, where $\mathrm{sgn}(\cdot)$ denotes the signum function. Now, since $|\!|\!| u |\!|\!|_{r,r} = [\sum_{k=0}^{\infty} |G(K-k)|^{\bar{r}}]^{1/r}$, it follows that

$$\begin{aligned} |y(K)| &= \left|\sum_{\kappa=0}^{\infty} G(K-\kappa)u(\kappa)\right| = \sum_{\kappa=0}^{\infty} |G(K-\kappa)|^{\bar{r}} \\ &= \left[\sum_{\kappa=0}^{\infty} |G(K-\kappa)|^{\bar{r}}\right]^{1/\bar{r}} |\!|\!| u |\!|\!|_{r,r} \\ &= \left[\sum_{\kappa=0}^{K} |G(\kappa)|^{\bar{r}}\right]^{1/\bar{r}} |\!|\!| u |\!|\!|_{r,r}. \end{aligned}$$

Hence,

$$\|y\|_{\infty,\infty} \geq \lim_{K\to\infty} |y(K)| = \|G\|_{\bar{r},\bar{r}}\|u\|_{r,r},$$

which, with (14), implies (13). □

Remark 3.1. Note that it follows from Lemma 3.1 that there exists $u \in \ell_{r,r}$ such that $\lim_{k\to\infty}(\mathcal{G} * u)(k) = \|G\|_{\bar{r},\bar{r}}\|u\|_{r,r}$.

Next define $\mathcal{P} \in \mathbb{R}^{m\times m}$ and $\mathcal{Q} \in \mathbb{R}^{l\times l}$ by

$$\mathcal{P} \triangleq \sum_{k=0}^{\infty} G^{\mathrm{T}}(k)G(k), \qquad \mathcal{Q} \triangleq \sum_{k=0}^{\infty} G(k)G^{\mathrm{T}}(k). \tag{15}$$

Note that $\mathcal{P} = B^{\mathrm{T}}PB$ and $\mathcal{Q} = CQC^{\mathrm{T}}$, where the observability and controllability Gramians P and Q, respectively, are the unique $n \times n$ nonnegative-definite solutions to the discrete-time Lyapunov equations

$$P = A^{\mathrm{T}}PA + C^{\mathrm{T}}C, \qquad Q = AQA^{\mathrm{T}} + BB^{\mathrm{T}}. \tag{16}$$

Furthermore, let $H(z) = C(zI - A)^{-1}B$ correspond to the transfer function of (1), (2) and let $G_{[p,q]}$ denote the $l \times m$ matrix whose (i,j)th element is $\|G_{(i,j)}\|_{(p,p),(q,q)}$.

Theorem 3.1. The following statements hold:

i) $\mathcal{G} : \ell_{2,2} \to \ell_{2,2}$, and

$$\|\mathcal{G}\|_{(2,2),(2,2)} = \sup_{\omega\in[0,2\pi]} \sigma_{\max}(H(e^{\jmath\omega})). \tag{17}$$

ii) Let $r \in [1,\infty]$. Then $\mathcal{G} : \ell_{1,r} \to \ell_{2,2}$, and

$$\|\mathcal{G}\|_{(2,2),(1,r)} = \|\mathcal{P}^{1/2}\|_{2,r}. \tag{18}$$

iii) Let $p \in [1,\infty]$. Then $\mathcal{G} : \ell_{2,2} \to \ell_{\infty,p}$, and

$$\|\mathcal{G}\|_{(\infty,p),(2,2)} = \|\mathcal{Q}^{1/2}\|_{2,\bar{p}}. \tag{19}$$

iv) Let $p, r \in [1,\infty]$. Then $\mathcal{G} : \ell_{1,r} \to \ell_{\infty,p}$, and

$$\|\mathcal{G}\|_{(\infty,p),(1,r)} = \sup_{k\in\mathcal{N}} \|G(k)\|_{p,r}. \tag{20}$$

v) Let $r \in [1,\infty]$. Then $\mathcal{G} : \ell_{r,r} \to \ell_{\infty,\infty}$, and

$$\|\mathcal{G}\|_{(\infty,\infty),(r,r)} = \max_{i=1,\ldots,l} \|\mathrm{row}_i(G_{[\bar{r},\bar{r}]})\|_{\bar{r}}. \tag{21}$$

vi) Let $p \in [1,\infty]$. Then $\mathcal{G} : \ell_{1,1} \to \ell_{p,p}$, and

$$\|\mathcal{G}\|_{(p,p),(1,1)} = \max_{j=1,\ldots,m} \|\mathrm{col}_j(G_{[p,p]})\|_p. \tag{22}$$

Proof. See Appendix A. □

The following corollary specializes Theorem 3.1 to provide analogous results to those given in [11] and [6, p. 26] for discrete-time signals.

