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Direct Adaptive Dynamic Compensation for Minimum Phase Systems with Unknown Relative Degree

Jesse B. Hoagg and Dennis S. Bernstein
Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2140, jhoagg@umich.edu

1. Introduction

High-gain adaptive stabilization methods typically invoke a minimum phase assumption since zeros attract poles under high gain [1, 2]. Adaptive high-gain proportional feedback can stabilize square multi-input, multi-output systems that are minimum phase and relative degree one with a positive high-frequency gain [1].

In [2], high-gain dynamic compensation is used to guarantee output convergence of single-input, single-output minimum phase systems with arbitrary known relative degree. This work is surprising since classical roots locus is not high-gain stable for plants with relative degree exceeding two. The approach of [2] uses dynamic compensation, which allows output stabilization of systems with relative degree higher than two. However, as we will show in Section 2, the results of [2] can fail when the relative degree of the plant is greater than four.

In the present paper, we adopt some of the techniques of [2] to develop lower order high-gain controllers that stabilize single-input single-output minimum phase systems with arbitrary known relative degree, correcting for the error encountered in [2] when the relative degree exceeds four (see Section 2). Furthermore, we develop a novel high-gain controller for minimum phase systems when the relative degree is unknown-but-bounded. This construction makes uses of the Fibonacci series and a variation of root locus. A parameter-monotonic adaptation law is shown to guarantee state convergence to zero for a large class of high-gain-stable closed-loop systems. Finally, this result is applied to the Fibonacci-based high-gain controllers. Thus, the main result of the paper is parameter-monotonic adaptive stabilization of single-input, single-output minimum phase systems with unknown-but-bounded relative degree.

2. Counterexample to the Results of [2]

In this section, we provide a counterexample to the controller given in [2]. In the notation of [2], consider the unstable plant

$$G(s) = \frac{q(s)}{p(s)} = \frac{1}{s^5 - 11s^4 - 7s^3 + 323s^2 - 186s - 2520}, \tag{2.1}$$

with poles at 5, 6, 7, -10, and -12, and relative degree 5. Lemma 4 and Figure 1 in [2] propose a 10th order controller to high-gain stabilize (2.1). To satisfy the hypotheses of Lemma 4, an upper bound on the high-frequency gain of the plant is chosen to be $g_0 = 2.5$. The gains of the controller

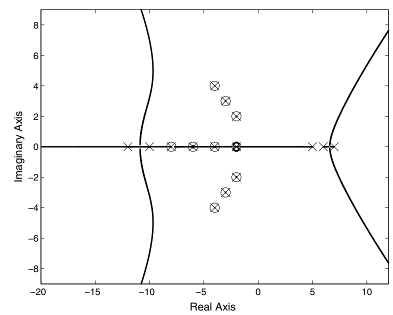

Fig. 1. Root locus for the closed-loop dynamics of the controller proposed in [2]. The system is not high-gain stable.

$g_1 = 2$, $g_2 = 5$, $g_3 = 3$, and $g_4 = 2$ are chosen so that the polynomial

$$s^5 + g_4 s^4 + g_3 s^3 + g_2 s^2 + g_1 s + g_0 \tag{2.2}$$

is Hurwitz. Furthermore, define the monic Hurwitz polynomials

$$r_1(s) = s^4 + 10s^3 + 40s^2 + 80s + 64, \tag{2.3}$$
$$r_2(s) = s^3 + 12s^2 + 54s + 108, \tag{2.4}$$
$$r_3(s) = s^2 + 8s + 32, \tag{2.5}$$
$$r_4(s) = s + 8. \tag{2.6}$$

The controller given in Lemma 4 of [2] yields a closed-loop characteristic polynomial $\tilde{p}(s)$, which depends on a parameter k. The claim of [2] is that there exists $k_{\rm s}$ such that, for all $k > k_{\rm s}$, $\tilde{p}(s)$ is asymptotically stable. Figure 1 provides a root locus for $\tilde{p}(s)$ as $k \to \infty$. The zero-gain $k = 0$ pole locations are shown by $\times$'s, and the zero locations, which attract certain poles, are shown by $\circ$'s. Ten of the closed-loop poles converge to the stable zero locations as $k \to \infty$. The real parts of three of the remaining five closed-loop poles go to minus infinity as $k \to \infty$. However, the real parts of the two remaining pole go to plus infinity as $k \to \infty$. Thus, $\tilde{p}(s)$ is not stable for sufficiently large k, and the closed-loop system is not stable for sufficiently large k.

The error in the result of [2] can be traced to the application of Lemma 3 to obtain Lemma 4. The Hurwitz hypothesis on (2.2) is not sufficient for stability of the closed-loop system. However, the hypothesis would be sufficient if it were required that the polynomial

$$s^5 + g_4 s^4 + g_3 s^3 + g_2 s^2 + g_1 s + \alpha g_0 \tag{2.7}$$

be Hurwitz for all $\alpha \in (0, 1]$. We will reconsider this example in Section 8.

3. Parameter-Dependent Dynamic Compensation

We consider the strictly proper single-input single-output linear time-invariant system

$$y = G(s)u, \qquad G(s) \triangleq \delta\beta\frac{z(s)}{p(s)}. \tag{3.1}$$

We make the following assumptions.

