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Combined L_2/H_∞ model reduction

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A model-reduction problem is considered which involves both L_2 (quadratic) and H_∞ (worst-case frequency-domain) aspects. Specifically, the goal of the problem is to minimize an L_2 model-reduction criterion subject to a prespecified H_∞ constraint on the model-reduction error. The principal result is a sufficient condition for characterizing reduced-order models with bounded L_2 and H_∞ approximation error. The sufficient condition involves a system of modified Riccati equations coupled by an oblique projection, i.e. idempotent matrix. When the H_∞ constraint is absent, the sufficient condition specializes to the L_2 model-reduction result given by Hyland and Bernstein (1985).

Notation and definitions

$\mathbb{R}, \mathbb{R}^{r\times s}, \mathbb{R}^r, \mathbb{E}$	real numbers, $r\times s$ real matrices, $\mathbb{R}^{r\times 1}$, expected value
$I_r, (\ )^{\mathrm{T}}, 0_{r\times s}, 0_r$	$r\times r$ identity matrix, transpose, $r\times s$ zero matrix, $0_{r\times r}$
$(\ )^*$	complex conjugate transpose
tr	trace
$\sigma_{\max}(Z)$	largest singular value of matrix Z
$\lambda_{\max}(Z)$	largest eigenvalue of matrix Z with real spectrum
$\lVert Z\rVert_F$	$[\operatorname{tr} ZZ^*]^{1/2}$ (Frobenius matrix norm)
$\lVert h(t)\rVert_2$	$[\int_0^\infty \lVert h(t)\rVert_F^2\, dt]^{1/2}$
$\lVert H(s)\rVert_2$	$[(1/2\pi)\int_{-\infty}^{\infty} \lVert H(j\omega)\rVert_F^2\, d\omega]^{1/2}$
$\lVert H(s)\rVert_\infty$	$\sup_{\omega\in\mathbb{R}} \sigma_{\max}[H(j\omega)]$
$\mathbb{S}^r, \mathbb{N}^r, \mathbb{P}^r$	$r\times r$ symmetric, non-negative-definite, positive-definite matrices
$Z_1 \leqslant Z_2, Z_1 < Z_2$	$Z_2 - Z_1 \in \mathbb{N}^r, Z_2 - Z_1 \in \mathbb{P}^r, Z_1, Z_2 \in \mathbb{S}^r$
$n, m, l, n_m, q, p; \tilde{n}$	positive integers; $n + n_m$
$x, y, y_m, x_m, \tilde{y}, \tilde{x}$	$n, l, l, n_m, l, \tilde{n}$-dimensional vectors
$\tilde{y}, \tilde{x}$	$y - y_m, \begin{bmatrix} x \\ x_m \end{bmatrix}$
A, B, C	$n\times n, n\times m, l\times n$ matrices
D, E	$m\times p, q\times l$ matrices
A_m, B_m, C_m	$n_m\times n_m, n_m\times m, l\times n_m$ matrices
$\tilde{A}, \tilde{B}, \tilde{C}$	$\begin{bmatrix} A & 0 \\ 0 & A_m \end{bmatrix}, \begin{bmatrix} B \\ B_m \end{bmatrix}, [C \quad -C_m]$

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$\tilde{D}$, $\tilde{E}$ $\quad \tilde{B}D = \begin{bmatrix} BD \\ B_m D \end{bmatrix}, \; E\tilde{C} = [EC \quad -EC_m]$

R $\quad E^{\mathrm{T}}E$, model-reduction error-weighting matrix in $\mathbb{P}^l$

$w(\cdot)$ $\quad p$-dimensional standard white noise process

V $\quad$ intensity of $Dw(\cdot)$, $V = DD^{\mathrm{T}} \in \mathbb{P}^m$

$\tilde{R}$, $\tilde{V}$ $\quad \begin{bmatrix} C^{\mathrm{T}}RC & -C^{\mathrm{T}}RC_m \\ -C_m^{\mathrm{T}}RC & C_m^{\mathrm{T}}RC_m \end{bmatrix}, \begin{bmatrix} BVB^{\mathrm{T}} & BVB_m^{\mathrm{T}} \\ B_m VB^{\mathrm{T}} & B_m VB_m^{\mathrm{T}} \end{bmatrix}$

γ $\quad$ positive constant

1. Introduction

One of the most fundamental problems in dynamic systems theory is to approximate a high-order, complex system with a low-order, relatively simpler model. The resulting reduced-order model can then be used to facilitate the analysis of complex systems as well as the design and implementation of feedback controllers and electronic filters. The model-reduction problem thus reflects the fundamental engineering desire for simplicity of implementation and parsimony of hardware.

In view of the practical motivations for the model-reduction problem, it is not surprising that significant effort has been devoted to this problem in recent years. Indeed, there now exists a well-developed theoretical foundation for model reduction under a variety of approximation criteria. Expanding on the original work of Adamjan *et al.* (1971), progress was achieved by Kung and Lin (1981), Lin and Kung (1982), Glover (1984), Latham and Anderson (1985), Hung and Glover (1986), Anderson (1986), Ball and Ran (1987) and Parker and Anderson (1987) for the Hankel-norm approximation criterion. Many of the cited works also present bounds for the closely related H_∞ approximation error, although the optimal H_∞ model-reduction problem remains open. Alternatively, early progress on the model-reduction problem with a quadratic (L_2) criterion was achieved by Wilson (1970) and further explored by Hyland and Bernstein (1985).

