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On the Zeros of Asymptotically Stable Serially Connected Structures

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***Abstract*— A serially interconnected N-mass system is considered. Specifically, we consider the single-input single-output (SISO) compliance from the force applied on any mass to the position of any mass. The main result of the paper is a result showing that all SISO compliances of serially connected structures are strictly minimum phase. Lastly, we present a dynamic compensator that is constructed using generalized root locus principles and the Fibonacci series. The compensator requires only limited knowledge of the plant and provides infinite upward gain margin.**

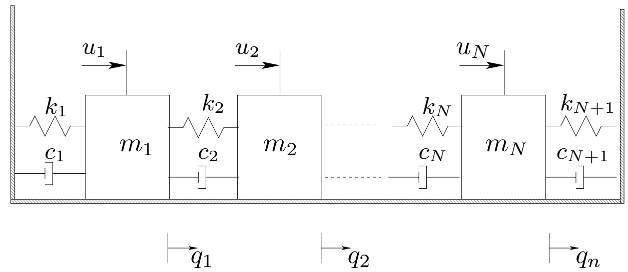

Fig. 1. N-Mass System

I. INTRODUCTION

It is well known that one of the main impediments to achievable performance in linear time-invariant control systems is the presence of nonminimum phase zeros [1, 2]. Open right half plane zeros cause peaking in the sensitivity function, and they limit gain margins for robust stability.

For noise and vibration control applications, stability robustness benefits from sensor/actuator colocation, although achievable performance can be improved by separating the control input from the measurement signal [3]. For colocated hardware, it is well known that the transfer function is minimum phase; in fact, force-to-velocity transfer functions are positive real. However, for the case of a noncolocated arrangement of control hardware it is of interest to know whether the resulting transfer function is minimum or nonminimum phase.

The role of nonminimum phase zeros in limiting both achievable performance and robust stability suggests the importance of understanding the mechanisms that give rise to such zeros in flexible structures. This issue is discussed in [4], where it is shown that nonminimum phase zeros arise in noncolocated transfer functions for beam models when multiple mechanisms are involved for energy transfer, for example, bending and torsion. Furthermore, it is shown in [5] that nonminimum phase zeros arise in noncolocated transfer functions for beam models when the dynamics are dispersive, as occurs in bending.

In the present paper we consider the existence of nonminimum phase zeros within the context of lumped structures. Specifically, we consider mass-spring-dashpot systems with serial connections, that is, in the form of a string of masses interconnected by springs and dashpots. This structural configuration provides an approximation for a beam in compression, and is also useful in modeling the dynamics of a string of vehicles with pairwise control loops. For this class of systems, we show that every force-to-motion transfer function between every pair of masses is minimum phase, while the relative degree of each such transfer function is a simple function of the number of intervening masses.

Since the relative degree of the noncolocated transfer functions can be as large as the number of masses, root locus analysis shows that high-gain feedback for improving structural response can be destabilizing. We therefore apply a high-gain controller that requires only a bound on the relative degree. This controller, developed in [6], is an extension of the approach of [7] to the case in which the relative degree is bounded but otherwise unknown.

The contents of the paper are as follows. In Section 2, we present the dynamics for a serially connected N-mass structure. Section 3 presents a simple formula for the relative degree of every force-to-position compliance transfer function. In Section 4, the zeros of the compliance are shown to be strictly minimum phase. A high-gain controller that improves structural response is presented in Section 5. Our conclusions are given in Section 7.

II. N-MASS SYSTEM

Consider the N-mass system shown in Figure 1 with force inputs $u_1, \ldots, u_N$. Let $q_1, \ldots, q_N$, denote the position of the mass $m_1, \ldots, m_N$, respectively. For all $i = 1, \ldots, N$, the mass m_i and m_{i+1} are connected by a spring

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with stiffness k_{i+1} and a damper with damping coefficient c_{i+1}. The equations of motion are

$$M\ddot{q} + C\dot{q} + Kq = u, \tag{2.1}$$

where $M \in \mathbb{R}^{N\times N}$, $C \in \mathbb{R}^{N\times N}$, $K \in \mathbb{R}^{N\times N}$, $q \in \mathbb{R}^N$, and $u \in \mathbb{R}^N$ are given by

$$M \triangleq \begin{bmatrix} m_1 & 0 & \cdots \\ \vdots & \ddots & \\ 0 & & m_N \end{bmatrix}, \tag{2.2}$$

$$K \triangleq \begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & \cdots & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -k_N & k_N+k_{N+1} \end{bmatrix}, \tag{2.3}$$

$$C \triangleq \begin{bmatrix} c_1+c_2 & -c_2 & 0 & \cdots & 0 & 0 \\ -c_2 & c_2+c_3 & -c_3 & \cdots & 0 & 0 \\ 0 & -c_3 & c_3+c_4 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -c_N & c_N+c_{N+1} \end{bmatrix}, \tag{2.4}$$

$$q \triangleq \begin{bmatrix} q_1 & \cdots & q_N \end{bmatrix}^{\mathrm{T}}, \quad u \triangleq \begin{bmatrix} u_1 & \cdots & u_N \end{bmatrix}^{\mathrm{T}}. \tag{2.5}$$

