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Output Feedback Adaptive Stabilization of Second-Order Systems[1]

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Abstract

We consider output feedback adaptive stabilization for second-order systems with no zeros. The assumptions we make are standard, namely, that the sign of the high frequency gain is known. However, we complement the existing literature by deriving an explicit expression for the adaptive controller. The controller has the form of a 6th-order dynamic compensator with quadratic, cubic and quartic nonlinearities. The proof of convergence is based on a variation of Lyapunov's method in which the Lyapunov derivative is shown to be asymptotically nonpositive. Application of the controller to the Van der Pol and Duffing oscillators shows that the controller is effective for nonlinear systems as well.

1 Introduction

In this paper we consider the problem of adaptive stabilization for second-order systems (with no zeros) under output feedback. As in [4, 5] we assume that the sign of the high frequency gain is known. However, our results extend the results of [4, 5] in two distinct ways. First, we derive an explicit expression for the adaptive controller which involves parameter estimates and filtered states. The overall controller has the form of a 6th-order dynamic compensator with quadratic, cubic and quartic nonlinearities. In addition, our proof of convergence is Lyapunov based. In particular, we develop a variation of Lyapunov's method in which the Lyapunov derivative is shown to be asymptotically nonpositive.

The contents of the paper are as follows. In Section 2 we introduce the stabilization problem for second order systems with relative degree 2. We state and prove Theorem 1 which provides the sixth-order dynamic compensator which stabilizes any second order system with relative degree 2 and known high frequency gain. In this section we provide a elaborate proof of convergence of states and boundedness of parameter estimates.

In Section 3 we present several numerical examples involving linear and nonlinear plants. In particular, we apply the controller to the Van der Pol and Duffing systems to show that the controller is effective for nonlinear systems as well.

2 Adaptive Stablization Problem

Consider the second -order system

$$\ddot{q} + a_1\dot{q} + a_2 q = bu, \tag{1}$$

where a_1, a_2 and b are constant. We make the following assumptions about (1). We assume that a_1 and a_2 are completely unknown. Furthermore, we assume that b is nonzero and sign(b) is known but b is otherwise unknown. Finally, we assume that only q is available for feedback. Let $m \triangleq 1/b$. The stabilization problem constitutes finding a control input u such that q, $\dot{q}$ converge to 0 as $t \to \infty$.

Theorem 1 Let λ, f_1, f_2, r_1, r_2, g_1 and g_2 be positive constants and let $p_1 = r_1/(2f_2)$ and $p_2 = (r_1 + f_2 r_2)/(2f_1 f_2)$. Consider the dynamic compensator

$$\begin{aligned} \dot{q}_{\rm f} &= -\lambda q_{\rm f} + q, & (2)\\ \ddot{\hat{q}}_{\rm f} &= -f_1\dot{\hat{q}}_{\rm f} - f_2\hat{q}_{\rm f} + \\ &\quad +(f_1 - \hat{a}_1)\dot{q}_{\rm f} + (f_2 - \hat{a}_2)q_{\rm f} + u_{\rm f}, & (3)\\ \dot{\hat{a}}_1 &= -(p_1 e + p_2\dot{e})\dot{q}_{\rm f}, & (4) \end{aligned}$$

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$$\dot{\hat{a}}_2 = -(p_1 e + p_2\dot{e})q_f, \quad (5)$$
$$\dot{\hat{m}} = -\text{sign}(b)(p_1 e + p_2\dot{e})u_f, \quad (6)$$

where

$$e \triangleq q_f - \hat{q}_f \quad (7)$$
$$u_f \triangleq (\hat{a}_1 - g_1)\dot{\hat{q}}_f + (\hat{a}_2 - g_2)\hat{q}_f. \quad (8)$$