(i) $z(s)$ is a monic Hurwitz polynomial but is otherwise unknown.

(ii) $p(s)$ is a monic polynomial but is otherwise unknown.

(iii) $z(s)$ and $p(s)$ are coprime.

(iv) The magnitude β of the high-frequency gain satisfies $0 < \beta \le b_0$, where b_0 is known.

(v) The sign $\delta = \pm 1$ of the high-frequency gain is known.

For later use, we define the notation

$$m \triangleq \deg z(s), \quad n \triangleq \deg p(s), \quad r \triangleq n - m. \tag{3.2}$$

Let $z(s,k)$ and $p(s,k)$ be parameter-dependent polynomials, that is, polynomials in s over the reals whose coefficients are functions of a parameter k. Furthermore, define the parameter-dependent transfer function

$$G(s,k) \triangleq \frac{z(s,k)}{p(s,k)}. \tag{3.3}$$

Note that the polynomials $z(s,k)$ and $p(s,k)$ need not be coprime for all $k \in \mathbb{R}$.

Definition 3.1. *The parameter-dependent polynomial $p(s,k)$ is* high-gain Hurwitz *if there exists $k_{\rm s} > 0$ such that $p(s,k)$ is Hurwitz for all $k \ge k_{\rm s}$.*

Definition 3.2. *The parameter-dependent transfer function $G(s,k)$ is* high-gain stable *if for all $k \in \mathbb{R}$, it can be expressed as the ratio of two parameter-dependent polynomials $z(s,k)$ and $p(s,k)$, where the denominator polynomial $p(s,k)$ is high-gain Hurwitz.*

Now, consider the system (3.1) and the input $u = v - u_c$ with the feedback

$$u_c = \hat{G}(s,k)y, \qquad \hat{G}(s,k) \triangleq \frac{\hat{z}(s,k)}{\hat{p}(s,k)}. \tag{3.4}$$

The polynomials $\hat{z}(s,k)$ and $\hat{p}(s,k)$ in s over the reals are also functions of a parameter k. Letting $\hat{z}(s,k) = \delta k$ and $\hat{p}(s,k) = 1$ yields $\hat{G}(s,k) = \delta k$, and the closed-loop poles can be determined by classical root locus. In general, (3.4) is a parameter-dependent dynamic compensator. The closed-loop transfer function from input v to output y is

$$\tilde{G}(s,k) \triangleq \frac{G(s)}{1+\hat{G}(s,k)G(s)} = \frac{\delta\beta z(s)\hat{p}(s,k)}{p(s)\hat{p}(s,k) + \delta\beta z(s)\hat{z}(s,k)}. \tag{3.5}$$

4. Dynamic Compensation for Systems with Known Arbitrary Relative Degree

In this section, we use parameter-dependent dynamic compensation to stabilize (3.1), where the relative degree is arbitrary but known. Consider the feedback (3.4) with the strictly proper controller

$$\hat{G}(s,k) \triangleq \frac{\delta k^{r+1}\hat{z}(s)}{s^r + kb_r s^{r-1} + \cdots + k^{r-1}b_2 s + k^r b_1}, \tag{4.1}$$

where $k > 0$, $b_1, \ldots, b_r$ are real numbers, and $\hat{z}(s)$ is a degree $r-1$ monic polynomial. The closed-loop system is

$$\tilde{G}(s,k) \triangleq \frac{G(s)}{1+\hat{G}(s,k)G(s)} = \frac{\tilde{z}(s,k)}{\tilde{p}(s,k)}, \tag{4.2}$$

where

$$\tilde{z}(s,k) \triangleq \delta\beta z(s)\left[s^r + kb_r s^{r-1} + \cdots + k^r b_1\right], \tag{4.3}$$

$$\tilde{p}(s,k) \triangleq p(s)s^r + kb_r p(s)s^{r-1} + \cdots + k^r b_1 p(s) + k^{r+1}\beta z(s)\hat{z}(s). \tag{4.4}$$

The following generalization of root locus analysis, similar to a result presented in [2], is used to analyze the stability of (4.2)-(4.4).

Lemma 4.1. *Let μ be a positive integer, and assume that the degree μ monic polynomial*

$$c(s) \triangleq s^\mu + c_{\mu-1}s^{\mu-1} + \cdots + c_1 s + \alpha c_0 \tag{4.5}$$

is Hurwitz for all $\alpha \in (0,1]$. Furthermore, let ν be a nonnegative integer, and, for all $i = 0, 1, \ldots, \mu$, let $q_i(s)$ be a monic polynomial of degree $\nu + i$, where $q_0(s)$ is Hurwitz. Then, for all $\alpha \in (0,1]$, the degree $\nu+\mu$ monic polynomial

$$\tilde{q}(s,k) \triangleq q_\mu(s) + kc_{\mu-1}q_{\mu-1}(s) + k^2 c_{\mu-2}q_{\mu-2}(s) + \cdots + k^\mu \alpha c_0 q_0(s) \tag{4.6}$$

is high-gain Hurwitz. Furthermore, as $k \to \infty$, ν roots of $\tilde{q}(s,k)$ converge to the roots of $q_0(s)$, and the real parts of the remaining μ roots approach $-\infty$.