Furthermore, (2.1) - (2.5) can be written in the first order form as

$$\dot{x} = Ax + Bu, \tag{2.6}$$

$$y = C_{\mathrm{pos}}x, \tag{2.7}$$

where $A \in \mathbb{R}^{2N\times 2N}$, $B \in \mathbb{R}^{2N\times N}$, $C_{\mathrm{pos}} \in \mathbb{R}^{N\times 2N}$, and $x \in \mathbb{R}^{2N}$ are defined by

$$\begin{aligned} A &\triangleq \begin{bmatrix} 0_N & I_N \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 0_N \\ M^{-1} \end{bmatrix}, \\ C_{\mathrm{pos}} &\triangleq \begin{bmatrix} I_N & 0_N \end{bmatrix}, \\ x &\triangleq \begin{bmatrix} q_1 & \cdots & q_N & \dot{q}_1 & \cdots & \dot{q}_N \end{bmatrix}^{\mathrm{T}}. \end{aligned} \tag{2.8}$$

We assume that M, K, and C are positive definite. Hence it can be shown that (2.6) is asymptotically stable (see [8]).

The compliance $G_{\mathrm{comp}}(s)$ has the realization

$$G_{\mathrm{comp}}(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C_{\mathrm{pos}} & 0 \end{array}\right], \tag{2.9}$$

with SISO entries

$$G_{\mathrm{adm}}(s) = \begin{bmatrix} G_{\mathrm{comp}_{1,1}}(s) & \cdots & G_{\mathrm{comp}_{1,N}}(s) \\ \vdots & \ddots & \vdots \\ G_{\mathrm{comp}_{N,1}}(s) & \cdots & G_{\mathrm{comp}_{N,N}}(s) \end{bmatrix}, \tag{2.10}$$

where $G_{\mathrm{comp}_{i,j}}(s)$ is the compliance from the force on mass m_j to the position of mass m_i. Furthermore, $G_{\mathrm{comp}_{i,j}}(s)$ can be expressed as

$$G_{\mathrm{comp}_{i,j}}(s) = C_{\mathrm{pos},i}\left(sI - A\right)^{-1} B_j, \tag{2.11}$$

where $C_{\mathrm{pos},i}$ is the ith row of C_{pos} and B_j is the jth column of B.

Proposition 2.1. For all $i = 1,\ldots,N$, and $j = 1,\ldots,N$, $G_{\mathrm{comp}_{i,j}}(s) = G_{\mathrm{comp}_{j,i}}(s)$.

Proof. Note that $G_{\mathrm{comp}}(s)$ can be expressed as

$$\begin{aligned} G_{\mathrm{comp}}(s) &= C_{\mathrm{pos}}(sI - A)^{-1}B \\ &= \frac{1}{s}\left(sI + M^{-1}C + \frac{1}{s}M^{-1}K\right)^{-1} M^{-1} \\ &= \left(Ms^2 + Cs + K\right)^{-1}. \end{aligned} \tag{2.12}$$

It follows from (2.9) that $G^{\mathrm{T}}_{\mathrm{comp}}(s)$ has the realization

$$G^{\mathrm{T}}_{\mathrm{comp}}(s) \sim \left[\begin{array}{c|c} A^{\mathrm{T}} & C^{\mathrm{T}}_{\mathrm{pos}} \\ \hline B^{\mathrm{T}} & 0 \end{array}\right] \tag{2.13}$$

and hence

$$\begin{aligned} G^{\mathrm{T}}_{\mathrm{comp}}(s) &= B^{\mathrm{T}}(sI - A^{\mathrm{T}})^{-1}C^{\mathrm{T}}_{\mathrm{pos}} \\ &= \left[C_{\mathrm{pos}}(sI - A)^{-1}B\right]^{\mathrm{T}} \\ &= \left[\left(Ms^2 + Cs + K\right)^{-1}\right]^{\mathrm{T}}. \end{aligned} \tag{2.14}$$

It follows from (2.2)-(2.4) that M, C, and K are symmetric, and hence, it follows from (2.12) and (2.14) that $G_{\mathrm{comp}}(s) = G^{\mathrm{T}}_{\mathrm{comp}}(s)$, which implies that $G_{\mathrm{comp}_{i,j}}(s) = G_{\mathrm{comp}_{j,i}}(s)$. □