Corollary 3.1. The following statements hold:

i) $\mathcal{G} : \ell_{1,2} \to \ell_{2,2}$, and

$$\|\mathcal{G}\|_{(2,2),(1,2)} = \sigma_{\max}^{1/2}(\mathcal{P}).$$

ii) $\mathcal{G} : \ell_{1,1} \to \ell_{2,2}$, and

$$\|\mathcal{G}\|_{(2,2),(1,1)} = \mathrm{d}_{\max}^{1/2}(\mathcal{P}).$$

iii) $\mathcal{G} : \ell_{2,2} \to \ell_{\infty,2}$, and

$$\|\mathcal{G}\|_{(\infty,2),(2,2)} = \sigma_{\max}^{1/2}(\mathcal{Q}).$$

iv) $\mathcal{G} : \ell_{2,2} \to \ell_{\infty,\infty}$, and

$$\|\mathcal{G}\|_{(\infty,\infty),(2,2)} = \mathrm{d}_{\max}^{1/2}(\mathcal{Q}).$$

v) $\mathcal{G} : \ell_{1,1} \to \ell_{\infty,\infty}$, and

$$\|\mathcal{G}\|_{(\infty,\infty),(1,1)} = \sup_{k\in\mathcal{N}} \|G(k)\|_{\infty}.$$

vi) $\mathcal{G} : \ell_{1,2} \to \ell_{\infty,2}$, and

$$\|\mathcal{G}\|_{(\infty,2),(1,2)} = \sup_{k\in\mathcal{N}} \sigma_{\max}(G(k)).$$

vii) $\mathcal{G} : \ell_{\infty,\infty} \to \ell_{\infty,\infty}$, and

$$\|\mathcal{G}\|_{(\infty,\infty),(\infty,\infty)} = \max_{i=1,\ldots,l} \|\mathrm{row}_i(G_{[1,1]})\|_1.$$

viii) $\mathcal{G} : \ell_{1,1} \to \ell_{1,1}$, and

$$\|\mathcal{G}\|_{(1,1),(1,1)} = \max_{j=1,\ldots,m} \|\mathrm{col}_j(G_{[1,1]})\|_1.$$

Remark 3.2. Recall that the H_2 norm of the system (1), (2) is given by $\|\mathcal{G}\|_{\mathrm{H}_2} = \|\mathcal{P}^{1/2}\|_{\mathrm{F}} = \|\mathcal{Q}^{1/2}\|_{\mathrm{F}}$. Hence, using the fact that $\|\cdot\|_{\mathrm{F}} = \sigma_{\max}(\cdot)$ for rank-one matrices, it follows from *i)* of Corollary 3.1 that if B (and hence $\mathcal{P}$) is a rank-one matrix then $\|\mathcal{G}\|_{\mathrm{H}_2} = \|\mathcal{G}\|_{(2,2),(1,2)}$. Similarly, it follows from *iii)* of Corollary 3.1 that if C (and hence $\mathcal{Q}$) is a rank-one matrix then $\|\mathcal{G}\|_{\mathrm{H}_2} = \|\mathcal{G}\|_{(\infty,2),(2,2)}$. Hence, in the single-input/multi-output and multi-output/single-input cases the H_2 norm of a dynamical system is induced.

Remark 3.3. Theorem 3.1 also applies to the more general case where $\mathcal{G}$ is a noncausal, time-invariant operator. In this case, the input-output spaces $\ell_{p,q}$ and $\ell_{r,s}$ are defined for $k \in \mathbb{Z}$, $H(e^{\jmath\omega})$ is the discrete-Fourier transform of $G(k)$, and the lower limit in the sums defining $\mathcal{P}$ and $\mathcal{Q}$ is replaced by $-\infty$.