Let the control input u be given by

$$u = \hat{m}(\hat{a}_1 - g_1)\ddot{\hat{q}}_f + \hat{m}[\dot{\hat{a}}_1 + \hat{a}_2 - g_2]\dot{\hat{q}}_f + (\dot{\hat{m}} + \lambda\hat{m})((\hat{a}_1 - g_1)\dot{\hat{q}}_f + (\hat{a}_2 - g_2)\hat{q}_f) + \hat{m}\dot{\hat{a}}_2\hat{q}_f. \quad (9)$$

Then q, $\dot{q}$, q_f, $\dot{q}_f$, $\hat{q}_f$ and $\dot{\hat{q}}_f \to 0$ as $t \to \infty$. Furthermore, $\hat{a}_1$, $\hat{a}_2$ and $\hat{m}$ are bounded. □

Proof: It follows from (9) that $\bar{u}_f \triangleq \hat{m}u_f$ satisfies

$$\dot{\bar{u}}_f = -\lambda\bar{u}_f + u. \quad (10)$$

Next, defining

$$H \triangleq \ddot{q}_f + a_1\dot{q}_f + a_2 q_f - b\bar{u}_f, \quad (11)$$

and using (1),(2) and (10), it follows that H satisfies $\dot{H} = -\lambda H$. Hence $H(t) = H(0)e^{-\lambda t}$. Equation (11) can be rewritten as

$$\ddot{q}_f + a_1\dot{q}_f + a_2 q_f = \frac{\hat{m}}{m}u_f + H. \quad (12)$$

Subtracting (3) from (12) yields

$$\ddot{e} + f_1\dot{e} + f_2 e = \tilde{a}_1\dot{q}_f + \tilde{a}_2 q_f + \frac{\tilde{m}}{m}u_f + H, \quad (13)$$

where $\tilde{a}_1 \triangleq \hat{a}_1 - a_1$, $\tilde{a}_2 \triangleq \hat{a}_2 - a_2$ and $\tilde{m} \triangleq \hat{m} - m$. Next, (13) can be rewritten as

$$\dot{E} = FE + \begin{bmatrix} 0 \\ \tilde{a}_2 q_f + \tilde{a}_1\dot{q}_f + \frac{\tilde{m}}{m}u_f + H \end{bmatrix}, \quad (14)$$

where $E \triangleq [e \quad \dot{e}]$ and $F \triangleq \begin{bmatrix} 0 & 1 \\ -f_2 & -f_1 \end{bmatrix}$. Let P denote the solution of the Lyapunov equation $F^{\mathrm{T}}P + PF = -R$ where $R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$. Then P is given by $P = \begin{bmatrix} p_0 & p_1 \\ p_1 & p_2 \end{bmatrix}$, where $p_0 = (f_1 r_1 + f_1 f_2 r_2 + f_1^2 r_2)/(2f_1 f_2)$.

Next, we define $\tilde{x} \triangleq [e, \dot{e}, \tilde{a}_1, \tilde{a}_2, \tilde{m}]^{\mathrm{T}}$ and the positive definite function

$$V(\tilde{x}) \triangleq \frac{1}{2}E^{\mathrm{T}}PE + \frac{1}{2}\tilde{a}_1{}^2 + \frac{1}{2}\tilde{a}_2{}^2 + \frac{1}{2|m|}\tilde{m}^2. \quad (15)$$

Then using parameter update laws (4), (5) and (6) and using $eHp_1 \le e^2 r_1/4 + H^2 p_1^2/r_1$ and $\dot{e}Hp_2 \le \dot{e}^2 r_2/4 + H^2 p_2^2/r_2$, we obtain