III. Relative Degree of the Compliance

In this section, we analyze the relative degree of the compliance. Without loss of generality, we consider $G_{\mathrm{comp}_{ij}}(s)$ for all $i \le j$. Partitioning A^n into four $N \times N$ matrices yields

$$A^n = \begin{bmatrix} A^{[n]}_k & A^{[n]}_c \\ A^{[n]}_{21} & A^{[n]}_{22} \end{bmatrix}, \tag{3.1}$$

where $A^{[n]}_k \in \mathbb{R}^{N\times N}$ and $A^{[n]}_c \in \mathbb{R}^{N\times N}$ have entries

$$A^{[n]}_k = \begin{bmatrix} a^{[n]}_{k_{1,1}} & \cdots & a^{[n]}_{k_{1,N}} \\ \vdots & \ddots & \vdots \\ a^{[n]}_{k_{N,1}} & \cdots & a^{[n]}_{k_{N,N}} \end{bmatrix}, \quad A^{[n]}_c = \begin{bmatrix} a^{[n]}_{c_{1,1}} & \cdots & a^{[n]}_{c_{1,N}} \\ \vdots & \ddots & \vdots \\ a^{[n]}_{c_{N,1}} & \cdots & a^{[n]}_{c_{N,N}} \end{bmatrix}. \tag{3.2}$$

Note that $A^{[0]}_k = I_N$ and $A^{[0]}_c = 0_N$. Furthermore, it follows from (2.8) that

$$\begin{aligned} A^{[1]}_k &= 0_N, \quad A^{[2]}_k = -M^{-1}K \\ A^{[1]}_c &= I_N, \quad A^{[2]}_c = -M^{-1}C. \end{aligned} \tag{3.3}$$

The following result is by observation.

Proposition 3.1. For all $n = 2,\ldots,N-1$, and for all $j = 1,\ldots,N-n$,

$$a^{[n]}_{k_{i,j}} = \begin{cases} \dfrac{k_{j+1}}{m_{j+1}} \displaystyle\prod_{p=j+2}^{j+n-1} \dfrac{c_p}{m_p}, & i = j+n-1, \\ 0, & i > j+n-1, \end{cases} \tag{3.4}$$

and

$$a_{c_{i,j}}^{[n]} = \begin{cases} \dfrac{c_{j+1}}{m_{j+1}} \displaystyle\prod_{p=j+2}^{j+n-1} \dfrac{c_p}{m_p}, & i = j+n-1, \\ 0, & i > j+n-1. \end{cases} \tag{3.5}$$

Proposition 3.2. For all $i = 1, \ldots, N$ and $j = 1, \ldots, i$, the relative degree $r_{i,j}$ of the compliance $G_{\mathrm{comp}_{i,j}}(s)$ is given by

$$r_{i,j} = i - j + 2. \tag{3.6}$$

Proof. Let $C_{\mathrm{pos},i} A^n B_j = 0$ for all $n < k$ and $C_{\mathrm{pos},i} A^k B_j \neq 0$, then the relative degree $r_{i,j}$ of $G_{\mathrm{comp}_{i,j}}(s)$ is given by $r_{i,j} \triangleq k+1$. It follows from (2.8) and (3.2) that

$$C_{\mathrm{pos},i} A^n B_j = \frac{a_{c_{i,j}}^{[n]}}{m_j}. \tag{3.7}$$

Proposition 3.1 implies that $C_{\mathrm{pos},i} A^n B_j = 0$ for all $n < i - j + 1$ and $C_{\mathrm{pos},i} A^n B_j \neq 0$ for $n = i - j + 1$. Thus the relative degree is $r_{i,j} = n + 1 = i - j + 2$. □

Note that the relative degree $r_{\min} = 2$ is minimum, when the sensor and actuator are colocated, that is, $i = j$.

IV. Zeros of the Compliance

Next, for all $G_{\mathrm{comp}_{i,j}}(s)$ with $i \geqslant j$, define $\tilde{G}_{\mathrm{comp}_{i,j}}(s)$ by

$$\tilde{G}_{\mathrm{comp}_{i,j}}(s) \triangleq G_{\mathrm{comp}_{i,j}}(s)(s+\delta)^{r_{i,j}}, \tag{4.1}$$

where $\delta > 0$. It follows from (2.9) that $\tilde{G}_{\mathrm{comp}_{i,j}}(s)$ has the realization

$$\tilde{G}_{\mathrm{comp}_{i,j}}(s) \sim \left[\begin{array}{c|c} A & B_j \\ \hline C_{\mathrm{pos},i}\left(A+\delta I\right)^{r_{i,j}} & \beta_{i,j} \end{array}\right], \tag{4.2}$$

where $\beta_{i,j} \triangleq C_{\mathrm{pos},i} A^{r_{i,j}-1} B_j$. Proposition 3.2 implies that $\beta_{i,j} > 0$. Hence, $\tilde{G}_{\mathrm{comp}_{i,j}}(s)$ is invertible with the realization