4. Upper Bounds for ℓ_1 Operator Norms

In this section we provide upper bounds for the ℓ_1 operator norm $\|\mathcal{G}\|_{(\infty,p),(\infty,r)}$. For $\rho > 1$, define the shifted impulse response function $G_\rho : \mathbb{Z} \to \mathbb{R}^{l\times m}$ by

$$G_\rho(k) \triangleq \begin{cases} 0 & k \leq 0, \\ \rho^{k/2}CA^{k-1}B, & k > 0, \end{cases} \tag{23}$$

and let $\mathcal{G}_\rho$ denote its convolution operator

$$y(k) = (\mathcal{G}_\rho * u)(k) \triangleq \sum_{k=0}^{\infty} G_\rho(k-\kappa)u(\kappa). \tag{24}$$

Theorem 4.1. Let $\rho > 1$ be such that $\sqrt{\rho}A$ is discrete-time asymptotically stable and let $Q_\rho \in \mathbb{R}^{n\times n}$ be the unique, nonnegative definite solution to the Lyapunov equation

$$Q_\rho = \rho A Q_\rho A^{\mathrm{T}} + \rho B B^{\mathrm{T}}. \qquad (25)$$

Then the following statements hold:

i) Let $p \in [1,\infty]$. Then $\mathcal{G} : \ell_{\infty,2} \to \ell_{\infty,p}$, and

$$|||\mathcal{G}|||_{(\infty,p),(\infty,2)} \le \frac{1}{\sqrt{\rho-1}} \|(CQ_\rho C^{\mathrm{T}})^{1/2}\|_{2,\bar{p}}. \qquad (26)$$

ii) Let $p, r \in [1,\infty]$. Then $\mathcal{G} : \ell_{\infty,r} \to \ell_{\infty,p}$, and

$$|||\mathcal{G}|||_{(\infty,p),(\infty,r)} \le \frac{1}{\sqrt{\rho}-1} \sup_{k\in\mathcal{N}} \|G_\rho(k)\|_{p,r}. \qquad (27)$$

Proof. See Appendix B. □

Next we specialize Theorem 4.1 to Euclidean and infinity spatial norms.

Corollary 4.1. Let $\rho > 1$ be such that $\sqrt{\rho}A$ is discrete-time asymptotically stable, let $G_\rho(\cdot)$ be given by (23), and let $Q_\rho \in \mathbb{R}^{n\times n}$ be the unique, nonnegative definite solution to (25). Then the following statements hold:

i) $\mathcal{G} : \ell_{\infty,2} \to \ell_{\infty,2}$, and

$$|||\mathcal{G}|||_{(\infty,2),(\infty,2)} \le \frac{1}{\sqrt{\rho-1}} \sigma_{\max}^{1/2}(CQ_\rho C^{\mathrm{T}}). \qquad (28)$$

ii) $\mathcal{G} : \ell_{\infty,2} \to \ell_{\infty,\infty}$, and

$$|||\mathcal{G}|||_{(\infty,\infty),(\infty,2)} \le \frac{1}{\sqrt{\rho-1}} \mathrm{d}_{\max}^{1/2}(CQ_\rho C^{\mathrm{T}}). \qquad (29)$$

Proof. The proof is a direct consequence of Lemma 2.1 and Theorem 4.1. □

Remark 4.1. Using set theoretic arguments involving closed convex sets and support functions the ℓ_1 norm bound in (28) was given by Schweppe [12]. Within the context of $\ell_{\infty,2}$ equi-induced norms, this ℓ_1 norm bound is referred to as the star-norm in [13].

A summary of the results of Sections 3 and 4 is given in Table 1.