$$\begin{aligned}
\dot{V}(\tilde{x}) &= -\frac{1}{2}E^{\mathrm{T}}RE + \tilde{a}_1(\dot{\hat{a}}_1 + p_1 e\dot{q}_f + p_2\dot{e}\dot{q}_f) \\
&\quad + \tilde{a}_2(\dot{\hat{a}}_2 + p_1 e q_f + p_2\dot{e}q_f) \\
&\quad + \frac{\tilde{m}}{m}(\text{sign}(b)\dot{\hat{m}} + (p_1 e + p_2\dot{e})u_f) + H(p_1 e + p_2\dot{e}) \\
&= -\frac{1}{2}r_1 e^2 - \frac{1}{2}r_2\dot{e}^2 + H(p_1 e + p_2\dot{e}) \\
&\le -\frac{1}{4}r_1 e^2 - \frac{1}{4}r_2\dot{e}^2 + \left(\frac{p_1^2}{r_1} + \frac{p_2^2}{r_2}\right)H^2 \\
&\le \left(\frac{p_1^2}{r_1} + \frac{p_2^2}{r_2}\right)H^2(0)e^{-2\lambda t}.
\end{aligned}$$

Let $V_0 \triangleq V + \frac{\alpha}{2\lambda}e^{-2\lambda t}$, where $\alpha = (\frac{p_1^2}{r_1} + \frac{p_2^2}{r_2})H^2(0)$. Then

$$\dot{V}_0 = \dot{V} - \alpha e^{-2\lambda t} \le \alpha e^{-2\lambda t} - \alpha e^{-2\lambda t} = 0$$

Since $\lambda > 0$ and $\alpha \ge 0$, it follows that V_0 is positive definite and monotonically decreasing. Hence V_0 has a limit as $t \to \infty$. Since $\lim_{t\to\infty} \frac{\alpha}{2\lambda}e^{-2\lambda t} = 0$, V has a limit as $t \to \infty$. Hence all components of $\tilde{x}$, namely e, $\dot{e}$, $\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{m}$ are bounded.

Next, note that

$$\int_0^T \frac{1}{4}r_1 e^2 dt + \int_0^T \frac{1}{4}r_2\dot{e}^2 dt \le V(\tilde{x}(0)) - V(\tilde{x}(T)) + \left(\frac{p_1^2}{r_1} + \frac{p_2^2}{r_2}\right)\int_0^T H(t)^2 dt, \quad (16)$$

along the trajectories of the closed-loop system. Noting that the right-hand-side of inequality (16) is bounded as $T \to \infty$, it follows that $e(t)$ and $\dot{e}(t)$ are square integrable on $[0, \infty)$. Since $\dot{e}(t)$ is bounded for all time, $e(t) \to 0$ as $t \to \infty$ (see Lemma A.1).

To prove convergence of q, $\dot{q}$, q_f, $\dot{q}_f$, $\hat{q}_f$ and $\dot{\hat{q}}_f$, we first prove boundedness. Using (8), (3) yields

$$\ddot{\hat{q}}_f + g_1\dot{\hat{q}}_f + g_2\hat{q}_f = (f_2 - \hat{a}_2)e + (f_1 - \hat{a}_1)\dot{e} \quad (17)$$

Note that since $\tilde{x}$ is bounded, the right-hand-side of (17) is bounded. Since the polynomial $p^2 + g_1 p + g_2$ is Hurwitz, $\hat{q}_f$ and $\dot{\hat{q}}_f$ are bounded. Furthermore, since e and $\dot{e}$ are bounded, the states q_f and $\dot{q}_f$ are bounded. Using (8), it can be seen that u_f is bounded as well. Therefore all terms on the right-hand-side of (13) are bounded, which implies that $\ddot{e}$ is bounded. Since $\dot{e}$ is square integrable, $\dot{e} \to 0$ as $t \to \infty$ (see Lemma A.1).