$$\tilde{G}_{\mathrm{comp}_{i,j}}^{-1}(s) \sim \left[\begin{array}{c|c} \tilde{A}_{\mathrm{inv}} & -\frac{1}{\beta_{i,j}} B_j \\ \hline \frac{1}{\beta_{i,j}} C_{\mathrm{pos},i}\left(A+\delta I\right)^{r_{i,j}} & \frac{1}{\beta_{i,j}} \end{array}\right], \tag{4.3}$$

where $\tilde{A}_{\mathrm{inv}} \in \mathbb{R}^{2N \times 2N}$ is given by

$$\tilde{A}_{\mathrm{inv}} \triangleq A - \frac{1}{\beta_{i,j}} B_j C_{\mathrm{pos},i}\left(A+\delta I\right)^{r_{i,j}}. \tag{4.4}$$

It follows from (4.1)-(4.3) that

$$\mathrm{spec}\left(\tilde{A}_{\mathrm{inv}}\right) = \mathrm{zeros}\left(G_{\mathrm{comp}_{ij}}(s)\right) \cup \{-\delta\}. \tag{4.5}$$

Since $\delta > 0$, $G_{\mathrm{comp}_{ij}}(s)$ is strictly minimum phase if and only if $\mathrm{Re}\,(\lambda) < 0$ for all $\lambda \in \mathrm{spec}(\tilde{A}_{\mathrm{inv}})$.

Partitioning $(A+\delta I)^{r_{i,j}}$ into four $N \times N$ matrices yields

$$(A+\delta I)^{r_{i,j}} = \begin{bmatrix} A_{\delta_k}^{[r_{i,j}]} & A_{\delta_c}^{[r_{i,j}]} \\ A_{\delta_{21}}^{[r_{i,j}]} & A_{\delta_{22}}^{[r_{i,j}]} \end{bmatrix}. \tag{4.6}$$

Expanding $(A+\delta I)^{r_{i,j}}$ into a binomial series yields

$$\begin{aligned}(A+\delta I)^{r_{i,j}} =& h_{(r_{i,j},0)} A^{r_{i,j}} + h_{(r_{i,j},1)} \delta A^{r_{i,j}-1} + \cdots \\ &+ h_{(r_{i,j},r_{i,j}-1)} \delta^{r_{i,j}-1} A + h_{(r_{i,j},r_{i,j})} \delta^{r_{i,j}} I,\end{aligned} \tag{4.7}$$

where for all $n = 0, \ldots, r_{i,j}$, $h_{(r_{i,j},n)} \triangleq \frac{r_{i,j}!}{n!(n-r_{i,j})!} > 0$. Substituting (3.1) and (4.6) into (4.7) yields

$$A_{\delta_k}^{[r_{i,j}]} = \sum_{n=0}^{r_{i,j}} h_{(r_{i,j},n)} \delta^n A_k^{[r_{i,j}-n]}, \tag{4.8}$$

$$A_{\delta_c}^{[r_{i,j}]} = \sum_{n=0}^{r_{i,j}} h_{(r_{i,j},n)} \delta^k A_c^{[r_{i,j}-n]}. \tag{4.9}$$

Let $A_{\delta_k}^{[n]}$ and $A_{\delta_c}^{[n]}$ have entries

$$\begin{aligned} A_{\delta_k}^{[n]} &= \begin{bmatrix} a_{\delta_{k_{1,1}}}^{[n]} & \cdots & a_{\delta_{k_{1,N}}}^{[n]} \\ \vdots & \ddots & \vdots \\ a_{\delta_{k_{N,1}}}^{[n]} & \cdots & a_{\delta_{k_{N,N}}}^{[n]} \end{bmatrix}, \\ A_{\delta_c}^{[k]} &= \begin{bmatrix} a_{\delta_{c_{1,1}}}^{[n]} & \cdots & a_{\delta_{c_{1,N}}}^{[n]} \\ \vdots & \ddots & \vdots \\ a_{\delta_{c_{N,1}}}^{[n]} & \cdots & a_{\delta_{c_{N,N}}}^{[n]} \end{bmatrix}. \end{aligned} \tag{4.10}$$

Substituting (4.10) into (4.8) yields

$$a_{\delta_{k_{p,q}}}^{[r_{i,j}]} = \sum_{n=0}^{r_{i,j}} h_{(r_{i,j},n)} \delta^n a_{k_{p,q}}^{[r_{i,j}-n]}, \tag{4.11}$$

$$a_{\delta_{c_{p,q}}}^{[r_{i,j}]} = \sum_{n=0}^{r_{i,j}} h_{(r_{i,j},n)} \delta^n a_{c_{p,q}}^{[r_{i,j}-n]}, \tag{4.12}$$

for all $p = 1, \ldots, N$, and $q = 1, \ldots, N$.