Appendix A. Proof of Theorem 3.1

i) is standard; see [14] for a proof.

ii) It follows from Lemma 2.4 that

$$\begin{aligned} |||y|||_{2,2} &\le \sum_{\kappa=0}^{\infty} |||G(k-\kappa)u(\kappa)|||_{2,2} = \sum_{\kappa=0}^{\infty} \|\mathcal{P}^{1/2}u(\kappa)\|_2 \\ &\le \sum_{\kappa=0}^{\infty} \|\mathcal{P}^{1/2}\|_{2,r}\|u(\kappa)\|_r = \|\mathcal{P}^{1/2}\|_{2,r}|||u|||_{1,r}, \end{aligned}$$

which implies that $|||\mathcal{G}|||_{(2,2),(1,r)} \le \|\mathcal{P}^{1/2}\|_{2,r}$.

Input	Output	Induced Norm	Upper Bound
$\ell_{2,2}$	$\ell_{2,2}$	$\sup_{\omega\in[0,2\pi]} \sigma_{\max}(H(e^{\jmath\omega}))$	
$\ell_{1,r}$	$\ell_{2,2}$	$\|\mathcal{P}^{1/2}\|_{2,r}$	
$\ell_{2,2}$	$\ell_{\infty,p}$	$\|\mathcal{Q}^{1/2}\|_{2,\bar{p}}$	
$\ell_{1,r}$	$\ell_{\infty,p}$	$\sup_{k\in\mathcal{N}} \|G(k)\|_{p,r}$	
$\ell_{r,r}$	$\ell_{\infty,\infty}$	$\max_{i=1,\ldots,l} \|\mathrm{row}_i(G_{[\bar{r},\bar{r}]})\|_{\bar{r}}$	
$\ell_{1,1}$	$\ell_{p,p}$	$\max_{j=1,\ldots,m} \|\mathrm{col}_j(G_{[p,p]})\|_p$	
$\ell_{\infty,2}$	$\ell_{\infty,p}$		$\frac{1}{\sqrt{\rho-1}}\|CQ_\rho C^{\mathrm{T}}\|_{2,\bar{p}}$
$\ell_{\infty,r}$	$\ell_{\infty,p}$		$\frac{1}{\sqrt{\rho}-1}\sup_{k\in\mathcal{N}} \|G_\rho(k)\|_{p,r}$

Table 1: Summary of Induced Operator Norms for $p, r \in [1,\infty]$ and $\rho > 1$

Next, let $u(\cdot) = \hat{u}v(\cdot)$, where $\hat{u} \in \mathbb{R}^d$ is such that $\|\hat{u}\|_r = 1$, $\|\mathcal{P}^{1/2}\hat{u}\|_2 = \|\mathcal{P}^{1/2}\|_{2,r}\|\hat{u}\|_r$, and $v : \mathcal{N} \to \mathbb{R}$ is such that $v(0) = 1$ and $v(k) = 0$, $k > 0$, so that $|||v|||_{1,1} = 1$ which implies that $y(k) = G(k)\hat{u}$, $k \in \mathcal{N}$. Hence,

$$\begin{aligned} |||\mathcal{G}|||_{(2,2),(1,r)} &\ge |||y|||_{2,2} = \left\{ \sum_{k=0}^{\infty} \hat{u}^{\mathrm{T}} G^{\mathrm{T}}(k) G(k) \hat{u} \right\}^{1/2} \\ &= (\hat{u}^{\mathrm{T}}\mathcal{P}\hat{u})^{1/2} = \|\mathcal{P}^{1/2}\hat{u}\|_2 = \|\mathcal{P}^{1/2}\|_{2,r}, \end{aligned}$$

which implies that $|||\mathcal{G}|||_{(2,2),(1,r)} = \|\mathcal{P}^{1/2}\|_{2,r}$.

iii) With $p = r = 2$ it follows from Lemma 2.2 that for all $k \in \mathcal{N}$,

