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Rotational Stabilization of a Rigid Body Using Two Torque Actuators

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Abstract The Hamilton-Jacobi-Bellman theorem is used to derive a control law that globally asymptotically stabilizes the Euler's equation to a prescribed state. It is shown that if all three components of the prescribed state are nonzero, then it is impossible to asymptotically stabilize the Euler's equation to that state using only two torque inputs along two principal axes. If two components of the prescribed state are nonzero and one of the two components is in the uncontrolled principal axes, then we obtain a family of optimal nonlinear stabilizing feedback control laws.

1 Introduction

Angular velocity stabilization of a rigid body has been studied by many researchers [1, 2, 7, 8, 12, 14, 17, 21, 22, 23]. If there are two torque inputs along two principal axes and the uncontrolled principal axis is not an axis of symmetry, then the system can be asymptotically stabilized by using a variety of design schemes. In [7], a locally asymptotically stabilizing control law was given. Later, Aeyels [1] applied center manifold theory to reduce the problem to one of lower dimension and thereby obtained another locally stabilizing control law. In [12], the authors applied the concept of finite gain developed by [7] and obtained the first globally stabilizing feedback control law. More recently, Byrnes and Isidori [8] used the general methodology of nonlinear zero dynamics to derive another globally stabilizing feedback control law for the system. Then, Krishnan, Reyhanoglu and McClamroch [14] globally asymptotically stabilized the system in finite time by using a piecewise analytic control law. In [22], Hamilton-Jacobi-Bellman theory [3] was used to generate a family of feedback control laws that globally asymptotically stabilize the system.

If there is only one torque input, asymptotic stabilization is still possible under conditions that depend on the orientation of the input torque and the symmetry of the rigid body. If the rigid body has no axis of symmetry and if the input torque does not lie in a principal plane, Aeyels and Szafranski [2] derived a linear control law to globally asymptotically stabilize the Euler's equation. If the rigid body has one axis of symmetry and if the input torque has nonzero components along the axis of symmetry and in any direction perpendicular to the axis of symmetry, Sontag and Sussmann [21] proved the existence of a stabilizing control law, while Outbib and Sallet [17] found such a control law explicitly.

Rotational stabilization of angular velocity of a rigid body has been considered in [5, 6, 24, 25]. In [6], the authors (Lyapunov) stabilized the unstable equilibrium of the Euler's equation, *i.e.*, the intermediate axis rotation, by applying a single torque along the major or minor axis. The design strategy of their approach is based on the Energy-Casimir method [11, 18]. In the presence of one symmetry axis, linear control law that (Lyapunov) stabilized one of the unsymmetric-principal-axis rotation is developed in [24]. In [25], the authors used the Energy-Momentum method to stabilize Euler's equation to an arbitrary point by applying three torque inputs.

In the present paper, we synthesize smooth control laws that globally asymptotically stabilize the Euler's equation to an arbitrarily prescribed state using only two torque inputs along two principal axes. It is shown that if all the three components of the prescribed state are nonzero, then it is impossible to asymptotically stabilize the Euler's equation to that state by merely using two torque inputs, a situation reminiscent of the nonzero set point regulation problem in linear systems [4, 10]. If two components of the prescribed state are nonzero and one of the two components is in the uncontrolled principal axes, then globally asymptotically stabilizing control laws are synthesized using the Hamilton-Jacobi-Bellman theorem.

2 Optimal Nonlinear Feedback Control

In this section we review the Hamilton-Jacobi-Bellman (HJB) theorem and state several related corollaries which were developed in [23]. Consider the controlled system

$$\dot{x}(t) = F(x(t), u(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{1}$$

where $x(t) \in \mathcal{D} \subset \mathcal{R}^n$ is the state variable, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in \mathcal{U} \subset \mathcal{R}^m$ is the control input, $\mathcal{U}$ is an arbitrary set with $0 \in \mathcal{U}$, and $F\colon \mathcal{D} \times \mathcal{U} \to \mathcal{R}^n$ satisfies $F(0,0) = 0$. The control $u(\cdot)$ in (1) is restricted to the class of ***admissible controls*** consisting of measurable functions $u(\cdot)$ such that $u(t) \in \Omega, t \geq 0$, where the control constraint set $\Omega \subset \mathcal{U}$ is given. We assume $0 \in \Omega$ and Ω is compact.

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A measurable mapping $\phi: \mathcal{D} \to \Omega$ satisfying $\phi(0) = 0$ is called a *control law*. If $u(t) = \phi(x(t))$, where ϕ is a control law and $x(t)$ satisfies (1), then $u(\cdot)$ is called a *feedback control law*. A feedback control law is admissible since the control law $\phi(\cdot)$ takes values in Ω, and $x(\cdot)$ is absolutely continuous.