Proposition 4.1 For all $i = 1, \ldots, N$, and $j = 1, \ldots, N$, $G_{\mathrm{comp}_{i,j}}(s)$ is strictly minimum phase.

Proof. Note that $r_{i,j} = i - j + 2$ and hence substituting $p = i$ and $q = j$ into (4.11) and using the fact that $h_{(r_{i,j},r_{i,j})} = h_{(r_{i,j},0)} = 1$ yields

$$\begin{aligned} a_{\delta_{k_{i,j}}}^{[r_{i,j}]} =& a_{k_{i,j}}^{[r_{i,j}]} + h_{(r_{i,j},1)} \delta a_{k_{i,j}}^{[r_{i,j}-1]} + \cdots \\ &+ h_{(r_{i,j},r_{i,j}-1)} \delta^{r_{i,j}-1} a_{k_{i,j}}^{[1]} + \delta^{r_{i,j}} a_{k_{i,j}}^{[0]}, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} a_{\delta_{c_{i,j}}}^{[r_{i,j}]} =& a_{c_{i,j}}^{[r_{i,j}]} + h_{(r_{i,j},1)} \delta a_{c_{i,j}}^{[r_{i,j}-1]} + \cdots \\ &+ h_{(r_{i,j},r_{i,j}-1)} \delta^{r_{i,j}-1} a_{c_{i,j}}^{[1]} + \delta^{r_{i,j}} a_{c_{i,j}}^{[0]}. \end{aligned} \tag{4.14}$$

Case 1. Let $r_{i,j} = 2$, which implies that $i = j$. Substituting $i = j$ into (4.13) and using (3.3) yields

$$\begin{aligned} a_{\delta_{k_{i,i}}}^{[2]} &= a_{k_{i,i}}^{[2]} + 2\delta a_{k_{i,i}}^{[1]} + \delta^2 a_{k_{i,i}}^{[0]} = a_{k_{i,i}}^{[2]} + \delta^2, \\ a_{\delta_{c_{i,i}}}^{[2]} &= a_{c_{i,i}}^{[2]} + 2\delta a_{c_{i,i}}^{[1]} + \delta^2 a_{c_{i,i}}^{[0]} = a_{c_{i,i}}^{[2]} + 2\delta. \end{aligned} \tag{4.15}$$

Hence, if $\delta > 0$ chosen sufficiently large, then $a^{[2]}_{\delta_{k_{i,i}}} > 0$ and $a^{[2]}_{\delta_{c_{i,i}}} > 0$.

Case 2. Let $r_{i,j} > 2$, which implies that $i > j$. In that case, it follows from (3.4) and (3.5) (Proposition 3.1), that there exists $n_r = i - j + 1$ such that

$$a^{[n]}_{k_{i,j}} \begin{cases} > 0, & n = n_r, \\ = 0, & n < n_r, \end{cases} \quad a^{[n]}_{c_{i,j}} \begin{cases} > 0, & n = n_r, \\ = 0, & n < n_r. \end{cases} \tag{4.16}$$

Substituting (4.16) into (4.13) yields

$$\begin{aligned} a^{[r_{i,j}]}_{\delta_{k_{i,j}}} = a^{[r_{i,j}]}_{k_{i,j}} + h_{(r_{i,j},1)}\delta a^{[r_{i,j}-1]}_{k_{i,j}} + \cdots \\ + h_{(r_{i,j},r_{i,j}-n_r)}\delta^{r_{i,j}-n_r} a^{[n_r]}_{k_{i,j}}, \end{aligned} \tag{4.17}$$

$$\begin{aligned} a^{[r_{i,j}]}_{\delta_{c_{i,j}}} = a^{[r_{i,j}]}_{c_{i,j}} + h_{(r_{i,j},1)}\delta a^{[r_{i,j}-1]}_{c_{i,j}} + \cdots \\ + h_{(r_{i,j},r_{i,j}-n_r)}\delta^{r_{i,j}-n_r} a^{[n_r]}_{c_{i,j}}. \end{aligned} \tag{4.18}$$

Hence, if δ is chosen sufficiently large, then $a^{[r_{i,j}]}_{\delta_{k_{i,j}}} > 0$ and $a^{[r_{i,j}]}_{\delta_{c_{i,j}}} > 0$. It follows from Case 1 and Case 2 that for all $i = 1, \ldots, N$ and $j = 1, \ldots, i$, if $\delta > 0$ is chosen sufficiently large, then

$$a^{[r_{i,j}]}_{\delta_{k_{i,j}}} > 0, \quad a^{[r_{i,j}]}_{\delta_{c_{i,j}}} > 0. \tag{4.19}$$