It follows that the right-hand-side of (17) converges to 0 as $t \to \infty$. Hence we conclude that estimator states $\hat{q}_{\mathrm{f}}$ and $\dot{\hat{q}}_{\mathrm{f}}$ converge to 0 as $t \to \infty$. Hence the filter states q_{f} and $\dot{q}_{\mathrm{f}}$ go to 0 as $t \to \infty$ and hence using (2) it follows that $q \to 0$. Using (8) it follows that $u_{\mathrm{f}} \to 0$ as $t \to \infty$. Furthermore, from (12) we note that $\ddot{q}_{\mathrm{f}} \to 0$ as $t \to \infty$. Differentiating (2) yields

$$\ddot{q}_{\mathrm{f}} = -\lambda \dot{q}_{\mathrm{f}} + \dot{q}, \tag{18}$$

which implies $\dot{q} \to 0$ as $t \to \infty$. ■

The controller (2)-(9) can be identified as the combination of three essential modules. Firstly, a stable filter represented by (2) filters the available feedback variable q. The filter state q_{f} mimics the second-order plant as indicated by (12). Secondly, the estimator with states $\hat{q}_{\mathrm{f}}$, $\dot{\hat{q}}_{\mathrm{f}}$ facilitates the use of a certainty-equivalence control input given by (8). Lastly, equations (4)-(6) constitute the parameter update laws.

Since q is available for feedback and q_{f} is obtained from the filter (2), $\dot{q}_{\mathrm{f}}$ can be computed using (2). Since $\hat{q}_{\mathrm{f}}$ and $\dot{\hat{q}}_{\mathrm{f}}$ are available from (3), the quantities e, $\dot{e}$ can be computed. Therefore, parameter update laws (4)-(6) and equation (8) are implementable. Lastly, $\ddot{\hat{q}}_{\mathrm{f}}$ required in (9) can be computed using (3). Hence, no differentiation or improper realization is required to implement the controller.

Finally, the controller implementation only requires q for feedback and does not need to know a_1, a_2 or the value of b. However, knowledge of sign(b) is required to implement (6).

The control input u explicitly in terms of the available states q, q_{f}, $\hat{q}_{\mathrm{f}}$, $\dot{\hat{q}}_{\mathrm{f}}$, $\hat{a}_1$, $\hat{a}_2$ and $\hat{m}$ is given by

$$\begin{aligned} u = {} & \hat{m}(\hat{a}_2 g_1 q_{\mathrm{f}} - f_2 g_1 q_{\mathrm{f}} - \hat{a}_2 g_1 \hat{q}_{\mathrm{f}} + f_2 g_1 \hat{q}_{\mathrm{f}} + \\ & g_1 g_2 \hat{q}_{\mathrm{f}} + \hat{a}_2 \lambda \hat{q}_{\mathrm{f}} - g_2 \lambda \hat{q}_{\mathrm{f}} - p_2 q \hat{q}_{\mathrm{f}}^2 - p_2 q_{\mathrm{f}} \hat{q}_{\mathrm{f}}^2 + \\ & p_2 \hat{q}_{\mathrm{f}}^3 + \hat{a}_2 \dot{\hat{q}}_{\mathrm{f}} - \hat{a}_2 g_1 \dot{\hat{q}}_{\mathrm{f}} + g_1^2 \hat{q}_{\mathrm{f}} - g_2 \dot{\hat{q}}_{\mathrm{f}} - \\ & g_1 \lambda \dot{\hat{q}}_{\mathrm{f}} + p_2 \hat{q}_{\mathrm{f}}^2 \dot{\hat{q}}_{\mathrm{f}} - p_2 q \dot{\hat{q}}_{\mathrm{f}}^2 - p_2 q_{\mathrm{f}} \dot{\hat{q}}_{\mathrm{f}}^2 + p_2 \hat{q}_{\mathrm{f}} \dot{\hat{q}}_{\mathrm{f}}^2 + \\ & p_2 \dot{\hat{q}}_{\mathrm{f}}^3 + p_2 q_{\mathrm{f}} \hat{q}_{\mathrm{f}}^2 r + p_2 q_{\mathrm{f}} \dot{\hat{q}}_{\mathrm{f}}^2 r + \hat{a}_1^2(-q + q_{\mathrm{f}} r) + \\ & f_1 g_1(-q + \dot{\hat{q}}_{\mathrm{f}} + q_{\mathrm{f}} r) + \hat{a}_1(g_1 q - \hat{a}_2 q_{\mathrm{f}} + f_2 q_{\mathrm{f}} + \\ & \hat{a}_2 \hat{q}_{\mathrm{f}} - f_2 \hat{q}_{\mathrm{f}} - g_2 \hat{q}_{\mathrm{f}} + \hat{a}_2 \dot{\hat{q}}_{\mathrm{f}} - g_1 \dot{\hat{q}}_{\mathrm{f}} + \lambda \dot{\hat{q}}_{\mathrm{f}} - \\ & g_1 q_{\mathrm{f}} r + f_1(q - \dot{\hat{q}}_{\mathrm{f}} - q_{\mathrm{f}} r))) + (\hat{a}_2 \hat{q}_{\mathrm{f}} - g_2 \hat{q}_{\mathrm{f}} + \\ & (\hat{a}_1 - g_1) \dot{\hat{q}}_{\mathrm{f}})(-g_1 q + \hat{a}_2 q_{\mathrm{f}} - g_2 q_{\mathrm{f}} + g_1 q_{\mathrm{f}} r + \\ & \hat{a}_1(q - q_{\mathrm{f}} r))(p_2(-q_{\mathrm{f}} + \hat{q}_{\mathrm{f}}) + p_2(-q + \dot{\hat{q}}_{\mathrm{f}} + \\ & q_{\mathrm{f}} r))\mathrm{sign}(b) \end{aligned}$$