Although the Hankel norm, H_∞, and L_2 model-reduction criteria represent distinct approximation objectives, there exist significant connections. For example, it was shown by Wilson (1985), that for systems which are either single input or single output, the input and output space topologies can be redefined so that the induced norm of the Hankel operator coincides with the L_2 system norm. In addition, the optimization technique utilized by Wilson (1970) was re-applied to the Hilbert–Schmidt Hankel operator topology by Wilson (1988). In a recent work, Wilson (1989) has shown that for single-input or single-output systems the quadratic model-reduction criterion is actually an induced norm of the convolution operator itself.

In the present paper we attempt a further unification of the L_2 and H_∞ model-reduction objectives. Specifically, we consider an L_2 model-reduction problem with a constraint on the H_∞ approximation error. The underlying idea involves the suitable application of a frequency-domain inequality due to Willems (1971), which has recently been applied to H_∞ control-design problems by Petersen (1987), Khargonekar *et al.* (1987) and Bernstein and Haddad (1989). The principal result of the present paper is a sufficient condition which characterizes reduced-order models satisfying an optimized L_2 bound as well as a pre-specified H_∞ bound. The sufficient condition is a direct generalization of the optimal projection approach developed by Hyland and Bernstein (1985) for the unconstrained L_2 problem. While the L_2-optimal reduced-order model was characterized by Hyland and Bernstein (1985) by means of a coupled system of two modified Lyapunov equations, the H_∞-constrained solution in the present paper involves a coupled system consisting of four modified Riccati equations. As in Hyland and Bernstein (1985), the coupling is due to the presence of an oblique projection (idempotent matrix) that determines the constrained reduced-order model. When the H_∞ constraint is sufficiently relaxed, we show that the conditions given herein specialize directly to those given by Hyland and Bernstein (1985). Although our result gives sufficient conditions for H_∞ approximation, we also state hypotheses under which these conditions are also necessary.

Although numerical algorithms were developed by Hyland and Bernstein (1985) for the 'pure' L_2 problem, computational methods for the H_∞-constrained problem are beyond the scope of the present paper. In view of the additional complexity engendered by the H_∞ constraint, more sophisticated algorithms appear necessary. Hence computational methods will focus on the homotopic continuation algorithm developed by Richter (1987) for reduced-order dynamic compensation.

2. Statement of the problem

In this section we introduce the model-reduction problem with constrained H_∞ norm of the model-reduction error. Specifically, we constrain the transfer function of the reduced-order model to lie within a specified H_∞ radius of the original system. In this paper we assume that the full-order model is asymptotically stable, i.e. the matrix A is asymptotically stable.

H_∞-Constrained L_2 model-reduction problem

Given the nth-order controllable and observable model

$$\dot{x}(t) = Ax(t) + BDw(t) \tag{2.1}$$

$$y(t) = Cx(t) \tag{2.2}$$

where $t \in [0, \infty)$, determine an n_mth-order model

$$\dot{x}_m(t) = A_m x_m(t) + B_m D w(t) \tag{2.3}$$

$$y_m(t) = C_m x_m(t) \tag{2.4}$$

which satisfies the following criteria:

(i) A_m is asymptotically stable;

(ii) the transfer function of the reduced-order model lies within a radius-γ H_∞ neighbourhood of the full-order model, i.e.

$$\|H(s) - H_m(s)\|_\infty \leqslant \gamma \tag{2.5}$$

where

$$H(s) \triangleq EC(sI_n - A)^{-1}BD, \quad H_m(s) \triangleq EC_m(sI_{n_m} - A_m)^{-1}B_m D \tag{2.6}$$

and $\gamma > 0$ is a given constant; and

(iii) the L_2 model-reduction criterion

$$J(A_m, B_m, C_m) \triangleq \lim_{t \to \infty} \mathbb{E}\{[y(t) - y_m(t)]^{\mathrm{T}} R [y(t) - y_m(t)]\} \tag{2.7}$$

is minimized.

Note that the full- and reduced-order systems (2.1)–(2.4) can be written as a single augmented system

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t), \quad t \in [0, \infty) \tag{2.8}$$

so that the $q \times p$ transfer function from $w(t)$ to $E\tilde{y}(t) = \tilde{E}\tilde{x}(t)$ is

$$\tilde{H}(s) = \tilde{E}(sI_{\tilde{n}} - \tilde{A})^{-1}\tilde{D} \tag{2.9}$$

and (2.7) can be written as

$$J(A_m, B_m, C_m) = \lim_{t \to \infty} \mathbb{E}\{[E\tilde{y}(t)]^{\mathrm{T}}[E\tilde{y}(t)]\} = \lim_{t \to \infty} \mathbb{E}[\tilde{x}^{\mathrm{T}}(t)\tilde{R}\tilde{x}(t)] \tag{2.10}$$

Before continuing it is useful to note that if A_m is asymptotically stable then the L_2 model-reduction criterion (2.7) is given by

$$J(A_m, B_m, C_m) = \operatorname{tr} \tilde{Q}\tilde{R} \tag{2.11}$$

where the steady-state covariance

$$\tilde{Q} \triangleq \lim_{t \to \infty} \mathbb{E}[\tilde{x}(t)\tilde{x}^{\mathrm{T}}(t)] \tag{2.12}$$

satisfies the augmented Lyapunov equation

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{V} \tag{2.13}$$

Using (2.11) and (2.13) it can be shown that the L_2 criterion (2.7) is an approximation measure involving the full- and reduced-order impulse responses with respect to an L_2 norm.