$$\begin{aligned} \|y(k)\|_p &= = \max_{\{\hat{u}\in\mathbb{R}^n:\ \|\hat{u}\|_{\bar{p}}=1\}} \sum_{\kappa=0}^{\infty} \hat{u}^{\mathrm{T}} G(k-\kappa) u(\kappa) \\ &\le \max_{\{\hat{u}\in\mathbb{R}^n:\ \|\hat{u}\|_{\bar{p}}=1\}} \left[\sum_{\kappa=0}^{\infty} \|G^{\mathrm{T}}(k-\kappa)\hat{u}\|_2^2 \right]^{1/2} |||u|||_{2,2} \\ &= \max_{\{\hat{u}\in\mathbb{R}^n:\ \|\hat{u}\|_{\bar{p}}=1\}} \left[\hat{u}^{\mathrm{T}} \sum_{\kappa=0}^{k} G(\kappa) G^{\mathrm{T}}(\kappa) \hat{u} \right]^{1/2} |||u|||_{2,2} \\ &\le \max_{\{\hat{u}\in\mathbb{R}^n:\ \|\hat{u}\|_{\bar{p}}=1\}} (\hat{u}^{\mathrm{T}}\mathcal{Q}\hat{u})^{1/2} |||u|||_{2,2} = \|\mathcal{Q}^{1/2}\|_{2,\bar{p}} |||u|||_{2,2}, \end{aligned}$$

which implies that $|||y|||_{\infty,p} \le \|\mathcal{Q}^{1/2}\|_{2,\bar{p}} |||u|||_{2,2}$ for all $y \in \ell_{\infty,p}$ and $u \in \ell_{2,2}$, and hence $|||\mathcal{G}|||_{(\infty,p),(2,2)} \le \|\mathcal{Q}^{1/2}\|_{2,\bar{p}}$.

Next, let $\hat{u} \in \mathbb{R}^d$ be such that $\|\hat{u}\|_{\bar{p}} = 1$ and $\|\mathcal{Q}^{1/2}\hat{u}\|_2 = \|\mathcal{Q}^{1/2}\|_{2,\bar{p}}$, and let $K > 0$ and

$$u(k) = \frac{1}{\|\mathcal{Q}^{1/2}\|_{2,\bar{p}}} G^{\mathrm{T}}(K-k)\hat{u},$$

so that $|||u|||_{2,2} \le 1$. Now, since $\|\cdot\|_{p\mathrm{D}} = \|\cdot\|_{\bar{p}}$, it follows that

$$\begin{aligned} \|y(k)\|_p &= \max_{\{\hat{y}\in\mathbb{R}^n:\|\hat{y}\|_{\bar{p}}=1\}} \hat{y}^{\mathrm{T}} y(k) \ge \hat{u}^{\mathrm{T}} y(k) \\ &= \frac{1}{\|\mathcal{Q}^{1/2}\|_{2,\bar{p}}} \sum_{\kappa=0}^{\infty} \hat{u}^{\mathrm{T}} G(K-\kappa) G^{\mathrm{T}}(K-\kappa) \hat{u}, \end{aligned}$$

which implies that, for every $K > 0$, there exists $u \in \ell_{2,2}$ such that $|||u|||_{2,2} \le 1$ and

$$|||y|||_{\infty,p} \ge \frac{1}{||\mathcal{Q}^{1/2}||_{2,\bar{p}}} \sup_{k\in\mathcal{N}} \sum_{\kappa=0}^{\infty} \hat{u}^{\mathrm{T}} G(K-\kappa) G^{\mathrm{T}}(K-\kappa)\hat{u}$$
$$\ge \frac{1}{||\mathcal{Q}^{1/2}||_{2,\bar{p}}} \sum_{\kappa=0}^{K} \hat{u}^{\mathrm{T}} G(\kappa) G^{\mathrm{T}}(\kappa)\hat{u},$$

or, equivalently,

$$|||\mathcal{G}|||_{(\infty,p),(2,2)} \ge \sup_{K>0} \frac{1}{||\mathcal{Q}^{1/2}||_{2,\bar{p}}} \sum_{\kappa=0}^{K} \hat{u}^{\mathrm{T}} G(\kappa) G^{\mathrm{T}}(\kappa)\hat{u}$$
$$= \frac{1}{||\mathcal{Q}^{1/2}||_{2,\bar{p}}} \sum_{\kappa=0}^{\infty} \hat{u}^{\mathrm{T}} G(\kappa) G^{\mathrm{T}}(\kappa)\hat{u}$$
$$= \frac{1}{||\mathcal{Q}^{1/2}||_{2,\bar{p}}} \hat{u}^{\mathrm{T}} \mathcal{Q}\hat{u}$$
$$= ||\mathcal{Q}^{1/2}||_{2,\bar{p}},$$

which further implies that $|||\mathcal{G}|||_{(\infty,p),(2,2)} = ||\mathcal{Q}^{1/2}||_{2,\bar{p}}$.

