# IS THERE MORE TO ROBUST CONTROL THEORY THAN SMALL GAIN?

by

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## 1. Introduction

The most fundamental and important principle in robust control theory, and perhaps all of feedback control theory, is without doubt the small gain theorem [1]. Essentially, bounded real/$H_\infty$ theory, structured singular value theory, and norm-based robustness theories [2-10] are based on small gain principles. This leads to the following question: Are there fundamental principles in robust feedback control theory that are distinct from the small gain theorem?

The purpose of this paper is to suggest that there is, in fact, more to robust control theory than small gain. Our main point, which was discussed previously in [11], is that it is possible to transcend small gain principles by exploiting knowledge of phase properties. Since $|e^{2\phi}| = 1$ regardless of the phase angle $\phi$, it can be expected that any norm-based technique will ignore phase information.

To illustrate the importance of phase information, it is useful to contrast the classical concepts of gain and phase stabilization. In terms of gain stabilization, stability of a single-input single-output closed-loop system is ensured by designing the controller so that the magnitude of the loop transfer function is less than unity in frequency regimes in which the phase is either known to be near 180° or is highly uncertain. In terms of phase stabilization, stability is achieved by ensuring that the phase of the loop transfer function is well behaved where the loop transfer function has gain greater than unity. Roughly speaking, phase stabilization can be used to allow high loop gain and thus achieve high performance in frequency regimes in which sufficient phase information is available, whereas gain stabilization (e.g., rolloff) is needed to insure stability in frequency regimes in which the phase of a system is very poorly known. For further discussion of the distinction between phase and gain stabilization, see [12].

There are many facets to the phase information question. Here we discuss a collection of ideas that are united by their common goal of circumventing the limitations of small gain theory. First we clarify these limitations and then discuss possible solutions.

## 2. Limitations to Small Gain Theory

The analysis and synthesis of robust feedback controllers entails a fundamental distinction between parametric and nonparametric uncertainty. Parameter uncertainty refers to plant uncertainty that is modeled as constant real parameters, whereas nonparametric uncertainty refers to uncertain transfer function gains that may be modeled as complex frequency-dependent quantities. In the time domain, nonparametric uncertainty is manifested as uncertain real parameters that may be time varying.

The distinction between parametric and nonparametric uncertainty is critical to the achievable performance of feedback control systems. For example, in the problem of vibration suppression for flexible space structures, if stiffness matrix uncertainty is modeled as a nonparametric uncertainty, then perturbations to the damping matrix will inadvertently be allowed. Predictions of stability and performance for given feedback gains will consequently be extremely conservative, thus limiting achievable performance [13]. Alternatively, this problem can be viewed by considering the classical analysis of Hill's equation (e.g., the Mathieu equation) which shows that time-varying parameter variations can destabilize a system even when the parameter variations are confined to a region in which constant variations are *non*destabilizing. Consequently, a feedback controller designed for time-varying parameter variations will unnecessarily sacrifice performance when the uncertain real parameters are actually constant.

## 3. Positivity Theory

Positivity theory [14-25] provides a classic example of a theory that overcomes small gain limitations. Essentially, positivity theory asserts that a feedback loop consisting of two transfer functions having phase less than 90° is stable regardless of gain magnitude. Positivity theory plays an invaluable role in certain applications such as the control of flexible structures [26,27].

One can argue, however, that positivity theory, in spite of its ability to capture knowledge of phase, is not based upon feedback principles that are fundamentally different from small gain theory. This argument is based upon the well-known small gain/positivity duality whose details follow from the bilinear transformation [16,24]. Hence small gain techniques can effectively yield positivity-type results by means of suitable application.

## 4. Constant Real Parameters and Phase Characterization

The constant real parameter uncertainty problem represents an extreme case of phase information due to the μμfact that every real parameter has phase 0° or 180°. By a simple shift of the nominal, however, one can usually consider 0° phase only for the case of positive real constants. It is clear that positivity approaches, which account for up to 90° phase, will be conservative with respect to 0° phase information.

The quintessential real parameter result is, of course, Kharitonov's Theorem [28], which appears to be quite independent of small gain principles. Beyond this result we can cite a broad range of results that seek to exploit phase information in some form [29-40]. Although it is difficult to discern a specific unifying principle among these techniques, it is clear that they all seek to go beyond the limitations of the small gain theorem.

The difficulties associated with phase information can be emphasized by the question: What is the phase of a matrix? While the gain of a matrix is well understood in terms of norms, phase properties are difficult to characterize. There are several possible starting points for an nxn real matrix $A$:

1) Write $A = A_1 + A_2$ where $A_1 = \frac{1}{2}(A + A^T)$ and $A_2 = \frac{1}{2}(A - A^T)$ and consider the imaginary eigenvalues of the skew-symmetric matrix $A_2$.
2) Write $A = MU$ where $M$ is nonnegative definite and $U$ is orthogonal and consider the phase of the complex eigenvalues of $U$ [29].
3) Consider the geometry of the numerical range of $A$ [30].
4) Consider the phase of each eigenvalue of $A$.