Proof. Write

$$q_0(s) = s^\nu + a_{\nu-1}s^{\nu-1} + \cdots + a_1 s + a_0. \tag{4.7}$$

The Hurwitz conditions for the stability of $\tilde{q}(s,k)$ are polynomials in k. For sufficiently large k, these are satisfied if

$$\Lambda_1 \triangleq kc_{\mu-1} > 0, \tag{4.8}$$

$$\Lambda_2 \triangleq \begin{vmatrix} kc_{\mu-1} & k^3 c_{\mu-3} \\ 1 & k^2 c_{\mu-2} \end{vmatrix} > 0, \tag{4.9}$$

$$\Lambda_3 \triangleq \begin{vmatrix} kc_{\mu-1} & k^3 c_{\mu-3} & k^5 c_{\mu-5} \\ 1 & k^2 c_{\mu-2} & k^4 c_{\mu-4} \\ 0 & kc_{\mu-1} & k^3 c_{\mu-3} \end{vmatrix} > 0, \tag{4.10}$$

$$\vdots$$

$$\Lambda_\mu \triangleq \begin{vmatrix} \Lambda_3 & & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & & k^\mu \alpha c_0 a_{\nu-1} \\ 0 & \cdots & k^{\mu-2}c_2 & k^\mu \alpha c_0 \end{vmatrix} > 0, \tag{4.11}$$

$$\vdots$$

$$\Lambda_{\mu+\nu} \triangleq \begin{vmatrix} \Lambda_\mu & & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & & 0 \\ 0 & \cdots & k^\mu \alpha c_0 a_2 & k^\mu \alpha c_0 a_0 \end{vmatrix} > 0. \tag{4.12}$$

Next, it can be seen that the first μ conditions, which are independent of k, are equivalent to the Hurwitz conditions for $c(s)$. The last ν conditions are equivalent to the Hurwitz conditions for $q_0(s)$. Therefore, $\tilde{q}(s,k)$ is high-gain Hurwitz.

The final statement of Lemma 4.1 follows from factoring (4.6) as

$$\tilde{q}(s,k) = q_\mu(s) + kc_{\mu-1}\left[q_{\mu-1}(s) + kc_{\mu-2}\left[q_{\mu-2}(s) + \cdots + kc_1\left[q_1(s) + k\alpha c_0 q_0(s)\right]\right]\right]. \quad (4.13)$$

Applying root locus techniques iteratively to relative degree one polynomials a total of μ times yields the result. □

The following result is an immediate consequence of Lemma 4.1 with $\mu = r+1$, $\nu = n-1$, $q_0(s) = z(s)\hat{z}(s)$, $\alpha c_0 = \beta$, $c_i = b_i$ for $i = 0,\ldots,r$, and $q_i(s) = p(s)s^{i-1}$ for $i = 1,\ldots,r+1$.

Theorem 4.1. *Consider the closed-loop system (4.2)-(4.4), and assume that the polynomials $\hat{z}(s)$ and*

$$b(s) \triangleq s^{r+1} + b_r s^r + b_{r-1}s^{r-1} + \cdots + b_1 s + \alpha b_0 \quad (4.14)$$

are Hurwitz for all $\alpha \in (0,1]$. Then $\tilde{p}(s,k)$ is high-gain Hurwitz and thus $\tilde{G}(s,k)$ is high-gain stable. Furthermore, as $k \to \infty$, $n-1$ roots of $\tilde{p}(s,k)$ converge to the roots of $z(s)\hat{z}(s)$, and the real parts of the remaining $r+1$ roots of $\tilde{p}(s,k)$ approach $-\infty$.

For implementation purposes, it is desirable that the controller $\hat{G}(s,k)$ be stable. The following result characterizes controllers that are stable for all $k > 0$.

Proposition 4.1. *The controller $\hat{G}(s,k)$, given by (4.1), is asymptotically stable for all $k > 0$ if and only if*

$$\hat{b}(s) \triangleq s^r + b_r s^{r-1} + b_{r-1}s^{r-2} + \cdots + b_2 s + b_1 \quad (4.15)$$

is Hurwitz.

Proof. Let $\lambda_1,\ldots,\lambda_r$ denote the roots of $\hat{b}(s)$. It follows that the poles of $\hat{G}(s,k)$ are given by $k\lambda_1,\ldots,k\lambda_r$. Therefore, for all $i = 1,\ldots,r$, $\mathrm{Re}(k\lambda_i) < 0$ if and only if $\mathrm{Re}(\lambda_i) < 0$. Thus $\hat{G}(s,k)$ is asymptotically stable if and only if $\hat{b}(s)$ is Hurwitz. □

The coefficients b_i for $i = 1,\ldots,r$ satisfy the assumptions of Proposition 4.1 if they satisfy the assumptions of Theorem 4.1.