iv) Note that for all $k \in \mathcal{N}$,

$$||y(k)||_p \le \sum_{\kappa=0}^{\infty} ||G(k-\kappa)u(\kappa)||_p$$
$$\le \sum_{\kappa=0}^{\infty} ||G(k-\kappa)||_{p,r}||u(\kappa)||_r$$
$$\le \sup_{k\in\mathcal{N}} ||G(k)||_{p,r}|||u|||_{1,r},$$

which implies that $|||\mathcal{G}|||_{(\infty,p),(1,r)} \le \sup_{k\in\mathcal{N}} ||G(k)||_{p,r}$.

Next, let $\varepsilon > 0$ and $k_0 \in \mathcal{N}$ be such that $||G(k_0)||_{p,r} > \sup_{k\in\mathcal{N}} ||G(k)||_{p,r} - \varepsilon$. In addition, let $u(\cdot) = v(\cdot)\hat{u}$, where $\hat{u} \in \mathbb{R}^n$ is such that $||\hat{u}||_r = 1$, $||G(k_0)\hat{u}||_p = ||G(k_0)||_{p,r}||\hat{u}||_r$, and $v : \mathcal{N} \to \mathbb{R}$ is such that $v(0) = 1$, $v(k) = 0$, $k > 0$, so that $|||v|||_{1,1} = 1$ and $y(k) = G(k)\hat{u}$. Hence,

$$|||\mathcal{G}|||_{(\infty,p),(1,r)} \ge \sup_{k\in\mathcal{N}} ||y(k)||_p = \sup_{k\in\mathcal{N}} ||G(k)\hat{u}||_p$$
$$\ge ||G(k_0)\hat{u}||_p = ||G(k_0)||_{p,r} > \sup_{k\in\mathcal{N}} ||G(k)||_{p,r} - \varepsilon,$$

which implies that

$$\sup_{k\in\mathcal{N}} ||G(k)||_{p,r} - \varepsilon < |||\mathcal{G}|||_{(\infty,p),(1,r)} \le \sup_{k\in\mathcal{N}} ||G(k)||_{p,r},\ \varepsilon > 0,$$

and hence (20) holds.

v) Note that for all $u \in \ell_{r,r}$ and $y \in \ell_{\infty,\infty}$ it follows that $|||u|||_{r,r} = ||\bar{u}||_r$ and $|||y|||_{\infty,\infty} = ||\bar{y}||_\infty$, where $\bar{u} \in \mathbb{R}^m$ and $\bar{y} \in \mathbb{R}^l$ with $\bar{u}_i = |||u_i|||_{r,r}$, $i = 1,\ldots,m$, and $\bar{y}_i = |||y_i|||_{\infty,\infty}$, $i = 1,\ldots,l$. Next, it follows from Lemma 3.1 that $|||\mathcal{G}_{(i,j)}|||_{(\infty,\infty),(r,r)} = |||G_{(i,j)}|||_{\bar{r},\bar{r}}$ and hence

$$|||y_i|||_{\infty,\infty} = |||\sum_{j=1}^{m} \mathcal{G}_{(i,j)} * u_j|||_{\infty,\infty} \le \sum_{j=1}^{m} |||\mathcal{G}_{(i,j)} * u_j|||_{\infty,\infty}$$
$$\le \sum_{j=1}^{m} |||\mathcal{G}_{(i,j)}|||_{\bar{r},\bar{r}}|||u_j|||_{r,r} \le ||\mathrm{row}_i(G_{[\bar{r},\bar{r}]})||_{\bar{r}}||\bar{u}||_r$$
$$\le \max_{i=1,\ldots,l} ||\mathrm{row}_i(G_{[\bar{r},\bar{r}]})||_{\bar{r}}||\bar{u}||_r,$$

which implies that

$$|||y|||_{\infty,\infty} = ||\bar{y}||_\infty \le \max_{i=1,\ldots,l} ||\mathrm{row}_i(G_{[\bar{r},\bar{r}]})||_{\bar{r}}|||u|||_{r,r}$$

and hence,

$$|||\mathcal{G}|||_{(\infty,\infty),(r,r)} \le \max_{i=1,\ldots,l} ||\mathrm{row}_i(G_{[\bar{r},\bar{r}]})||_{\bar{r}}. \tag{30}$$