Next, we analyze $B_j C_{\mathrm{pos},i}(A+\delta I)^{r_{i,j}}$. It follows from (2.8) that for all $p = 1, \ldots, 2N$, and $q = 1, \ldots, 2N$, the (p,q)th entry of $B_j C_{\mathrm{pos},i} \in \mathbb{R}^{2N\times 2N}$ is given by

$$(B_j C_{\mathrm{pos},i})_{p,q} = \begin{cases} \frac{1}{m_j}, & (p,q) = (N+j, i) \\ 0, & (p,q) \neq (N+j, i) \end{cases}. \tag{4.20}$$

Hence, it follows from (4.20) and (4.6) that

$$B_j C_{\mathrm{pos},i}(A+\delta I)^{r_{i,j}} = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_j}\hat{K} & \frac{1}{m_j}\hat{C} \end{bmatrix}, \tag{4.21}$$

where for all $p = 1, \ldots, N$,

$$\mathrm{row}_p(\hat{K}) = \begin{cases} \mathrm{row}_i(A^{[r_{i,j}]}_{\delta_k}), & p = j, \\ 0, & p \neq j, \end{cases} \tag{4.22}$$

$$\mathrm{row}_p(\hat{C}) = \begin{cases} \mathrm{row}_i(A^{[r_{i,j}]}_{\delta_c}), & p = j, \\ 0, & p \neq j, \end{cases} \tag{4.23}$$

which implies that the jth row of $\hat{K} \in \mathbb{R}^{N\times N}$ and $\hat{C} \in \mathbb{R}^{N\times N}$ is the only non-zero row. Since, M in (2.2) is diagonal, $B_j C_{\mathrm{pos},i}(A+\delta I)^{r_{i,j}}$ in (4.21) can also be expressed as

$$B_j C_{\mathrm{pos},i}(A+\delta I)^{r_{i,j}} = \begin{bmatrix} 0 & 0 \\ M^{-1}\hat{K} & M^{-1}\hat{C} \end{bmatrix}. \tag{4.24}$$

Hence, (4.10) and (4.22) imply that

$$\mathrm{spec}(\hat{K}) = \{a^{[r_{i,j}]}_{\delta_{k_{i,j}}}\} \cup \{0\}, \ \mathrm{spec}(\hat{C}) = \{a^{[r_{i,j}]}_{\delta_{c_{i,j}}}\} \cup \{0\}. \tag{4.25}$$

Substituting (4.24) into (4.4) yields

$$\tilde{A}_{\mathrm{inv}} = \begin{bmatrix} 0 & I \\ M^{-1}\tilde{K} & M^{-1}\tilde{C} \end{bmatrix}, \tag{4.26}$$

where $\tilde{K} \in \mathbb{R}^{N\times N}$ and $\tilde{C} \in \mathbb{R}^{N\times N}$ are defined by

$$\tilde{K} = K + \frac{1}{\beta_{i,j}}\hat{K}, \quad \tilde{C} = C + \frac{1}{\beta_{i,j}}\hat{C}. \tag{4.27}$$

It follows from (4.19), (4.22) and (4.25) that $\hat{K}$ and $\hat{C}$ can be made positive semi-definite by choosing a large $\delta > 0$. Furthermore, since K and C are positive definite and $\beta_{i,j} > 0$, (4.27) implies that for a sufficiently large $\delta > 0$, $\tilde{K}$ and $\tilde{C}$ are positive definite. Hence, it follows from [8] that for all $\lambda \in \mathrm{spec}(\tilde{A}_{\mathrm{inv}})$, $\mathrm{Re}(\lambda) < 0$, and hence (4.5) implies that for all $i = 1, \ldots, N$, and $j = 1, \ldots, i$, $G_{\mathrm{comp}_{i,j}}(s)$ is strictly minimum phase. □

V. High-Gain Dynamic Compensation

In this section, we consider a high-gain stable dynamic compensator for the single-input single-output compliance

$$y_i(t) = G_{\mathrm{comp}_{i,j}}(s) u_j(t), \tag{5.1}$$

where $y_i(t)$ is the position of the ith mass and $u_j(t)$ is the force on the jth mass. Furthermore, let $\delta_{\mathrm{s}} = \pm 1$ be the sign of the high-frequency gain, and let $\overline{\beta}$ be the magnitude of the high-frequency gain.

The results of [8] and Proposition 4.1 implies that (5.1) is asymptotically stable and strictly minimum phase. Nevertheless, active control is frequently used on asymptotically stable structures to add damping and/or stiffness. Under this motivation, we consider a high-gain stable dynamic compensator that is constructed using generalized root locus principles and the Fibonacci series [6]. The novel aspect of this construction is that the compensator requires limited information of the transfer function $G_{\mathrm{comp}_{i,j}}(s)$ to yield the closed-loop system high-gain stable. We assume the following information.