Proposition 4.2. *Consider the controller (4.1) with $r = 3$. If the relative degree of $G(s)$ is 2, then $\tilde{G}(s,k)$ is not high-gain stable.*

Proof. The Hurwitz conditions for the stability of $\tilde{p}(s,k)$ are polynomials in k. For sufficiently large k, the Hurwitz conditions for $\tilde{p}(s,k)$ are satisfied if

$$\Lambda_1 \triangleq kb_3 > 0, \quad \Lambda_2 \triangleq \begin{vmatrix} kb_3 & k^4 b_0 \\ 1 & k^2 b_2 \end{vmatrix} > 0, \quad \ldots \quad (4.16)$$

The second condition is violated for all sufficiently large k. Therefore, $\tilde{p}(s,k)$ is not high-gain Hurwitz, and $\tilde{G}(s,k)$ is not high-gain stable. □

Motivated by Proposition 4.2, we consider alternative controller structures that are robust to errors in relative degree when the relative degree is greater than two.

5. Dynamic Compensation for Systems with Unknown Relative Degree

In this section, we use parameter-dependent dynamic compensation to stabilize (3.1) with unknown relative degree. We assume that the bound $\rho > 0$ on the relative degree is known. Hence $0 < r \le \rho$.

For all $j \ge 0$ let F_j be the jth Fibonacci number, where $F_0 = 0$, $F_1 = 1$, $F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, \ldots$, and define

$$f_{\rho,h} \triangleq F_{\rho+2} - F_{h+1}, \quad (5.1)$$

where h satisfies $1 \le h \le \rho$.

Consider the feedback (3.4) with the strictly proper controller

$$\hat{G}(s,k) \triangleq \frac{\delta k^{F_{\rho+2}}\hat{z}(s)}{s^\rho + k^{f_{\rho,\rho}}b_\rho s^{\rho-1} + \cdots + k^{f_{\rho,2}}b_2 s + k^{f_{\rho,1}}b_1}, \quad (5.2)$$

where $k > 0$, $b_1,\ldots,b_\rho$ are real numbers, and $\hat{z}(s)$ is a degree $\rho-1$ monic polynomial. The closed-loop system is

$$\tilde{G}(s,k) \triangleq \frac{G(s)}{1+\hat{G}(s,k)G(s)} = \frac{\tilde{z}(s,k)}{\tilde{p}(s,k)}, \quad (5.3)$$

where

$$\tilde{z}(s,k) \triangleq \delta\beta z(s)\left[s^\rho + k^{f_{\rho,\rho}}b_\rho s^{\rho-1} + k^{f_{\rho,\rho-1}}b_{\rho-1}s^{\rho-2} + \cdots + k^{f_{\rho,2}}b_2 s + k^{f_{\rho,1}}b_1\right], \quad (5.4)$$

$$\tilde{p}(s,k) \triangleq p(s)s^\rho + k^{f_{\rho,\rho}}b_\rho p(s)s^{\rho-1} + k^{f_{\rho,\rho-1}}b_{\rho-1}p(s)s^{\rho-2} + \cdots + k^{f_{\rho,1}}b_1 p(s) + k^{F_{\rho+2}}\beta z(s)\hat{z}(s). \quad (5.5)$$

The following result is used to analyze the stability of (5.3)-(5.5). The result can be viewed as a robust version of Lemma 4.1.

Lemma 5.1. *Let $\rho \ge 2$ be a positive integer, and, for all $i = 0,1,\ldots,\rho-2$, let $C_i(s)$ be the Hurwitz polynomial*

$$C_i(s) \triangleq c_{i+3}s^3 + c_{i+2}s^2 + c_{i+1}s + c_0, \quad (5.6)$$

where $c_{\rho+1} = 1$. Furthermore, let ν be a positive integer, and, for all $i = 1,2,\ldots,\rho+1$, let $q_i(s)$ be a monic polynomial of degree $\nu-1+i$. Finally, let $0 \le j \le \rho$ and let $q_0(s)$ be a monic Hurwitz polynomial of degree $\nu-1+j$. Then, for all $\alpha \in (0,1]$ the degree $\nu+\rho$ monic polynomial

$$\tilde{q}(s,k) \triangleq q_{\rho+1}(s) + k^{f_{\rho,\rho}}c_\rho q_\rho(s) + k^{f_{\rho,\rho-1}}c_{\rho-1}q_{\rho-1}(s) + k^{f_{\rho,\rho-2}}c_{\rho-2}q_{\rho-2}(s) + \cdots + k^{f_{\rho,2}}c_2 q_2(s) + k^{f_{\rho,1}}c_1 q_1(s) + k^{F_{\rho+2}}\alpha c_0 q_0(s) \quad (5.7)$$

is high-gain Hurwitz. Furthermore, as $k \to \infty$, $\nu-1+j$ roots of $\tilde{q}(s,k)$ converge to the roots of $q_0(s)$, and the real parts of the remaining $\rho+1-j$ roots approach $-\infty$.

Before proving Lemma 5.1, we show that there exist coefficients $c_1,\ldots,c_\rho$ such that the polynomials

$C_0(s),\dots,C_{\rho-2}(s)$ are Hurwitz. First, let $c_\rho > 0$ and $c_{\rho-1} > 0$ be such that $c_{\rho-1}c_\rho > c_0c_{\rho+1} = c_0$, which implies that $C_{\rho-2}(s)$ is Hurwitz. Next, let $c_{\rho-2} > \frac{c_0c_\rho}{c_{\rho-1}}$, which implies that $C_{\rho-3}(s)$ is Hurwitz. In the same manner, for $i = 4,5,\dots,\rho$, let $c_{\rho-i+1} > \frac{c_0c_{\rho-i+3}}{c_{\rho-i+2}}$ so that $C_{\rho-i}(s)$ is Hurwitz. Thus $C_0(s),\dots,C_{\rho-2}$ are Hurwitz.