Next, let $I \in \{1,\ldots,l\}$ be such that $||\mathrm{row}_I(G_{[\bar{r},\bar{r}]})||_{\bar{r}} = \max_{i=1,\ldots,l} ||\mathrm{row}_i(G_{[\bar{r},\bar{r}]})||_{\bar{r}}$. Now, let $\hat{u} \in \mathbb{R}^m$ be such that $||\hat{u}||_r = 1$, let $\mathrm{row}_I(G_{[\bar{r},\bar{r}]})\hat{u} = ||\mathrm{row}_I(G_{[\bar{r},\bar{r}]})||_{\bar{r}}$, and let $u_j \in \ell_{r,r}$, $j = 1,\ldots,m$, be such that $|||u_j|||_{r,r} = \hat{u}_j$ and $\lim_{k\to\infty}(\mathcal{G}_{(I,j)} * u_j)(k) = |||G_{(I,j)}|||_{\bar{r},\bar{r}}|||u_j|||_{r,r}$. Note that existence of such a $u_j(\cdot)$ follows from Lemma 3.1 and Remark 3.1. Now,

$$|||y|||_{\infty,\infty} \ge |||y_I|||_{\infty,\infty} \ge \lim_{k\to\infty} |\sum_{j=1}^{m}(\mathcal{G}_{(I,j)} * u_j)(k)|$$
$$= \sum_{j=1}^{m} |||G_{(I,j)}|||_{\bar{r},\bar{r}}|||u_j|||_{r,r} = \mathrm{row}_I(G_{[\bar{r},\bar{r}]})\hat{u} = ||\mathrm{row}_I(G_{[\bar{r},\bar{r}]})||_{\bar{r}},$$

which, with (30), implies (21).

vi) For $p = \infty$, (22) is a direct consequence of *iv)* or *v)*. Now, let $p \in [1,\infty)$ and note that it follows from Lemma 2.3 that $|||y|||_{p,p} = \sup_{\{\hat{y}\in\ell_{\bar{p},\bar{p}}:\ |||\hat{y}|||_{\bar{p},\bar{p}}=1\}} \langle y,\hat{y}\rangle$. Hence, with $p = r = 1$ it follows from Lemma 2.2 that

$$|||y|||_{p,p} = \sup_{|||\hat{y}|||_{\bar{p},\bar{p}}=1} \sum_{k=0}^{\infty} y^{\mathrm{T}}(k)\hat{y}(k)$$
$$= \sup_{|||\hat{y}|||_{\bar{p},\bar{p}}=1} \sum_{k=0}^{\infty} \left(\sum_{\kappa=0}^{\infty} u^{\mathrm{T}}(\kappa)G^{\mathrm{T}}(k-\kappa)\right)\hat{y}(k)$$
$$= \sup_{|||\hat{y}|||_{\bar{p},\bar{p}}=1} \sum_{\kappa=0}^{\infty} u^{\mathrm{T}}(\kappa)\left(\sum_{k=0}^{\infty} G^{\mathrm{T}}(k-\kappa)\hat{y}(k)\right)$$
$$= \sup_{|||\hat{y}|||_{\bar{p},\bar{p}}=1} \langle u,\hat{u}\rangle$$
$$\le |||u|||_{1,1} \sup_{|||\hat{y}|||_{\bar{p},\bar{p}}=1} |||\hat{u}|||_{\infty,\infty},$$

where $\hat{u}(k) \triangleq \sum_{\kappa=0}^{\infty} G^{\mathrm{T}}(\kappa-k)\hat{y}(\kappa)$. Now, with $r = \bar{p}$, it follows from *v)* that

$$|||\mathcal{G}|||_{(p,p),(1,1)} \le \sup_{|||\hat{y}|||_{\bar{p},\bar{p}}=1} |||\hat{u}|||_{\infty,\infty} = \max_{j=1,\ldots,m} ||\mathrm{col}_j(G_{[p,p]})||_p. \tag{31}$$