(i) The magnitude of the high-frequency gain satisfies $0 \le \overline{\beta} \le b_0$, where b_0 is known.
(ii) The sign of the high-frequency gain is known.
(iii) The relative degree satisfies $0 < r_{i,j} \le \rho$, where ρ is known.

For all $j \ge 0$ let F_j be the jth Fibonacci number, where $F_0 = 0$, $F_1 = 1$, $F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, \ldots$, and define

$$f_{\rho,h} \triangleq F_{\rho+2} - F_{h+1}, \tag{5.2}$$

where h satisfies $1 \le h \le \rho$.

Consider the input $u_j = v - u_c$ with the feedback

$$u_c = \hat{G}(s,k) y_i, \tag{5.3}$$

and the strictly proper controller

$$\hat{G}(s,k) \triangleq \frac{\delta_s k^{F_\rho+2}\hat{z}(s)}{s^\rho + k^{f_{\rho,\rho}} b_\rho s^{\rho-1} + k^{f_{\rho,\rho-1}} b_{\rho-1} s^{\rho-2} + \cdots + k^{f_{\rho,1}} b_1}, \quad (5.4)$$

where $k > 0$, $b_1, \ldots, b_\rho$ are real numbers, and $\hat{z}(s)$ is a degree $\rho - 1$ monic polynomial. The closed-loop system is

$$\tilde{G}(s,k) \triangleq \frac{1}{1 + \hat{G}(s,k)G_{\mathrm{comp}_{i,j}}(s)}. \quad (5.5)$$

The following two results are specializations of results presented in [6].

Theorem 5.1: Consider the closed-loop system (5.5). Assume that the polynomials $\hat{z}(s)$,

$$B_{\rho-2}(s) \triangleq s^3 + b_\rho s^2 + b_{\rho-1}s + b_0, \quad (5.6)$$

and

$$B_i(s) \triangleq b_{i+3}s^3 + b_{i+2}s^2 + b_{i+1}s + b_0, \quad (5.7)$$

for all $i = 0, 1, \ldots, \rho - 3$ are Hurwitz. Then $\tilde{G}(s,k)$ is high-gain stable for all $\beta \in (0, b_0]$. Furthermore, as $k \to \infty$, $2N - r_{i,j} + \rho - 1$ poles of the closed-loop system converge to the union of the roots of $\hat{z}(s)$ and open-loop zeros. The real parts of the remaining $r_{i,j} + 1$ roots approach $-\infty$.

For implementation purposes, it is desirable that the controller $\hat{G}(s,k)$ be stable.

Proposition 5.1. Consider the controller given by (5.4), and assume that

$$\hat{B}(s) \triangleq s^\rho + b_\rho s^{\rho-1} + b_{\rho-1}s^{\rho-2} + \cdots + b_2 s + b_1 \quad (5.8)$$

is Hurwitz. Then the controller (5.4) is stable for all $k > 1$.

VI. EXAMPLE

Consider a serially connected 3-mass system as shown in Figure 1 when $N = 3$. Let $G_{\mathrm{comp}_{i,j}}(s)$ be the single-input single-output compliance from the force of the ith mass to the position of the jth mass. We assume that the magnitude of the high-frequency gain is positive and the upper bound on the high frequency gain is $b_0 = 4$. Proposition 3.2 implies that the relative degree of $G_{\mathrm{comp}_{i,j}}(s)$ must satisfy $r_{i,j} \leq \rho = 4$.

Now, we use the results of Theorem 5.1 to design a dynamic compensator for $G_{\mathrm{comp}_{i,j}}(s)$ that will yield the closed-loop high-gain stable. Consider the dynamic compensator

$$\hat{G}(s,k) \triangleq \frac{k^8\hat{z}(s)}{s^4 + k^3 b_4 s^3 + k^5 b_3 s^2 + k^6 b_2 s + k^7 b_1}, \quad (6.1)$$

where $k > 0$ and $\hat{z}(s)$ is a degree 3 monic Hurwitz polynomial. The controller parameters are chosen to be

$$\hat{z}(s) = (s+5)(s+4)(s+3), \quad (6.2)$$

$$b_1 = 5, \quad b_2 = 10, \quad b_3 = 10, \quad b_4 = 5, \quad (6.3)$$

to satisfy the assumptions of Theorem 5.1 and Proposition 5.1.

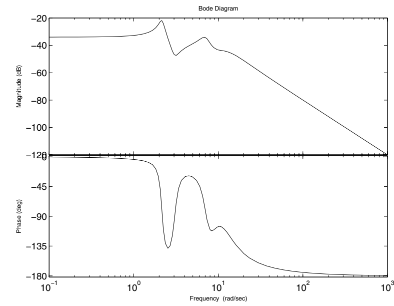

Fig. 2. Bode plot of the open-loop system for $i = 1$ and $j = 1$.