Proposition 2.1

The L_2 model-reduction criterion (2.11) can be written as

$$J(A_m, B_m, C_m) = \int_0^\infty \|EC \exp(At)BD - EC_m \exp(A_m t)B_m D\|_F^2 \, dt \tag{2.14 a}$$

or, equivalently

$$J(A_m, B_m, C_m) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|H(j\omega) - H_m(j\omega)\|_F^2 \, d\omega \tag{2.14 b}$$

Proof

It need only be noted that (2.11) is equivalent to

$$\begin{aligned} \operatorname{tr} \int_0^\infty \exp(\tilde{A}t)\tilde{V}\exp(\tilde{A}^{\mathrm{T}}t)\, dt \tilde{R} &= \operatorname{tr} \int_0^\infty \tilde{E}\exp(\tilde{A}t)\tilde{D}\tilde{D}^{\mathrm{T}}\exp(\tilde{A}^{\mathrm{T}}t)\tilde{E}^{\mathrm{T}}\, dt \\ &= \operatorname{tr} \int_0^\infty (\tilde{E}\exp(\tilde{A}t)\tilde{D})(\tilde{E}\exp(\tilde{A}t)\tilde{D})^{\mathrm{T}}\, dt \\ &= \int_0^\infty \|\tilde{E}\exp(\tilde{A}t)\tilde{D}\|_F^2\, dt \end{aligned}$$

which is equivalent to (2.14 *a*). Finally, (2.14 *b*) follows from Plancherel's Theorem. □

The key step in enforcing (2.5) is to replace the algebraic Lyapunov equation (2.13) by an algebraic Riccati equation. Justification for this technique is provided by the following result.

Lemma 2.1

Let (A_m, B_m, C_m) be given and assume there exists $\mathcal{Q} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ satisfying

$$\mathcal{Q} \in \mathbb{N}^{\tilde{n}} \tag{2.15}$$

and

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q} + \tilde{V} \tag{2.16}$$

Then

$$(\tilde{A}, \tilde{D}) \text{ is stabilizable} \tag{2.17}$$

if and only if

$$A_m \text{ is asymptotically stable} \tag{2.18}$$

Furthermore, in this case

$$\|H(s) - H_m(s)\|_\infty \leqslant \gamma \tag{2.19}$$

$$\tilde{Q} \leqslant \mathcal{Q} \tag{2.20}$$

and

$$J(A_m, B_m, C_m) \leqslant \mathcal{J}(A_m, B_m, C_m, \mathcal{Q}) \tag{2.21}$$

where

$$\mathcal{J}(A_m, B_m, C_m, \mathcal{Q}) \triangleq \operatorname{tr} \mathcal{Q}\tilde{R} \tag{2.22}$$

Proof

Using the assumed existence of a non-negative-definite solution to (2.16) and the stabilizability condition (2.17), it follows from the dual of Lemma 12.2 of Wonham (1979) that $\tilde{A}$ is asymptotically stable. Since $\tilde{A}$ is block diagonal, A_m is also asymptotically stable. Conversely, since A is assumed to be asymptotically stable, (2.18) implies (2.17). The proof of (2.19) follows from a standard manipulation of (2.16); for details see Lemma 1 of Willems (1971). To prove (2.20), subtract (2.13) from (2.16) to obtain

$$0 = \tilde{A}(\mathcal{Q} - \tilde{Q}) + (\mathcal{Q} - \tilde{Q})\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q} \tag{2.23}$$

which, since $\tilde{A}$ is asymptotically stable, is equivalent to

$$\mathcal{Q} - \tilde{Q} = \int_0^\infty \exp(\tilde{A}t)[\gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q}] \exp(\tilde{A}^{\mathrm{T}}t)\, dt \geqslant 0 \tag{2.24}$$

Finally, (2.21) follows immediately from (2.20). □

Lemma 2.1 shows that the H_∞ constraint is automatically enforced when a non-negative-definite solution to (2.16) is known to exist. Furthermore, the solution $\mathcal{Q}$ provides an upper bound for the actual state covariance $\tilde{Q}$ along with a bound on the L_2 model-reduction criterion. Next, we present a partial converse of Lemma 2.1 which guarantees the existence of a non-negative definite solution to (2.16) when (2.19) is satisfied.

Lemma 2.2

Let (A_m, B_m, C_m) be given, suppose $\tilde{A}$ is asymptotically stable, and assume the H_∞ approximation constraint (2.19) is satisfied. Then there exists a unique non-negative-definite solution $\mathcal{Q}$ satisfying (2.16) and such that $\tilde{A}+\gamma^{-2}\mathcal{Q}\tilde{R}$ is asymptotically stable. Furthermore, this solution is minimal.

Proof

The result is an immediate consequence of Theorems 3 and 2, of Brockett (1970; pp. 150, 167) and the dual of Lemma 12.2 of Wonham (1979). □

Finally, we show that the quadratic term $\gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q}$ in (2.16) also constrains the Hankel norm of the approximation error $E\tilde{y}$ when $\mathcal{Q}$ is positive-definite. To show this, let $\tilde{P}\in\mathbb{N}^{\tilde{n}}$ be the observability Gramian for the augmented system $(\tilde{A}, \tilde{D}, \tilde{E})$ which satisfies

$$0=\tilde{A}^{\mathrm{T}}\tilde{P}+\tilde{P}\tilde{A}+\tilde{R} \tag{2.25}$$

Furthermore, note that $\tilde{Q}$ satisfying (2.13) is the dual controllability Gramian.