## 5. Parameter-Dependent Lyapunov Functions and the Popov Criterion

An alternative approach to the phase information/real parameter uncertainty problem is to construct refined Lyapunov functions that are functions of the uncertain parameters. This idea has been proposed in [41,42]. The idea behind parameter-dependent Lyapunov functions is to allow the matrix $P$ of the Lyapunov function $V(x) = x^TPx$ to be a function of the uncertainty $\Delta A$. In the usual case $P$ is a single, fixed matrix, whereas the parameter-dependent Lyapunov function $V(x,\Delta A) = x^TP(\Delta A)x$ represents *a family* of Lyapunov functions.

In [41], Barmish and DeMarco propose a parameter-dependent Lyapunov function $V(x,\lambda_1,...,\lambda_r) = x^TP(\lambda_1,...,\lambda_r)x$, where $P(\lambda_1,...,\lambda_r) = \Sigma_{i=1}^r \lambda_i P_i$, which they call an "adaptive" Lyapunov function. In this case the matrices $P_i$ correspond to the vertices of a polytope of uncertain matrices $A_i$. In [42], Leal and Gibson consider a Lyapunov function with matrix $P(\sigma_1,...,\sigma_r) = P_0 + \Sigma_{i=1}^r \sigma_i P_i$, where $P_0$ corresponds to the nominal system and $P_i$ are "first-order perturbations" of $P_0$. Numerical techniques are used to determine $P_i$ and the range of robust stability. In both [41] and [42] there is considerable evidence that parameter-dependent Lyapunov functions offer significant advantages over "fixed" Lyapunov functions.

A related fact, which was clarified and generalized in [43,44], is that the classical Popov criterion is actually based upon a parameter-dependent Lyapunov function. To see this, recall that the Popov criterion is based upon the Lur'e-Postnikov Lyapunov function

$$V(x,\phi) = x^TPx + \alpha \int_0^y \phi(\sigma)d\sigma,$$

where $\phi(\cdot)$ is a scalar memoryless time-invariant nonlinearity in the sector $[0,k]$, that is, $0 \le \phi(y)y \le ky^2$. **Specializing to the linear case** $\phi(y) = Fy$, where the real, constant uncertain gain $F$ satisfies $0 \le F \le k$, and letting $y = Cx$, we see that

$$\begin{aligned} V(x,F) &= x^{\mathrm{T}}Px + \alpha \int_0^y F\sigma d\sigma \\ &= x^{\mathrm{T}}Px + \alpha F\frac{y^2}{2} \\ &= x^{\mathrm{T}}Px + \frac{1}{2}\alpha F(Cx)^2 \\ &= x^{\mathrm{T}}Px + \frac{1}{2}\alpha F(x^{\mathrm{T}}C^{\mathrm{T}}Cx) \\ &= x^{\mathrm{T}}[P + \frac{1}{2}\alpha FC^{\mathrm{T}}C]x \\ &= x^{\mathrm{T}}P(F)x, \end{aligned}$$

an observation pointed out in [43].

For practical purposes the form of the parameter-dependent Lyapunov function $V(x,F)$ is critical since the presence of $F$ severely restricts the allowable time-varying uncertain parameters. That is, if $F(t)$ were permitted, then terms involving $\dot{F}(t)$ might subvert the negative definiteness of $V(x,F)$. Hence parameter-dependent Lyapunov functions possess the potential for exploiting phase information .

The Popov criterion is a special case of the class of absolute stability criteria involving frequency-dependent multipliers. Once the frequency-dependent multiplier is chosen, the resulting criterion is usually cast as a positive real condition, which, in turn, is equivalent to a small gain condition. The presence of the frequency-dependent multiplier, however, entails a parameter-dependent Lyapunov function which distinguishes the robustness test from the usual small gain conditions. Besides the Popov criterion the off-axis circle criterion and parabola test possess these features [20, 45-48].

An alternative approach developed in [49] utilizes irrational frequency-domain transformations to capture constant real parameter uncertainty. The relationship between this approach and classical frequency domain criteria remains to be explored.

## 6. Maximum Entropy Theory

Another approach to overcoming the limitations of small gain theory is the Maximum Entropy theory originally developed by Hyland [11, 50-53]. Using insights from the analysis of structural vibrations, this approach is based upon a model that captures the statistical effects of uncertain modal parameters. A concise discussion of the rationale behind this approach is given in [45]. One interpretation for this technique has been given in [54] in terms of covariance averaging, while a parameter-dependent Lyapunov function basis is given in [55].

## 7. Conclusion

Small gain theory guarantees robustness with respect to complex or time-varying uncertainty and thus is conservative with respect to constant real parameter uncertainty. In this paper we reviewed a variety of approaches that seek to overcome the limitations of small gain theory. These approaches include Kharitonov theory, positivity theory, phase characterization, parameter-dependent Lyapunov functions, the Popov, parabola, and off-axis circle criteria, and Maximum Entropy theory.


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