Note that the control contains square, cubic and quartic nonlinear terms.

3 Numerical Examples

Consider the second-order unstable system

$$\ddot{q} - 4\dot{q} + 10.5q = -0.5u. \tag{19}$$

For adaptive control, we choose $\lambda = 10$, $f_1 = g_1 = 11$ and $f_2 = g_2 = 36$. The closed-loop response shown in Figure 1 indicates that the algorithm successfully stabilizes the unstable system. The controller is turned on at $t = 2.0$ sec. The time-history of the parameter estimates $\hat{a}_1$, $\hat{a}_2$ and $\hat{m}$ (Figure 2) shows that the estimates converge to a constant value which are not the true values of the parameters a_1, a_2 and m. This is consistent with Theorem 1.

Next, we investigate the effects of a change in the value of b by changing b from $b = -0.5$ to $b = -0.04$ at time $t = 3.0$ sec. The controller is turned on at $t = 1.6$ sec (Figure 3). After the change in value of b, we observe a small transient due to a reduction of control authority, following which the output of the system converges to 0. The parameter estimates $\hat{a}_1$, $\hat{a}_2$ and $\hat{m}$ (Figure 4) converge to constant values.

Next, we consider Van der Pol's oscillator given by

$$\ddot{q} + \varepsilon(q^2 - 1)\dot{q} + \omega^2 q = bu. \tag{20}$$

The controller is turned on when the system approaches the limit cycle as indicated by the phase portrait of the system (Figure 6). Although Theorem 1 applies to second-order systems with constant coefficients, Figure 5 shows that the controller is able to suppress the limit cycle oscillations.

Lastly, we consider a second order mass-spring-damper system with a nonlinear (Duffing) spring described by the dynamical equation

$$\ddot{q} + \varepsilon\dot{q} + (q^2 - 1)q = bu. \tag{21}$$

The uncontrolled system has three equilibria, namely $(q, \dot{q}) = (0,0), (-1,0), (1,0)$. The origin is an unstable (saddle) equilibrium, whereas the equilibria $(\pm 1, 0)$ are stable (foci). In the simulation, the system is allowed to approach to one of the stable equilibria before the controller is turned on (Figure 7, Figure 8). The phase portrait of the system indicates that the controller is able to bring the system to the origin.

4 Conclusions

A sixth-order adaptive controller is developed for stabilizing second-order plants with relative degree 2.

The controller requires the knowledge of the sign of high frequency gain b. The proof of convergence involves a positive definite function with an asymptotically non-positive time derivative. As an extension of this work, current research focuses on generalizing this method of proof to higher order systems with arbitrary relative degree.