Letting $L(x,u)$ be the performance integrand, where $L: \mathcal{D} \times \mathcal{U} \to \mathcal{R}$, the corresponding Hamiltonian is defined as

$$H(x,u,p) \triangleq L(x,u) + p^{\mathrm{T}} F(x,u),$$

where $p \in \mathcal{R}^n$. Furthermore, we define the set of asymptotically stabilizing admissible control laws $\mathcal{S}(x_0)$ for each initial condition $x_0 \in \mathcal{D}$, that is,

$$\mathcal{S}(x_0) \triangleq \{u(\cdot):\ u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (1) satisfies } x(t) \to 0 \text{ as } t \to \infty\}.$$

Although this set plays a role in the following theorem, an explicit characterization of this set is not required.

Theorem 2.1. Consider the controlled system (1) with performance functional

$$J(x_0, u(\cdot)) \triangleq \int_0^\infty L(x(t), u(t))\mathrm{d}t. \tag{2}$$

Assume that there exists a C^1 function $V: \mathcal{D} \to \mathcal{R}$ and a control law $\phi: \mathcal{D} \to \Omega$ such that

$$V(0) = 0, \tag{3}$$

$$V(x) > 0, \quad x \in \mathcal{D}, \quad x \neq 0, \tag{4}$$

$$\phi(0) = 0, \tag{5}$$

$$V'(x)F(x,\phi(x)) < 0, \quad x \in \mathcal{D}, \quad x \neq 0, \tag{6}$$

$$H(x,\phi(x),V'^{\mathrm{T}}(x)) = 0, \quad x \in \mathcal{D}, \tag{7}$$

$$H(x,u,V'^{\mathrm{T}}(x)) \geq 0, \quad x \in \mathcal{D}, \quad u \in \Omega. \tag{8}$$

Then, with the feedback control law $u(\cdot) = \phi(x(\cdot))$, the solution $x(t) = 0$, $t \geq 0$, of the closed-loop system

$$\dot{x}(t) = F(x(t), \phi(x(t))), \quad x(0) = x_0, \quad t \geq 0, \tag{9}$$

is locally asymptotically stable, and

$$J(x_0, \phi(x(\cdot))) = V(x_0), \quad \text{for all } x_0 \in \mathcal{D}. \tag{10}$$

Furthermore, the feedback control law $u(\cdot) = \phi(x(\cdot))$ minimizes $J(x_0, u(\cdot))$ in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} J(x_0, u(\cdot)), \quad \text{for all } x_0 \in \mathcal{D}. \tag{11}$$

Finally, if $\mathcal{D} = \mathcal{R}^n$ and

$$V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty, \tag{12}$$

then the asymptotic stability is global.

Proof. See [3]. □

We now consider a nonlinear system that is affine in control given by

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{13}$$

where $\mathcal{D} = \mathcal{R}^n, \Omega = \mathcal{U} = \mathcal{R}^m$. Let the performance integrand $L(x,u)$ be given by

$$L(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}} R u, \tag{14}$$

where $L_1: \mathcal{R}^n \to \mathcal{R}$, $L_2: \mathcal{R}^n \to \mathcal{R}^{1\times m}$ satisfies $L_2(0) = 0$, and $R \in \mathcal{R}^{m\times m}$ is positive definite. With the specialization (13), (14), we have the following corollary of Theorem 2.1.

Corollary 2.2. Consider the controlled system (13), and assume that there exists a C^1 function $V: \mathcal{R}^n \to \mathcal{R}$ and a function $L_2: \mathcal{R}^n \to \mathcal{R}^{1\times m}$, such that

$$V(0) = 0, \tag{15}$$

$$V(x) > 0, \quad x \in \mathcal{R}^n, \quad x \neq 0, \tag{16}$$

$$V'(x)[f(x) - \frac{1}{2}g(x)R^{-1}L_2^{\mathrm{T}}(x) - \frac{1}{2}g(x)R^{-1}g^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)] < 0, \tag{17}$$

for all $x \in \mathcal{R}^n$, $x \neq 0$, and

$$V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty. \tag{18}$$

Furthermore, define the feedback control law $u = \phi(x)$, where

$$\phi(x) \triangleq -\frac{1}{2}R^{-1}[L_2^{\mathrm{T}}(x) + g^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)]. \tag{19}$$

Then the solution $x(t) = 0$, $t \geq 0$, of the closed-loop system

$$\dot{x}(t) = f(x(t)) + g(x(t))\phi(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{20}$$

is globally asymptotically stable, and the performance functional (2) with $L(x,u)$ defined by (14) and

$$L_1(x) \triangleq \phi^{\mathrm{T}}(x)R\phi(x) - V'(x)f(x), \quad x \in \mathcal{R}^n, \tag{21}$$

is minimized in the sense of (10) and (11).