Next, let $J \in \{1,\ldots,m\}$ be such that $||\mathrm{col}_J(G_{[p,p]})||_p = \max_{j=1,\ldots,m} ||\mathrm{col}_j(G_{[p,p]})||_p$ and let $u(\cdot) \triangleq v(\cdot)e_J$, where $v : \mathcal{N} \to \mathbb{R}$ is such that $v(0) = 1$, $v(k) = 0$, $k > 0$, so that $|||v|||_{1,1} = 1$ and $y(k) = \mathrm{col}_J(G(k))$, $k \in \mathcal{N}$. Hence,

$$|||\mathcal{G}|||_{(p,p),(1,1)} \ge |||y|||_{p,p} = |||\mathrm{col}_J(G)|||_{p,p}$$
$$= ||\mathrm{col}_J(G_{[p,p]})||_p = \max_{j=1,\ldots,m} ||\mathrm{col}_j(G_{[p,p]})||_p,$$

which, with (31), implies (22). □

Appendix B. Proof of Theorem 4.1

Let $K > 0$, $u \in \ell_{\infty,2}$, and define

$$u_K(k) \triangleq \begin{cases} \rho^{\frac{k-K}{2}} u(k), & 0 \le k < K, \\ 0, & k \ge K. \end{cases} \quad (32)$$

Now, note that

$$\|u_K\|_{2,2}^2 = \sum_{k=0}^{K-1} \rho^{k-K} \|u(k)\|_2^2 \le \frac{1}{\rho - 1} \|u\|_{\infty,2}^2,$$

or, equivalently, $\|u_K\|_{2,2} \le \frac{1}{\sqrt{\rho-1}} \|u\|_{\infty,2}$. Next, define $y_K(k) \triangleq \rho^{\frac{k-K}{2}} y(k)$ and note that

$$\begin{aligned} y_K(k) &= \sum_{\kappa=0}^{\infty} \rho^{\frac{k-K}{2}} G(k-\kappa) u(\kappa) \\ &= \sum_{\kappa=0}^{\infty} \rho^{\frac{k-\kappa}{2}} G(k-\kappa) \rho^{\frac{\kappa-K}{2}} u(\kappa) \\ &= \sum_{k=0}^{\infty} G_\rho(k-\kappa) u_K(\kappa) = (\mathcal{G}_\rho * u_K)(k). \end{aligned}$$

Hence,

$$\begin{aligned} \|y_K(k)\|_p &\le \|\mathcal{G}_\rho\|_{(\infty,p),(2,2)} \|u_K\|_{2,2} \\ &\le \frac{1}{\sqrt{\rho-1}} \|\mathcal{G}_\rho\|_{(\infty,p),(2,2)} \|u\|_{\infty,2}. \end{aligned}$$

Now, noting that $y(K) = y_K(K)$ it follows that

$$\|y(K)\|_p \le \frac{1}{\sqrt{\rho-1}} \|\mathcal{G}_\rho\|_{(\infty,p),(2,2)} \|u\|_{\infty,2}, \qquad K \ge 0,$$

which implies (26).

To show (27) let $K > 0$, $u \in \ell_{\infty,r}$, and let $u_K(\cdot)$ be given by (32). Then

$$\|u_K\|_{1,r} = \sum_{k=0}^{K-1} \rho^{\frac{k-K}{2}} \|u(k)\|_r \le \frac{1}{\sqrt{\rho}-1} \|u\|_{\infty,r}.$$

Thus,

$$\begin{aligned} \|y_K(k)\|_p &\le \|\mathcal{G}_\rho\|_{(\infty,p),(1,r)} \|u_K\|_{1,r} \\ &\le \frac{1}{\sqrt{\rho}-1} \|\mathcal{G}_\rho\|_{(\infty,p),(1,r)} \|u\|_{\infty,r}. \end{aligned}$$

Hence, since $y(K) = y_K(K)$,

$$\|y(K)\|_p \le \frac{1}{\sqrt{\rho}-1} \|\mathcal{G}_\rho\|_{(\infty,p),(1,r)} \|u\|_{\infty,r}, \qquad K \ge 0,$$

which implies (27). □

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