A Proofs and Lemmas

Lemma A.1 *Let $x : [0,\infty) \to \mathbb{R}$ be C^1 and square integrable on $[0,\infty)$, and assume that $\dot{x}$ is bounded. Then $x(t) \to 0$ as $t \to \infty$.*

Proof: Let $M > 0$ satisfy $|\dot{x}(t)| \le M$ for all $t \ge 0$. Next, note that

$$\begin{aligned} |x^3(t+h) - x^3(t)| &\le \int_t^{t+h} 3|x^2(t)\dot{x}(t)|dt \\ &\le 3M \int_t^{t+h} |x^2(t)|dt \qquad (22) \end{aligned}$$

Since x is square integrable, it follows that, for all $\varepsilon > 0$, we can choose sufficiently large $T > 0$ such that

$$\int_T^{T+h} |x^2(t)| \le \frac{\varepsilon}{3M}. \qquad (23)$$

Hence $|x^3(t+h) - x^3(t)| \le \varepsilon$ for all $t > T$. Since h is arbitrary, it follows that $x^3(t)$ has a limit L as $t \to \infty$. Hence $x(t) \to L^{\frac{1}{3}}$ as $t \to \infty$. Since x is square integrable, L must be 0. ■

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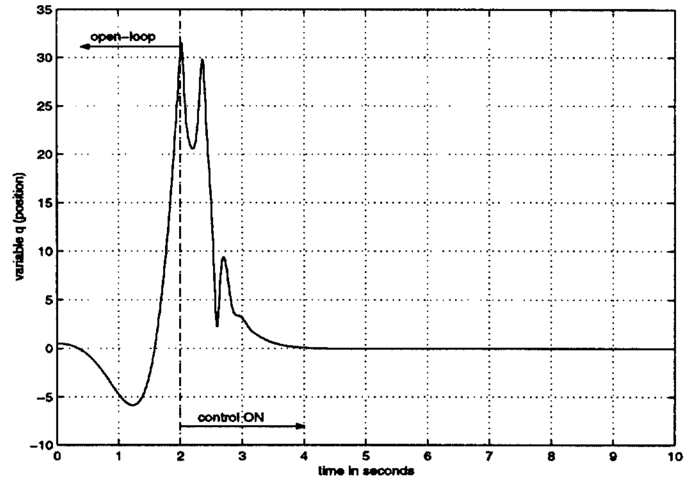

Figure 1: Adaptive stabilization. Closed-loop response of the unstable second-order plant (19).

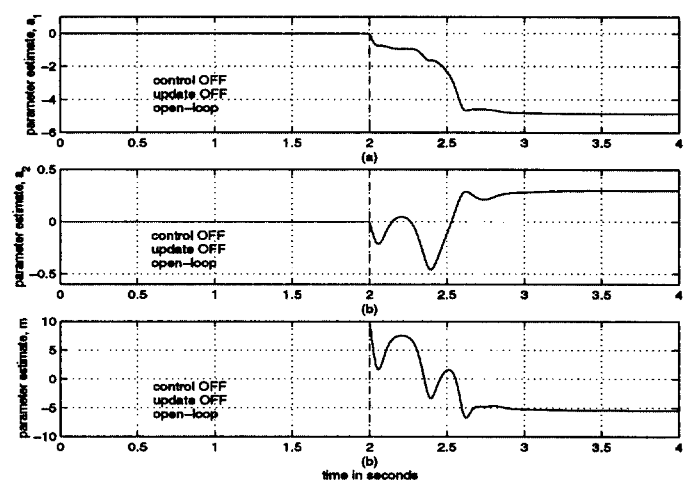

Figure 2: Time-history of parameter estimates (a) $\hat{a}_1$, (b) $\hat{a}_2$ and (c) $\hat{m}$ corresponding to Figure 1.