Proof. See Wan and Bernstein [22]. □

In the following we apply Corollary 2.2 to minimum phase systems satisfying some additional conditions [13, 16]. We consider the affine control system

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{22}$$

$$y(t) = h(x(t)), \tag{23}$$

where $y(t) \in \mathcal{R}^m$ is an artificial output function, and $h(x) = (h_1(x), h_2(x), \ldots, h_m(x))^{\mathrm{T}}$.

Lemma 2.3. Assume that the nonlinear system (22), (23) is minimum phase with relative degree $\{1,1,\ldots,1\}$. If the vector field $g(\mathrm{L}_g h)^{-1}$ is complete, then there exists a global diffeomorphism $\mathcal{C}: \mathcal{R}^n \to \mathcal{R}^n$, a C^∞ function $f_0 : \mathcal{R}^{n-m} \to \mathcal{R}^{n-m}$, and a C^∞ function $\mathrm{r} : \mathcal{R}^{n-m} \times \mathcal{R}^m \to \mathcal{R}^{(n-m)\times m}$ such that, in the new coordinates

$$\begin{bmatrix} z \\ y \end{bmatrix} \triangleq \mathcal{C}(x), \tag{24}$$

the differential equation (22) is equivalent to the normal form

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} f_0(z) + \mathrm{r}(z,y)y \\ \mathrm{L}_f h(x) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathrm{L}_g h(x) \end{bmatrix} u. \tag{25}$$

Proof. See Byrnes and Isidori [8]. □

Since the system is minimum phase, the zero dynamics $\dot{z} = f_0(z)$ are asymptotically stable. Hence the converse Lyapunov theorem [15, 20] implies that there exists a C^1 Lyapunov function $V_0(z)$ such that

$$\frac{\partial V_0(z)}{\partial z} f_0(z) < 0, \quad z \in \mathcal{R}, \quad z \neq 0. \tag{26}$$

In the following result, we use the HJB framework to derive an optimal nonlinear feedback control law for such systems. This control law is a generalization of the results obtained by Byrnes and Isidori [8].

Proposition 2.4. Consider the nonlinear system defined by equations (22), (23). Assume that the system is minimum phase with relative degree $\{1, 1, \ldots, 1\}$ and the vector field $g(L_g h)^{-1}$ is complete. Furthermore, let

$$L_2^{\mathrm{T}}(x) = R(\mathrm{L}_g h)^{-1}[P^{-1}\mathbf{r}^{\mathrm{T}}(z,y)(\frac{\partial V_0(z)}{\partial z})^{\mathrm{T}} + 2\mathrm{L}_f h] \tag{27}$$

and

$$V(x) = V_0(z) + y^{\mathrm{T}} P y, \tag{28}$$

where P is an arbitrary $m \times m$ positive-definite matrix. Then the optimal nonlinear feedback control law

$$\begin{aligned} \phi(x) &= -\frac{1}{2}[\mathrm{L}_g h(x)]^{-1}[P^{-1}\mathbf{r}^{\mathrm{T}}(z,y)(\frac{\partial V_0(z)}{\partial z})^{\mathrm{T}} + 2\mathrm{L}_f h(x)] \\ &\quad - R^{-1}[\mathrm{L}_g h(x)]^{\mathrm{T}} P^{\mathrm{T}} h(x) \end{aligned} \tag{29}$$

globally stabilizes (22). Finally, this control law minimizes $J(x_0, u(\cdot))$ in the sense of (10) and (11).

Proof. See Wan and Bernstein [23]. □

3 Rotational Stabilization with Two Torque Inputs

Consider the rotational stabilization problem of Euler's equation at an arbitrary angular velocity state ω_s by using three independent torque inputs. It can be shown that any stabilizing control law that asymptotically stabilizes Euler's equation to the null equilibrium can be modified to asymptotically stabilize the equation to an arbitrary prescribed state ω_s. An almost robust control law was developed by using the Energy-Momentum method [25]. Other possible choices of control laws include linear control law and sublinear finite time control law [9, 19].