Proof. Suppose $j = 0$ and write

$$q_0(s) = s^{\nu-1} + a_{\nu-2}s^{\nu-2} + \cdots + a_1 s + a_0. \tag{5.8}$$

The Hurwitz conditions for the stability of $\tilde{q}(s,k)$ are polynomials in k. For sufficiently large k, the Hurwitz conditions for $\tilde{q}(s,k)$ are satisfied if

$$\Lambda_1 \triangleq k^{f_{\rho,\rho}}c_\rho > 0, \tag{5.9}$$

$$\Lambda_2 \triangleq \begin{vmatrix} k^{f_{\rho,\rho}}c_\rho & k^{f_{\rho,\rho-2}}c_{\rho-2} \\ 1 & k^{f_{\rho,\rho-1}}c_{\rho-1} \end{vmatrix} > 0, \tag{5.10}$$

$$\Lambda_3 \triangleq \begin{vmatrix} k^{f_{\rho,\rho}}c_\rho & k^{f_{\rho,\rho-2}}c_{\rho-2} & k^{f_{\rho,\rho-4}}c_{\rho-4} \\ 1 & k^{f_{\rho,\rho-1}}c_{\rho-1} & k^{f_{\rho,\rho-3}}c_{\rho-3} \\ 0 & k^{f_{\rho,\rho}}c_\rho & k^{f_{\rho,\rho-2}}c_{\rho-2} \end{vmatrix} > 0, \tag{5.11}$$

$$\vdots$$

$$\Lambda_{\rho+1} \triangleq \begin{vmatrix} \Lambda_3 & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ & & & k^{F_{\rho+2}}\alpha c_0 a_{\nu-2} \\ 0 & \cdots & k^{f_{\rho,2}}c_2 & k^{F_{\rho+2}}\alpha c_0 \end{vmatrix} > 0, \tag{5.12}$$

$$\Lambda_{\rho+2} \triangleq \begin{vmatrix} \Lambda_3 & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ & & & k^{F_{\rho+2}}\alpha c_0 a_{\nu-3} \\ 0 & \cdots & k^{f_{\rho,1}}c_1 & k^{F_{\rho+2}}\alpha c_0 a_{\nu-2} \end{vmatrix} > 0, \tag{5.13}$$

$$\vdots$$

$$\Lambda_{\rho+\nu} \triangleq \begin{vmatrix} \Lambda_{\rho+2} & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ & & & 0 \\ 0 & \cdots & k^{F_{\rho+2}}\alpha c_0 a_2 & k^{F_{\rho+2}}\alpha c_0 a_0 \end{vmatrix} > 0. \tag{5.14}$$

The first $\rho+1$ conditions are equivalent to $c_i > 0$ for $i = 0,1,\dots,\rho$, and

$$c_2c_1 - \alpha c_3 c_0 > 0, \tag{5.15}$$

which is satisfied since $C_0(s)$ is Hurwitz. The last $\nu - 1$ conditions are equivalent to the Hurwitz conditions for $q_0(s)$. Therefore, $\tilde{q}(s,k)$ is high-gain Hurwitz.

Suppose $j = 1$ and write $q_0(s) = s^\nu + a_{\nu-1}s^{\nu-1} + \cdots + a_1 s + a_0$. For sufficiently large k, the first ρ Hurwitz conditions for $\tilde{q}(s,k)$ are satisfied if $c_i > 0$ for all $i = 0,2,3,\dots,\rho$, and $c_3c_2 - \alpha c_4 c_0 > 0$, which is satisfied since $C_1(s)$ is Hurwitz. The last ν conditions are equivalent to the Hurwitz conditions for $q_0(s)$. Therefore, $\tilde{q}(s,k)$ is high-gain Hurwitz.

The same argument holds for deg $q_0(s) = \nu+1,\dots,\nu+\rho-3$.

Suppose $j = \rho-1$ and let $q_0(s)$ be a degree $\nu+\rho-2$ polynomial. For sufficiently large k, the first two Hurwitz conditions for $\tilde{q}(s,k)$ are satisfied if $c_0 > 0$ and $c_\rho > 0$. The Hurwitz assumption for $q_0(s)$ implies that the remaining $\nu+\rho-2$ Hurwitz conditions for $\tilde{q}(s,k)$ are satisfied for sufficiently large k. Therefore, $\tilde{q}(s,k)$ is high-gain Hurwitz.

Suppose $j = \rho$ and let $q_0(s)$ be a degree $\nu+\rho-1$ polynomial. For sufficiently large k, the first Hurwitz conditions for $\tilde{q}(s,k)$ is satisfied if $c_0 > 0$. The Hurwitz assumption for $q_0(s)$ implies that the remaining $\nu+\rho-1$ Hurwitz conditions for $\tilde{q}(s,k)$ are satisfied for sufficiently large k. Therefore, $\tilde{q}(s,k)$ is high-gain Hurwitz.