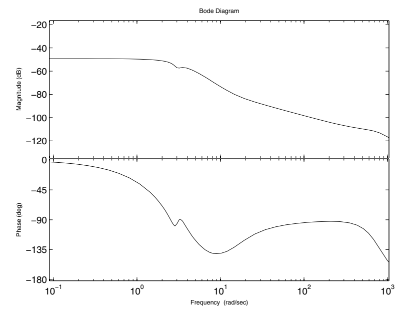

Fig. 3. Bode plot of the closed-loop system for $i = 1$ and $j = 1$, and the gain $k = 15$.

Now, assume that the 3-mass system is given by

$$\dot{x} = \begin{bmatrix} 0 & I_3 \\ -M^{-1}K & -M^{-1}C \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} u, \quad (6.4)$$

$$y = \begin{bmatrix} I_3 & 0 \end{bmatrix} x, \quad (6.5)$$

where M, C and K are defined by (2.2)-(2.5) and x is defined by (2.8) with $N = 3$. The masses are $m_1 = 1$ kg, $m_2 = 2$ kg, and $m_3 = 3$ kg. The stiffness coefficients are $k_1 = 50$ N/m, $k_2 = 70$ N/m, and $k_3 = 60$ N/m. The structure is lightly damped with the damping coefficients given by $c_1 = 3$ Ns/m, $c_2 = 6$ Ns/m, and $c_3 = 2$ Ns/m.

Assume that $G_{\mathrm{comp}_{i,j}}(s)$ is the compliance from the force of the first mass to the position of the first mass. Hence, the relative degree $r_{i,j} = 2$. The Bode plot for this transfer function is shown in Figure 2. To improve performance, we implement the high-gain controller (6.1)-(6.3) with the gain set at $k = 15$. The Bode plot of the closed-loop system is shown in Figure 3. Note that the addition of high-gain feedback has attenuated the resonant peaks.

Now, we assume that $G_{\mathrm{comp}_{i,j}}(s)$ is the compli-

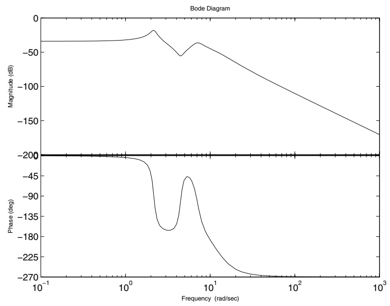

Fig. 4. Bode plot of the open-loop system for $i = 2$ and $j = 1$.

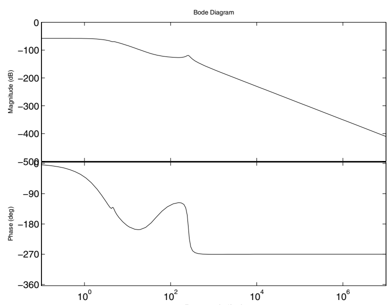

Fig. 5. Bode plot of the closed-loop system for $i = 2$ and $j = 1$, and the gain $k = 60$.

ance from the force of the second mass to the position of the first mass so that the relative degree $r_{i,j} = 3$. The Bode plot of the open-loop system is shown in Figure 4. As seen in Figure 5 the high-gain control with $k = 60$ improves performance.

Lastly, we assume that $G_{\mathrm{comp}_{i,j}}(s)$ is the compliance from the force of the third mass to the position of the first mass so that the relative degree $r_{i,j} = 4$. The Bode plot of the open-loop system and the closed-loop system are given in Figure 6 and Figure 7, respectively. The gain $k = 150$.

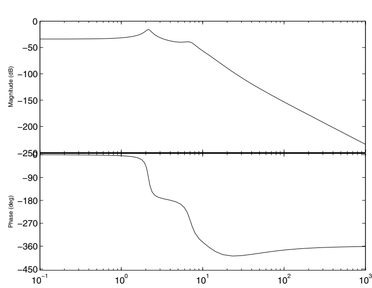

Fig. 6. Bode plot of the open-loop system for $i = 3$ and $j = 1$.

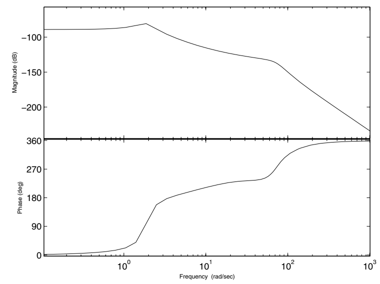

Fig. 7. Bode plot of the closed-loop system for $i = 3$ and $j = 1$, and the gain $k = 150$.

VII. CONCLUSIONS

In this paper, we examined a serially connected N-mass structure. The relative degree of all SISO force-to-position transfer functions was shown to be a simple function of the number of masses between the sensor and actuator. Furthermore, we showed that all SISO force-to-position transfer functions are strictly minimum phase. Lastly, we apply a specially constructed controller that provides infinite upward gain margin. Thus high-gain feedback can be used to improve structural response.

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