Proposition 2.2

Let (A_m, B_m, C_m) be given and assume there exists $\mathcal{Q}\in\mathbb{P}^{\tilde{n}}$ satisfying (2.16) and (2.17) or, equivalently, (2.18). Then

$$\lambda_{\max}^{1/2}(\tilde{P}\tilde{Q})\leqslant\gamma \tag{2.26}$$

Proof

Since $\mathcal{Q}$ is invertible, (2.16) implies

$$0=\gamma^2\tilde{A}^{\mathrm{T}}\mathcal{Q}^{-1}+\gamma^2\mathcal{Q}^{-1}\tilde{A}+\gamma^2\mathcal{Q}^{-1}\tilde{V}\mathcal{Q}^{-1}+\tilde{R} \tag{2.27}$$

Next, subtract (2.25) from (2.27) to obtain

$$0=\tilde{A}^{\mathrm{T}}(\gamma^2\mathcal{Q}^{-1}-\tilde{P})+(\gamma^2\mathcal{Q}^{-1}-\tilde{P})\tilde{A}+\gamma^2\mathcal{Q}^{-1}\tilde{V}\mathcal{Q}^{-1} \tag{2.28}$$

which, since $\tilde{A}$ is asymptotically stable, is equivalent to

$$\gamma^2\mathcal{Q}^{-1}-\tilde{P}=\int_0^\infty \exp(\tilde{A}^{\mathrm{T}}t)[\gamma^2\mathcal{Q}^{-1}\tilde{V}\mathcal{Q}^{-1}]\exp(\tilde{A}t)\,dt\geqslant 0 \tag{2.29}$$

Thus, (2.29) implies $\tilde{P}\leqslant\gamma^2\mathcal{Q}^{-1}$, or, equivalently, $\mathcal{Q}^{1/2}\tilde{P}\mathcal{Q}^{1/2}\leqslant\gamma^2 I_{\tilde{n}}$. Hence, $\lambda_{\max}^{1/2}(\tilde{P}\mathcal{Q})\leqslant\gamma$. Finally, (2.26) follows immediately from (2.20). □

3. Auxiliary minimization problem and necessary conditions for optimality

As discussed in the previous section, the replacement of (2.13) by (2.16) enforces the H_∞ approximation constraint between the full- and reduced-order systems and results in an upper bound for the L_2 model-reduction criterion. That is, if (2.16) is solvable then the reduced-order model (A_m, B_m, C_m) satisfies the H_∞ approximation constraint (2.5) while the actual L_2 model-reduction criterion is guaranteed to be no worse than the bound given by $\mathscr{J}(A_m, B_m, C_m, \mathscr{Q})$. Hence, $\mathscr{J}(A_m, B_m, C_m, \mathscr{Q})$ can be interpreted as an auxiliary cost that leads to the following mathematical programming problem.

Auxiliary minimization problem

Determine $(A_m, B_m, C_m, \mathscr{Q})$ that minimizes $\mathscr{J}(A_m, B_m, C_m, \mathscr{Q})$ subject to (2.15) and (2.16).

It follows from Lemma 2.1 that the satisfaction of (2.15)–(2.17) leads to (i) A_m stable; (ii) a bound on the H_∞ distance between the full-order and reduced-order systems; and (iii) an upper bound for the L_2 model-reduction criterion. Hence, it remains to determine (A_m, B_m, C_m) that minimizes $\mathscr{J}(A_m, B_m, C_m, \mathscr{Q})$ and thus provides an optimized bound for the actual L_2 criterion $J(A_m, B_m, C_m)$. Rigorous derivation of the necessary conditions for the auxiliary minimization problem requires additional technical assumptions. Specifically, we restrict $(A_m, B_m, C_m, \mathscr{Q})$ to the open set

$$\mathscr{S} \triangleq \{(A_m, B_m, C_m, \mathscr{Q}) : \mathscr{Q} \in \mathbb{P}^{\tilde{n}}, \tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R} \text{ is asymptotically stable,}$$
$$\text{and } (A_m, B_m, C_m) \text{ is controllable and observable}\} \tag{3.1}$$

Remark 1

The set $\mathscr{S}$ constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the auxiliary minimization problem. Specifically, the requirement that $\mathscr{Q}$ be positive-definite replaces (2.15) by an open set constraint, the stability of $\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}$ serves as a normality condition and (A_m, B_m, C_m) minimal is a non-degeneracy condition.

The following lemma is needed for the statement of the main result.

Lemma 3.1

Let $\hat{Q}, \hat{P} \in \mathbb{N}^n$ and suppose rank $\hat{Q}\hat{P} = n_m$. Then there exist $n_m \times n$ G, Γ and $n_m \times n_m$ invertible M, unique except for a change of basis in $\mathbb{R}^{n_m}$, such that

$$\hat{Q}\hat{P} = G^{\mathrm{T}} M \Gamma \tag{3.2}$$

$$\Gamma G^{\mathrm{T}} = I_{n_m} \tag{3.3}$$

Furthermore, the $n \times n$ matrices

$$\tau \triangleq G^{\mathrm{T}} \Gamma \tag{3.4}$$

$$\tau_\perp \triangleq I_n - \tau \tag{3.5}$$

are idempotent and have rank n_m and $n - n_m$, respectively. If, in addition

$$\text{rank } \hat{Q} = \text{rank } \hat{P} = n_m \tag{3.6}$$

then

$$\hat{Q} = \tau \hat{Q}, \quad \hat{P} = \hat{P}\tau \tag{3.7), (3.8}$$

Finally, if $P \in \mathbb{N}^n$ then the inverse

$$S \triangleq (I_n + \gamma^{-2}\hat{Q}P)^{-1} \tag{3.9}$$

exists.