The final statement of Lemma 4.1 follows from factoring (5.7) in k in a similar fashion to (4.13). Applying root locus techniques iteratively to relative degree one polynomials a total of $\rho+1$ times yields the asymptotic result. □

The following result is an immediate consequence of Lemma 5.1 with $\nu = n$, $j = \rho - r$, $q_0(s) = z(s)\hat{z}(s)$, $\alpha c_0 = \beta$, $c_i = b_i$ for $i = 0,\dots,\rho$, and $q_i(s) = p(s)s^{i-1}$ for $i = 1,\dots,\rho+1$.

Theorem 5.1. *Consider the closed-loop system (5.3)-(5.5). Assume that the polynomials $\hat{z}(s)$,*

$$B_{\rho-2}(s) \triangleq s^3 + b_\rho s^2 + b_{\rho-1}s + b_0, \tag{5.16}$$

and

$$B_i(s) \triangleq b_{i+3}s^3 + b_{i+2}s^2 + b_{i+1}s + b_0, \tag{5.17}$$

for all $i = 0,1,\dots,\rho-3$ are Hurwitz. Then $\tilde{p}(s,k)$ is high-gain Hurwitz and thus $\tilde{G}(s,k)$ is high-gain stable. Furthermore, as $k \to \infty$, $m+\rho-1$ roots of $\tilde{p}(s,k)$ converge to the roots of $z(s)\hat{z}(s)$ and the real parts of the remaining $r+1$ roots approach $-\infty$.

For implementation purposes, it is desirable that the controller $\hat{G}(s,k)$ be stable. We present the following stability result, which is a consequence of the Hurwitz conditions for $\hat{G}(s,k)$.

Proposition 5.1. *Consider the controller given by (5.2), and assume that*

$$\hat{B}(s) \triangleq s^\rho + b_\rho s^{\rho-1} + b_{\rho-1}s^{\rho-2} + \cdots + b_2 s + b_1 \tag{5.18}$$

is Hurwitz. Then the controller is stable for all $k > 1$.

6. Parameter-Monotonic Adaptive Control

In Section 5 we presented the strictly proper compensator (5.2), where the stabilizing parameter k_s is unknown. In this section, we consider parameter-monotonic adaptive stabilization for a general class of high-gain stable systems.

Consider the parameter-dependent transfer function

$$G(s,k) = \frac{z(s,k)}{p(s,k)}, \tag{6.1}$$

where $z(s,k)$ and $p(s,k)$ are polynomials in s over the reals with coefficients that are polynomial functions in k. Furthermore, for all $k > 0$, the degree of $z(s,k)$ in s is less than or equal to $\nu \triangleq \deg p(s,k)$ in s, where ν is assumed to be independent of k. The transfer function $G(s,k)$ is thus a parameter-dependent transfer function.

The following result is an immediate consequence of forming an observable canonical realization.

Proposition 6.1. *For all $k > 0$, there exist matrices $A(k) \in \mathbb{R}^{\nu\times\nu}$, $B(k) \in \mathbb{R}^{\nu\times 1}$, and $D(k) \in \mathbb{R}$ whose entries are polynomial functions of k such that,*

$$G(s,k) = C\left[sI - A(k)\right]^{-1} B(k) + D(k), \tag{6.2}$$

where $C \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{1\times\nu}$. If, in addition, $p(s,k)$ is high-gain Hurwitz, then there exists $k_{\rm s} > 0$ such that, for all $k \geq k_{\rm s}$, $A(k)$ is asymptotically stable.

The following lemma will be used to prove a general result for parameter-monotonic adaptive control.

Lemma 6.1. *Let $(A(k), C)$ be an observable pair whose entries are polynomial functions of k, and assume that there exists $k_{\rm s} > 0$ such that, for all $k \geq k_{\rm s}$, $A(k)$ is asymptotically stable. Then for all $k \geq k_{\rm s}$, there exists a positive semi-definite matrix $P(k)$, whose entries are real rational functions of k, satisfying*

$$A^{\rm T}(k)P(k) + P(k)A(k) = -C^{\rm T}C. \tag{6.3}$$

The following result concerns parameter-monotonic adaptive stabilization.

Theorem 6.1. *Let $A(k) \in \mathbb{R}^{\nu\times\nu}$ have polynomial entries in k, and let $C \in \mathbb{R}^{1\times\nu}$, where $(A(k), C)$ is observable for all $k > 0$, and assume there exists $k_{\rm s} > 0$ such that, for all $k \geq k_{\rm s}$, $A(k)$ is asymptotically stable. Consider the system*

$$\dot{x}(t) = A(k)x(t), \qquad y(t) = Cx(t), \tag{6.4}$$

and the parameter-monotonic adaptive law

$$\dot{k}(t) = \gamma y^2(t), \tag{6.5}$$

where $\gamma > 0$ and $k(0) > 0$. Then, for all initial conditions $x(0)$, $k_\infty \triangleq \lim_{t\to\infty} k(t)$ exists and $\lim_{t\to\infty} x(t) = 0$.