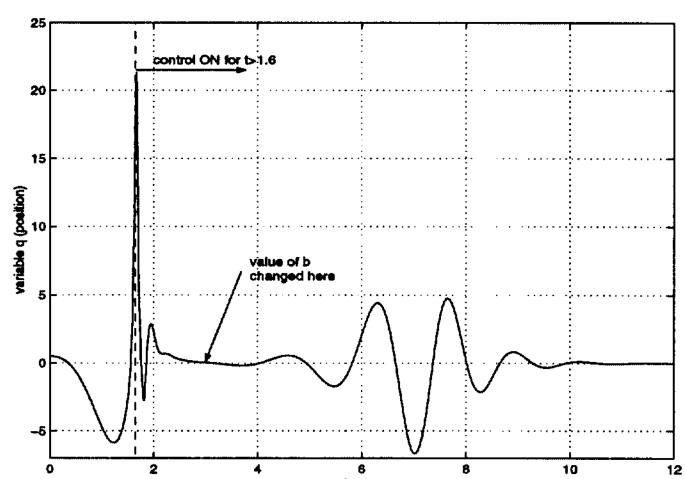

Figure 3: Adaptive stabilization with change in b. Closed-loop response of the unstable second-order plant (19) with change in b from $b = -0.5$ to $b = -0.04$ at arbitrary time $t = 3.0$ sec.

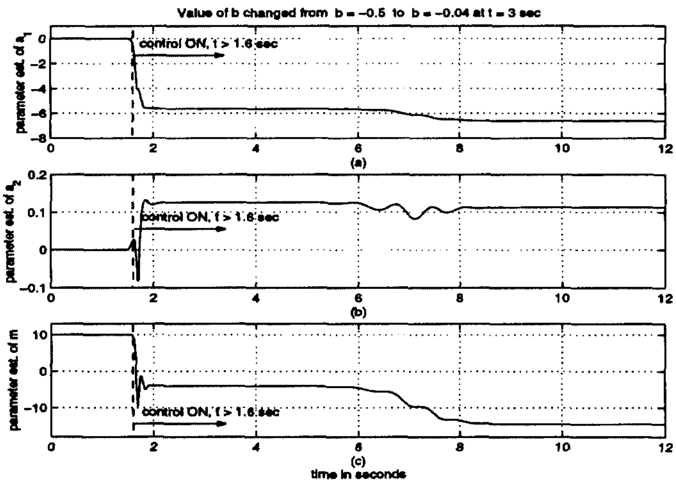

Figure 4: Time-history of parameter estimates (a) $\hat{a}_1$, (b) $\hat{a}_2$ and (c) $\hat{m}$ corresponding to Figure 3.

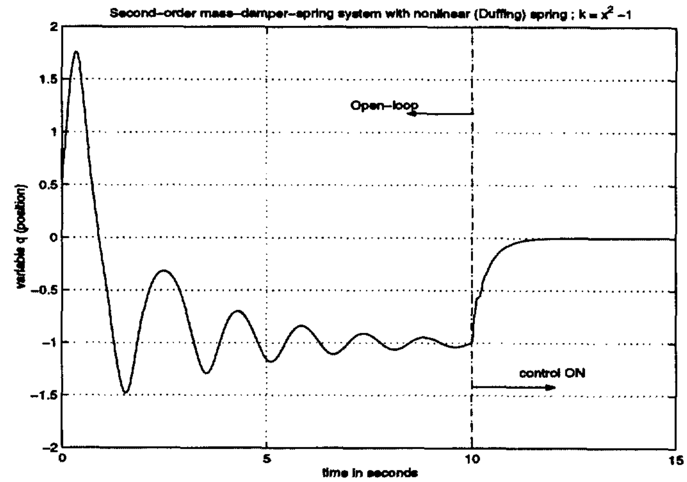

Figure 7: Response (q) of Duffing's mass-spring-damper system with nonlinear spring (21). The controller is turned on at $t = 10.0$ sec.