Proof

Conditions (3.2)–(3.8) are a direct consequence of Theorem 6.2.5 of Rao and Mitra (1971). To prove that the inverse in (3.9) exists, note that since the eigenvalues of $\hat{Q}P$ coincide with the eigenvalues of the non-negative-definite matrix $P^{1/2}\hat{Q}P^{1/2}$, it follows that $\hat{Q}P$ has non-negative eigenvalues. Thus, the eigenvalues of $I_n + \gamma^{-2}\hat{Q}P$ are all greater than one so that the above inverse exists. □

Finally, for convenience define

$$\Sigma \triangleq BVB^{\mathrm{T}}, \quad \bar{\Sigma} \triangleq C^{\mathrm{T}}RC$$

Theorem 3.1

If $(A_m, B_m, C_m, \mathscr{Q}) \in \mathscr{S}$ solves the auxiliary minimization problem then there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ such that

$$A_m = \Gamma(A - \gamma^{-4}Q\bar{\Sigma}QPS)G^{\mathrm{T}} \tag{3.10}$$

$$B_m = \Gamma B \tag{3.11}$$

$$C_m = C(I_n + \gamma^{-2}QPS)G^{\mathrm{T}} \tag{3.12}$$

$$\mathscr{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^{\mathrm{T}} \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^{\mathrm{T}} \end{bmatrix} \tag{3.13}$$

and such that $Q, P, \hat{Q}, \hat{P}$ satisfy

$$0 = AQ + QA^{\mathrm{T}} + \gamma^{-2}Q\bar{\Sigma}Q + \tau_\perp\Sigma\tau_\perp^{\mathrm{T}} \tag{3.14}$$

$$0 = A^{\mathrm{T}}P + PA - \gamma^{-4}S^{\mathrm{T}}PQ\bar{\Sigma}QPS + \tau_\perp^{\mathrm{T}}(I_n + \gamma^{-2}QPS)^{\mathrm{T}}\bar{\Sigma}(I_n + \gamma^{-2}QPS)\tau_\perp \tag{3.15}$$

$$\begin{aligned} 0 = {} & (A - \gamma^{-4}Q\bar{\Sigma}QPS)\hat{Q} + \hat{Q}(A - \gamma^{-4}Q\bar{\Sigma}QPS)^{\mathrm{T}} + \gamma^{-6}\hat{Q}S^{\mathrm{T}}PQ\bar{\Sigma}QPS\hat{Q} \\ & + \Sigma - \tau_\perp\Sigma\tau_\perp^{\mathrm{T}} \end{aligned} \tag{3.16}$$

$$\begin{aligned} 0 = {} & (A + \gamma^{-2}Q\bar{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A + \gamma^{-2}Q\bar{\Sigma}) + (I_n + \gamma^{-2}QPS)^{\mathrm{T}}\bar{\Sigma}(I_n + \gamma^{-2}QPS) \\ & - \tau_\perp^{\mathrm{T}}(I_n + \gamma^{-2}QPS)^{\mathrm{T}}\bar{\Sigma}(I_n + \gamma^{-2}QPS)\tau_\perp \end{aligned} \tag{3.17}$$

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_m \tag{3.18}$$

Furthermore, the auxiliary cost is given by

$$\mathscr{J}(A_m, B_m, C_m, \mathscr{Q}) = \operatorname{tr} \bar{\Sigma}(Q + \gamma^{-4}QPS\hat{Q}S^{\mathrm{T}}PQ) \tag{3.19}$$

Conversely, if there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying (3.14)–(3.18), then $(A_m, B_m, C_m, \mathscr{Q})$ given by (3.10)–(3.13) satisfy (2.15) and (2.16) with the auxiliary cost (2.22) given by (3.19).

Proof

See Appendix A.

Remark 2

Theorem 3.1 presents necessary conditions for the auxiliary minimization problem that explicitly synthesize extremal reduced-order models (A_m, B_m, C_m). As a check of these conditions, consider the extreme case $n_m = n$. Then $G = \Gamma^{-1}$ and thus, without loss of generality, $G = \Gamma = \tau = I_n$ and $\tau_\perp = 0$. Furthermore, (3.14) implies that $Q = 0$ and (3.15) implies that $P = 0$. Hence the H_∞-constrained full-order model is given (as expected) by (A, B, C) regardless of γ. Furthermore, note that $\mathcal{Q}$ given by (3.13) becomes

$$\mathcal{Q} = \begin{bmatrix} \hat{Q} & \hat{Q} \\ \hat{Q} & \hat{Q} \end{bmatrix} \tag{3.20}$$

so that the quadratic term $\gamma^{-2}\mathcal{Q}\tilde{R}\mathcal{Q}$ in (2.16) vanishes. Thus, (2.16) reduces to (2.13) so that $\mathcal{Q}$ coincides with the controllability Gramian $\tilde{Q}$. If, alternatively, the reduced-order constraint is retained but the transfer function approximation constraint (2.5) is sufficiently relaxed, i.e. $\gamma \to \infty$, then $S = I_n$ so that the reduced-order model (3.10)–(3.12) is given by $(A_m, B_m, C_m) = (\Gamma A G^{\mathrm{T}}, \Gamma B, C G^{\mathrm{T}})$. In this case, (3.14) and (3.15) are superfluous and (3.16) and (3.17) reduce to the optimal projection equations obtained by Hyland and Bernstein (1985) for the unconstrained L_2 problem.