Proof. We first show that $k(t)$ converges. For all $k \geq k_{\rm s}$, define

$$V(x,k) \triangleq x^{\rm T}P(k)x, \tag{6.6}$$

where $P(k)$ is given by Lemma 6.1. Note that $V(x,k) \geq 0$ for all $k \geq k_{\rm s}$ and for all $x \in \mathbb{R}^\nu$. Taking the derivative of $V(x,k)$ along trajectories of (6.4) yields

$$\begin{aligned}\dot{V}(x,k) &= -x^{\rm T}C^{\rm T}Cx + \dot{k}x^{\rm T}\frac{\partial P(k)}{\partial k}x \\ &= \left[x^{\rm T}\frac{\partial P(k)}{\partial k}x - \frac{1}{\gamma}\right]\frac{dk}{dt},\end{aligned} \tag{6.7}$$

which implies

$$\dot{V}(x,k)dt = \left[x^{\rm T}\frac{\partial P(k)}{\partial k}x - \frac{1}{\gamma}\right]dk. \tag{6.8}$$

Next, we show that if $x(t)$ escapes at finite time t_1, then $k(t)$ escapes at finite time t_1. Assume that $k(t)$ does not escape at finite time t_1. We can therefore consider (6.4) to be a linear time-varying differential equation, where $A(k(t))$ is continuous in t. The solution to a linear time-varying system, where $A(t)$ is continuous in t, exists and is unique globally [3]. Therefore, $x(t)$ does not escape at finite time t_1. Hence, $x(t)$ escapes at finite time t_1 only if $k(t)$ escapes at finite time t_1.

Now suppose that $k(t)$ diverges to infinity in either finite or infinite time. Then there exists t_s such that $k(t_s) = k_{\rm s}$. Sine $k(t)$ does not escape at t_s, it follows that $x(t)$ does not escape at time t_s. Therefore, integrating (6.8) from t_s to t and from $k_{\rm s}$ to k, then solving for $k(t)$ yields

$$\begin{aligned}k(t) &= \gamma V(x(t_s), k_{\rm s}) + k_{\rm s} - \gamma x^{\rm T}(t)P(k_{\rm s})x(t) \\ &\leq \gamma V(x(t_s), k_{\rm s}) + k_{\rm s}.\end{aligned} \tag{6.9}$$

Hence $k(t)$ is bounded, which is a contradiction. Since $k(t)$ is non-decreasing, $k_\infty \triangleq \lim_{t\to\infty} k(t)$ exists.

To show that $A(k_\infty)$ is asymptotically stable, assume that $A(k_\infty)$ is not asymptotically stable and write

$$\dot{x}(t) = A(k_\infty)x(t), \qquad y(t) = Cx(t), \tag{6.10}$$

so that $y(t) = Ce^{A(k_\infty)t}x(0)$. Since $A(k_\infty)$ is not asymptotically stable and $(A(k_\infty), C)$ is observable, it follows that there exists an initial state $x(0)$ such that $\lim_{t\to\infty}\int_0^t y^2(\tau)d\tau = \infty$. The adaptive law (6.5) implies

$$k_\infty - k(0) = \lim_{t\to\infty} k(t) - k(0) = \lim_{t\to\infty}\gamma\int_0^t y^2(\tau)d\tau = \infty, \tag{6.11}$$

which is a contradiction. Hence $A(k_\infty)$ is asymptotically stable.

Next, to prove that $x(t)$ converges to zero, we write (6.4) as

$$\dot{x}(t) = \left[A(k_\infty) + \Delta(t)\right]x(t), \tag{6.12}$$

where $\Delta(t) \triangleq A(k(t)) - A(k_\infty)$. Note that $A(k_\infty)$ is asymptotically stable and $\Delta(t) \to 0$ as $t \to \infty$. Consider the Lyapunov candidate

$$V(x) \triangleq x^{\rm T}Px, \tag{6.13}$$

where $A^{\rm T}(k_\infty)P + PA(k_\infty) = -Q$, $P = P^{\rm T} > 0$, and $Q = Q^{\rm T} > 0$. Taking the derivative along trajectories yields

$$\dot{V}(x(t)) = -x^{\rm T}(t)Qx(t) + x^{\rm T}(t)\left[\Delta^{\rm T}(t)P + P\Delta(t)\right]x(t) \tag{6.14}$$

$$\leq \left[-c_1 + c_2(t)\right]\|x(t)\|_2^2, \tag{6.15}$$

where $c_1 > 0$, and $c_2(t) \to 0$ as $t \to \infty$. Therefore, there exists $t_0 \geq 0$ such that, for all $t \geq t_0$,

$$\dot{V}(x(t)) \leq -\frac{c_1}{2}\|x(t)\|_2^2. \tag{6.16}$$

Hence, $V_\infty \triangleq \lim_{t\to\infty} V(x(t))$ exists. Assume $V_\infty > 0$. For all $t \geq t_0$,

$$V_\infty \leq x^{\rm T}(t)Px(t) \leq \sigma_{\max}(P)\|x(t)\|_2^2. \tag{6.17}$$

Combining (6.16) and (6.17) yields

$$\dot{V}(x(t)) \leq -\frac{c_1 V_\infty}{2\sigma_{\max}(P)}. \tag{6.18}$$

Integrating (6.18) shows that $V(x(t)) \to -\infty$ as $t \to \infty$, which is a contradiction. Hence $\lim_{t\to\infty} V(x(t)) = 0$, and thus $\lim_{t\to\infty} x(t) = 0$. □

7. Parameter-Monotonic Adaptive Stabilization for Systems with Unknown Relative Degree

Now, we apply Theorem 6.1 to the strictly proper parameter-dependent dynamic compensator (5.2) which stabilizes minimum phase systems with unknown relative degree.