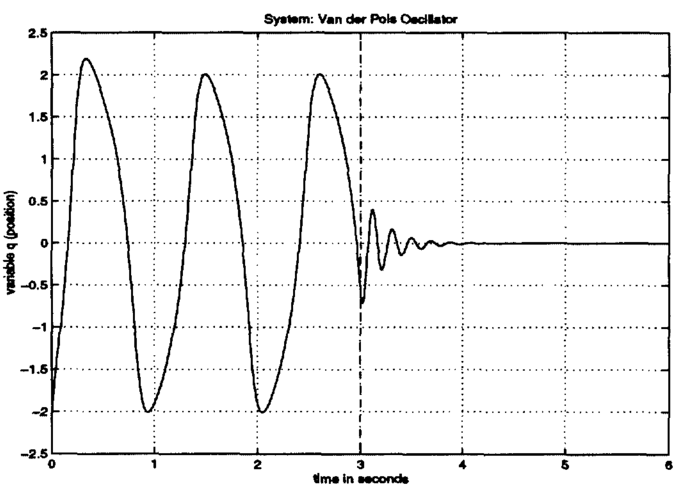

Figure 5: Response (q) of Van der Pol's oscillator (20). The controller is turned on at $t = 3.0$ sec.

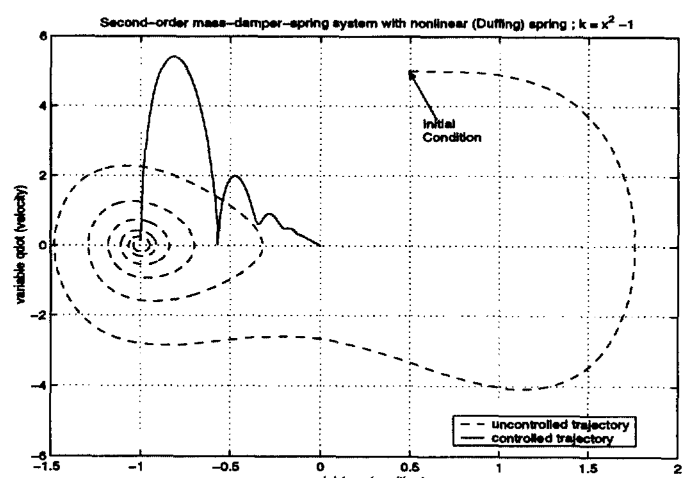

Figure 8: Phase portrait of mass-spring-damper system with nonlinear spring (21). The system is allowed to approach the equilibrium point $(-1, 0)$ before the controller is turned on at $t = 10.0$ sec.

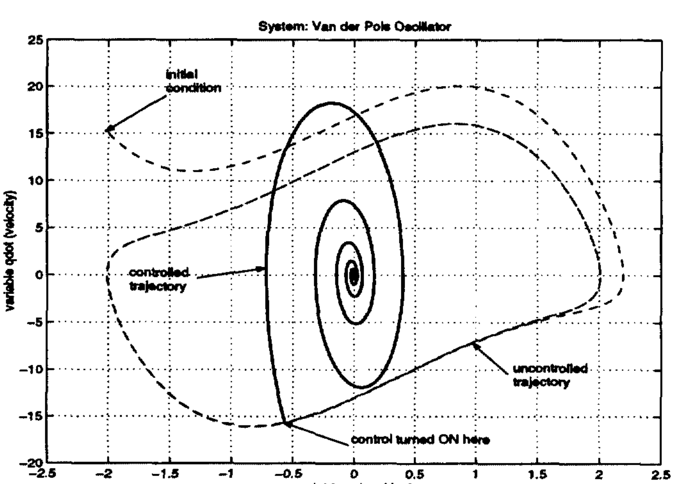

Figure 6: Phase portrait of Van der Pol's oscillator (20). The system is allowed to reach the limit cycle before the controller is turned on at $t = 3.0$ sec.