4. Sufficient conditions for combined L_2/H_∞ approximation

In this section we combine Lemma 2.1 with the converse of Theorem 3.1 to obtain our main result guaranteeing constrained H_∞ approximation along with an optimized L_2 model-reduction bound.

Theorem 4.1

Suppose there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying (3.14)–(3.18) and let $(A_m, B_m, C_m, \mathcal{Q})$ be given by (3.10)–(3.13). Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if A_m is asymptotically stable. In this case, the reduced-order transfer function $H_m(s)$ satisfies the H_∞ approximation constraint

$$\|H(s) - H_m(s)\|_\infty \leqslant \gamma \tag{4.1}$$

and the L_2 approximation bound

$$\|H(s) - H_m(s)\|_2 \leqslant [\operatorname{tr} \Sigma(Q + \gamma^{-4} QPS\hat{Q}S^{\mathrm{T}}PQ)]^{1/2} \tag{4.2}$$

Proof

The converse portion of Theorem 3.1 implies that $\mathcal{Q}$ given by (3.13) satisfies (2.15) and (2.16) with auxiliary cost given by (3.19). It now follows from Lemma 2.1 that the stabilizability condition (2.17) is equivalent to the asymptotic stability of A_m, the H_∞ approximation condition (2.19) holds, and the L_2 model-reduction criterion satisfies the bound (2.21) which is equivalent to (4.2). □

In applying Theorem 4.1, the principal issue concerns conditions on the problem data under which the coupled Riccati equations (3.14)–(3.17) possess non-negative-definite solutions. Clearly, for γ sufficiently large, (3.14)–(3.17) approximate the 'pure' L_2 solution obtained by Hyland and Bernstein (1985). In practice, we would numerically solve (3.14)–(3.17) for successively smaller values of γ until solutions are no longer obtainable. The important case of interest, however, involves small γ so that accurate H_∞ approximation is enforced. Thus, if (4.1) can be satisfied for a given $\gamma > 0$ by a class of reduced-order models, it is of interest to know whether one such reduced-order model can be obtained by solving (3.14)–(3.17). Lemma 2.2 guarantees that (2.16) possesses a solution for any model satisfying (4.1). Thus our sufficient conditions will also be necessary so long as the auxiliary minimization problem possesses at least one extremal over $\mathscr{S}$. When this is the case we have the following immediate result.

Proposition 4.1

Let γ^* denote the infimum of $\|H(s) - H_m(s)\|_\infty$ over all asymptotically stable reduced-order models and suppose that the auxiliary minimization problem has a solution for all $\gamma > \gamma^*$. Then for all $\gamma > \gamma^*$ there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying (3.14)–(3.17).

Remark 3

As in Hyland and Bernstein (1985), it can be expected that (3.14)–(3.17) possess multiple solutions. Theorem 4.1 guarantees, however, that the bounds (4.1) and (4.2) are enforced for all such extremals obtained by solving (3.14)–(3.17).

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Appendix A

Proof of Theorem 3.1

To optimize (2.22) over the open set $\mathscr{S}$ subject to the constraint (2.16), form the lagrangian

$$\mathscr{L}(A_m, B_m, C_m, \mathscr{Q}, \mathscr{P}, \lambda) \triangleq \operatorname{tr}\{\lambda\mathscr{Q}\tilde{R} + [\tilde{A}\mathscr{Q} + \mathscr{Q}\tilde{A}^{\mathrm{T}} + \gamma^{-2}\mathscr{Q}\tilde{R}\mathscr{Q} + \tilde{V}]\mathscr{P}\} \quad \text{(A 1)}$$

where the Lagrange multipliers $\lambda \geqslant 0$ and $\mathscr{P} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ are not both zero. We thus obtain

$$\frac{\partial \mathscr{L}}{\partial \mathscr{Q}} = (\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R})^{\mathrm{T}}\mathscr{P} + \mathscr{P}(\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}) + \lambda\tilde{R} \quad \text{(A 2)}$$

Setting $\partial\mathscr{L}/\partial\mathscr{Q} = 0$ yields

$$0 = (\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R})^{\mathrm{T}}\mathscr{P} + \mathscr{P}(\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}) + \lambda\tilde{R} \quad \text{(A 3)}$$

Since $\tilde{A} + \gamma^{-2}\mathscr{Q}\tilde{R}$ is assumed to be stable, $\lambda = 0$ implies $\mathscr{P} = 0$. Hence, it can be assumed without loss of generality that $\lambda = 1$. Furthermore, $\mathscr{P}$ is non-negative-definite.