Proposition 6.1 implies that $\tilde{G}(s,k)$ has the observable canonical realization

$$\dot{\tilde{x}}(t) = \tilde{A}(k)\tilde{x}(t) + \tilde{B}(k)v(t), \qquad \tilde{x}(0) = \tilde{x}_0, \tag{7.1}$$

$$y(t) = \tilde{C}\tilde{x}(t), \tag{7.2}$$

where $(\tilde{A}(k), \tilde{C})$ is observable for all $k \in \mathbb{R}$. Furthermore, Theorem 5.1 implies that there exists $k_{\mathrm{s}} > 0$ such that, for all $k \geq k_{\mathrm{s}}$, $A(k)$ is asymptotically stable.

Therefore, the following result is an immediate consequence of Theorem 6.1.

Theorem 7.1. *Consider the closed-loop system (7.1)-(7.2) with unknown relative degree r satisfying $0 < r \leq \rho$. Furthermore, consider the parameter-monotonic adaptive law*

$$\dot{k}(t) = \gamma y(t)^2, \tag{7.3}$$

where $\gamma > 0$ and $k(0) > 0$. Let $v(t) \equiv 0$. Then, for all initial conditions, $k(t)$ converges and $\lim_{t\to\infty} \tilde{x}(t) = 0$.

Theorem 7.1 presents an adaptive compensator for systems with bounded relative degree. If the relative degree of the system is known, then the controller (4.1) can be used with the adaptive law $\dot{k}(t) = \gamma y(t)^2$. The conclusions and proof of Theorem 7.1 remain unchanged.

8. Counterexample to the results of [2] revisited

In this section, we consider the unstable plant (2.1). In Section 2, we demonstrated that the 10th-order controller proposed in [2] can fail to stabilize (2.1). In contrast, consider the parameter-dependent dynamic compensator (4.1) with $r = 5$

$$\hat{G}(s,k) = \frac{\delta k^6 \hat{z}(s)}{s^5 + kb_5 s^4 + k^2 b_4 s^3 + k^3 b_3 s^2 + k^4 b_2 s + k^5 b_1}, \tag{8.1}$$

where $\hat{z}(s)$ is a degree 4 monic Hurwitz polynomial. We assume that the high-frequency gain of (2.1) is known to be positive and that $b_0 = 2.5$ as in Section 2. To satisfy the assumptions of Theorem 4.1, the numerator polynomial is chosen to be

$$\hat{z}(s) = (s+10)(s+15)(s+20)(s+25), \tag{8.2}$$

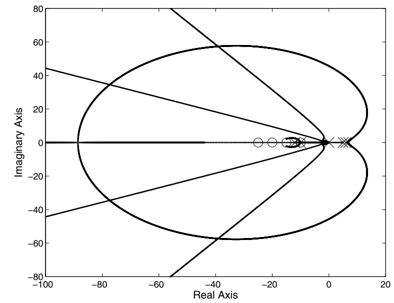

Fig. 2. Root locus of the closed-loop dynamics using parameter-dependent dynamic compensation. The closed-loop system is high-gain stable.

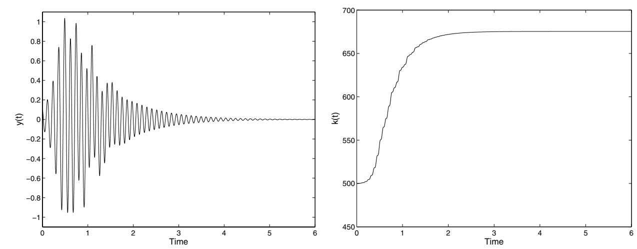

Fig. 3. Time history of the output $y(t)$ of the closed-loop system (left) and of the adaptive parameter $k(t)$ (right).

and the design parameters are chosen to be $b_1 = 13.5$, $b_2 = 30$, $b_3 = 35$, $b_4 = 22.5$, and $b_5 = 7.5$. Figure 2 illustrates the root locus for the closed-loop characteristic polynomial as $k \to \infty$. The zero-gain $k = 0$ pole locations are shown by $\times$'s, and the zero locations, which attract certain poles, are shown by $\circ$'s. Four of the closed-loop poles converge to the stable zero locations as $k \to \infty$. The remaining six closed-loop poles diverge to infinity through the left-half plane. Thus the closed-loop system is high-gain stable.

Theorem 7.1 yields an adaptive controller given by a state-space realization of (8.1) and the adaptive law $\dot{k}(t) = \gamma y(t)^2$, where we choose $\gamma = 500$. The system (2.1) is simulated with the initial condition $x(0) = \begin{bmatrix} 0.1 & -0.1 & -0.5 & 1.5 & 2.0 \end{bmatrix}^{\mathrm{T}}$. The adaptive controller is implemented in the feedback loop with the initial conditions $k(0) = 500$ and $\hat{x}(0) = 0$. Figure 3 shows that the output y of the closed-loop system converges to zero, and the adaptive parameter $k(t)$ converges to approximately 675.3.

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