Now partition $\tilde{n} \times \tilde{n}$, $\mathscr{Q}$, $\mathscr{P}$, into $n \times n$, $n \times n_m$, and $n_m \times n_m$ sub-blocks as

$$\mathscr{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_2 \end{bmatrix}, \quad \mathscr{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\mathrm{T}} & P_2 \end{bmatrix}$$

and for notational convenience define

$$\mathscr{P}\mathscr{Q} = \begin{bmatrix} Z_1 & Z_{12} \\ Z_{21} & Z_2 \end{bmatrix}$$

where

$$Z_1 \triangleq P_1 Q_1 + P_{12} Q_{12}^{\mathrm{T}}, \quad Z_{12} \triangleq P_1 Q_{12} + P_{12} Q_2$$
$$Z_{21} \triangleq P_{12}^{\mathrm{T}} Q_1 + P_2 Q_{12}^{\mathrm{T}}, \quad Z_2 \triangleq P_{12}^{\mathrm{T}} Q_{12} + P_2 Q_2$$

Thus, with $\lambda = 1$ the stationarity conditions are given by

$$\frac{\partial \mathscr{L}}{\partial \mathscr{Q}} = (\tilde{A} + \gamma^{-2} \mathscr{Q} \tilde{R})^{\mathrm{T}} \mathscr{P} + \mathscr{P}(\tilde{A} + \gamma^{-2} \mathscr{Q} \tilde{R}) + \tilde{R} = 0 \tag{A 4}$$

$$\frac{\partial \mathscr{L}}{\partial A_m} = Z_2 = 0 \tag{A 5}$$

$$\frac{\partial \mathscr{L}}{\partial B_m} = P_{12}^{\mathrm{T}} B V + P_2 B_m V = 0 \tag{A 6}$$

$$\frac{\partial \mathscr{L}}{\partial C_m} = 2RC_m Q_2 + 2\gamma^{-2} RC_m Z_{12}^{\mathrm{T}} Q_{12} - 2RCQ_{12} - \gamma^{-2} RCZ_1^{\mathrm{T}} Q_{12}$$
$$- \gamma^{-2} RCQ_1 Z_{12} - \gamma^{-2} RCZ_{21}^{\mathrm{T}} Q_2 = 0 \tag{A 7}$$

Expanding (2.16) and (A 4) yields

$$0 = AQ_1 + Q_1 A^{\mathrm{T}} + \gamma^{-2}(Q_1 C^{\mathrm{T}} - Q_{12} C_m^{\mathrm{T}}) R (Q_1 C^{\mathrm{T}} - Q_{12} C_m^{\mathrm{T}})^{\mathrm{T}} + BVB^{\mathrm{T}} \tag{A 8}$$

$$0 = AQ_{12} + Q_{12} A_m^{\mathrm{T}} + \gamma^{-2} Q_1 C^{\mathrm{T}} RCQ_{12} - \gamma^{-2} Q_{12} C_m^{\mathrm{T}} RCQ_{12} - \gamma^{-2} Q_1 C^{\mathrm{T}} RC_m Q_2$$
$$+ \gamma^{-2} Q_{12} C_m^{\mathrm{T}} RC_m Q_2 \tag{A 9}$$

$$0 = A_m Q_2 + Q_2 A_m^{\mathrm{T}} + \gamma^{-2}(Q_{12}^{\mathrm{T}} C^{\mathrm{T}} - Q_2 C_m^{\mathrm{T}}) R (Q_{12}^{\mathrm{T}} C^{\mathrm{T}} - Q_2 C_m^{\mathrm{T}})^{\mathrm{T}} + B_m V B_m^{\mathrm{T}} \tag{A 10}$$

$$0 = A^{\mathrm{T}} P_1 + P_1 A + \gamma^{-2} C^{\mathrm{T}} RCZ_1^{\mathrm{T}} - \gamma^{-2} C^{\mathrm{T}} RC_m Z_{12}^{\mathrm{T}} + \gamma^{-2} Z_1 C^{\mathrm{T}} RC$$
$$- \gamma^{-2} Z_{12} C_m^{\mathrm{T}} RC + C^{\mathrm{T}} RC \tag{A 11}$$

$$0 = A^{\mathrm{T}} P_{12} + P_{12} A_m + \gamma^{-2} C^{\mathrm{T}} RCZ_{21}^{\mathrm{T}} - \gamma^{-2} Z_1 C^{\mathrm{T}} RC_m + \gamma^{-2} Z_{12} C_m^{\mathrm{T}} RC_m$$
$$- C^{\mathrm{T}} RC_m \tag{A 12}$$

$$0 = A_m^{\mathrm{T}} P_2 + P_2 A_m - \gamma^{-2} C_m^{\mathrm{T}} RCZ_{21}^{\mathrm{T}} - \gamma^{-2} Z_{21} C^{\mathrm{T}} RC_m + C_m^{\mathrm{T}} RC_m \tag{A 13}$$

Now define the $n \times n$ matrices

$$Q \triangleq Q_1 - Q_{12} Q_2^{-1} Q_{12}^{\mathrm{T}}, \quad P \triangleq P_1 - P_{12} P_2^{-1} P_{12}^{\mathrm{T}}$$
$$\hat{Q} \triangleq Q_{12} Q_2^{-1} Q_{12}^{\mathrm{T}}, \quad \hat{P} \triangleq P_{12} P_2^{-1} P_{12}^{\mathrm{T}}$$
$$\tau \triangleq -Q_{12} Q_2^{-1} P_2^{-1} P_{12}^{\mathrm{T}}$$

and the $n_m \times n$, $n_m \times n_m$ and $n_m \times n$ matrices

$$G \triangleq Q_2^{-1} Q_{12}^{\mathrm{T}}, \quad M \triangleq Q_2 P_2, \quad \Gamma \triangleq -P_2^{-1} P_{12}^{\mathrm{T}}$$

The existence of Q_2^{-1} and P_2^{-1} follows from the fact that (A_m, B_m, C_m) is minimal. See Bernstein and Haddad (1989) and Hyland and Bernstein (1985) for details. Note that $\tau = G^{\mathrm{T}} \Gamma$. Clearly, Q, P, $\hat{Q}$ and $\hat{P}$ are symmetric and non-negative-definite.

Next note that with the above definitions, (A 5) implies (3.3) and that (3.2) holds. Hence, $\tau = G^{\mathrm{T}}\Gamma$ is idempotent, i.e. $\tau^2 = \tau$. Sylvester's inequality yields (3.18). Note also that (3.7) and (3.8) hold.

The components of $\mathcal{Q}$ and $\mathcal{P}$ can be written in terms of Q, P, $\hat{Q}$, $\hat{P}$, G and Γ as

$$Q_1 = Q + \hat{Q}, \quad P_1 = P + \hat{P} \tag{A 14}$$

$$Q_{12} = \hat{Q}\Gamma^{\mathrm{T}}, \quad P_{12} = -\hat{P}G^{\mathrm{T}} \tag{A 15}$$

$$Q_2 = \Gamma\hat{Q}\Gamma^{\mathrm{T}}, \quad P_2 = G\hat{P}G^{\mathrm{T}} \tag{A 16}$$

Next note that by using (A 14)–(A 16), (A 7) becomes

$$C_m\hat{S} = C[I_n + \gamma^{-2}(Q + \hat{Q})P]G^{\mathrm{T}}$$

where

$$\hat{S} \triangleq I_{n_m} + \gamma^{-2}\Gamma\hat{Q}PG^{\mathrm{T}}$$

To prove that $\hat{S}$ is invertible use (3.7) and (3.4) and note that

$$\begin{aligned} I_{n_m} + \gamma^{-2}\Gamma\hat{Q}PG^{\mathrm{T}} &= I_{n_m} + \gamma^{-2}\Gamma\hat{Q}\tau^{\mathrm{T}}PG^{\mathrm{T}} \\ &= I_{n_m} + \gamma^{-2}(\Gamma\hat{Q}\Gamma^{\mathrm{T}})(GPG^{\mathrm{T}}) \end{aligned}$$

Since $\Gamma\hat{Q}\Gamma^{\mathrm{T}}$ and GPG^{T} are non-negative-definite, their product has non-negative eigenvalues. Thus each eigenvalue of $I_{n_m} + \gamma^{-2}\Gamma\hat{Q}PG^{\mathrm{T}}$ is real and is greater than unity. Hence $\hat{S}$ is invertible. Now note that by using (3.3) and (3.4) it can be shown that

$$G^{\mathrm{T}}\hat{S}^{-1}\Gamma = S\tau$$

The expressions (3.11), (3.12) and (3.13) follow from (A 6), (A 7), (3.9) and the definition of $\mathcal{Q}$ by using the above identities. Next, computing either Γ (A 9) $-$ (A 10) or G(A 12) + (A 13) yields (3.10). Substituting this expression for A_m into (A 8)–(A 13) it follows that (A 10) = Γ(A 9) and (A 13) = G(A 12). Thus, (A 10) and (A 13) are superfluous and can be omitted. Next, using (A 8) + $G^{\mathrm{T}}\Gamma$(A 9)G $-$ (A 9)G $-$ [(A 9)G]$^{\mathrm{T}}$ and $G^{\mathrm{T}}\Gamma$(A 9)G $-$ (A 9)G $-$ [(A 9)G]$^{\mathrm{T}}$ yields (3.14) and (3.16). Using (A 11) + $\Gamma^{\mathrm{T}}G$(A 12)Γ $-$ (A 12)Γ $-$ [(A 12)Γ]$^{\mathrm{T}}$ and $\Gamma^{\mathrm{T}}G$(A 12)Γ $-$ (A 12)Γ $-$ [(A 12)Γ]$^{\mathrm{T}}$ yields (3.15) and (3.17).

Finally, to prove the converse we use (3.10)–(3.18) to obtain (2.16) and (A 4)–(A 7). Let A_m, B_m, C_m, G, Γ, τ, Q, P, $\hat{Q}$, $\hat{P}$, $\mathcal{Q}$ be as in the statement of Theorem 3.1 and define $Q_1, Q_{12}, Q_2, P_1, P_{12}, P_2$ by (A 14)–(A 16). Using (3.3), (3.11) and (3.12) it is easy to verify (A 6) and (A 7). Finally, substitute the definitions of Q, P, $\hat{Q}$, $\hat{P}$, G, Γ and τ into (3.14)–(3.17) along with (3.3), (3.4), (3.7) and (3.8) to obtain (2.16) and (A 4). Finally, note that

$$\mathcal{Q} = \begin{bmatrix} Q & 0_{n \times n_m} \\ 0_{n_m \times n} & 0_{n_m} \end{bmatrix} + \begin{bmatrix} I_n \\ \Gamma \end{bmatrix} \hat{Q} [I_n \quad \Gamma^{\mathrm{T}}]$$

which shows that $\mathcal{Q} \geqslant 0$.

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