In this section we consider rigid body rotational stabilization with two torque inputs along two principal axes. We can, without loss of generality, express the dynamical equation as

$$\begin{aligned} \dot{\omega}_1 &= J_{23}\omega_2\omega_3 + u_1, \\ \dot{\omega}_2 &= J_{31}\omega_3\omega_1 + u_2, \\ \dot{\omega}_3 &= J_{12}\omega_1\omega_2, \end{aligned} \tag{30}$$

where $J_{23} = (J_2 - J_3)/J_1, J_{31} = (J_3 - J_1)/J_2, J_{12} = (J_1 - J_2)/J_3$ and J_1, J_2, J_3 are the principal moment of inertia. For the null solution of the above equation, if the uncontrolled principal axis is an axis of symmetry, *i.e.*, $J_{12} = 0$, or $J_1 = J_2$, then the system cannot be asymptotically stabilized. If the uncontrolled principal axis is not an axis of symmetry, *i.e.*, $J_{12} \neq 0$, or $J_1 \neq J_2$, then the system can be globally asymptotically stabilized, see [22] and the references therein for detail.

For the rotational stabilization problem, it is required that the rigid body be stabilized to an arbitrary prescribed angular velocity state $\omega_s = (\omega_{s1}, \omega_{s2}, \omega_{s3})^{\mathrm{T}}$. Letting

$$\bar{\omega} \triangleq \omega - \omega_s, \tag{31}$$

equation (30) becomes

$$\begin{aligned} \dot{\bar{\omega}}_1 &= J_{23}(\bar{\omega}_2\bar{\omega}_3 + \omega_{s2}\bar{\omega}_3 + \omega_{s3}\bar{\omega}_2 + \omega_{s2}\omega_{s3}) + u_1, \\ \dot{\bar{\omega}}_2 &= J_{31}(\bar{\omega}_3\bar{\omega}_1 + \omega_{s3}\bar{\omega}_1 + \omega_{s1}\bar{\omega}_3 + \omega_{s3}\omega_{s1}) + u_2, \\ \dot{\bar{\omega}}_3 &= J_{12}(\bar{\omega}_1\bar{\omega}_2 + \omega_{s1}\bar{\omega}_2 + \omega_{s2}\bar{\omega}_1 + \omega_{s1}\omega_{s2}). \end{aligned} \tag{32}$$

In equation (32), if the uncontrolled principal axis is an axis of symmetry, *i.e.*, $J_{12} = 0$, or $J_1 = J_2$, then it cannot be asymptotically stabilized to $\bar{\omega} = 0$. Henceforth, we assume that *the uncontrolled principal axis is not an axis of symmetry, i.e.*, $J_{12} \neq 0$, or $J_1 \neq J_2$. Moreover, in (32), if $\omega_{s1}\omega_{s2} \neq 0$, then it is impossible for the closed-loop of (32) to have an equilibrium at the origin. Hence, we consider the case in which $\omega_{s1}\omega_{s2} = 0$. Without loss of generality we let $\omega_{s1} = 0$, so that (32) becomes

$$\begin{aligned} \dot{\bar{\omega}}_1 &= J_{23}(\bar{\omega}_2\bar{\omega}_3 + \omega_{s2}\bar{\omega}_3 + \omega_{s3}\bar{\omega}_2 + \omega_{s2}\omega_{s3}) + u_1, \\ \dot{\bar{\omega}}_2 &= J_{31}(\bar{\omega}_3\bar{\omega}_1 + \omega_{s3}\bar{\omega}_1) + u_2, \\ \dot{\bar{\omega}}_3 &= J_{12}(\bar{\omega}_1\bar{\omega}_2 + \omega_{s2}\bar{\omega}_1). \end{aligned} \tag{33}$$

Note that in equation (33), if $J_{23} = 0$, or $J_2 = J_3$, then we can redefine the body coordinate and control vectors such that the new orientation of u_2 has the same direction as the prescribed state. Thus in this special case, (33) can be simplified and ω_s will correspond to an equilibrium state of a single axis spin in the new coordinate. However, it is not necessarily easier to design an asymptotically stabilizing control law in the new coordinates. Hence, whether $J_2 = J_3$ or not, equation (33) will be used to design the control law.

Remark. If there is only one torque input, *i.e.* $u_2 = 0$, (if $u_1 = 0$, the closed-loop equation of (33) won't have an equilibrium at the origin), equation (33) cannot be asymptotically stabilized. This can be shown by the function

$$V(\bar{\omega}) = J_{12}(\bar{\omega}_2 + \omega_{s2})^2 - J_{31}(\bar{\omega}_3 + \omega_{s3})^2,$$

which has time derivative

$$\dot{V}(\bar{\omega}) = 0.$$

This implies $V(\bar{\omega}) =$ constant is an invariant manifold for all u_1, and thus precludes the possibility of asymptotic stabilization. □

Define $x_1 \triangleq \bar{\omega}_1, x_2 \triangleq \bar{\omega}_2, x_3 \triangleq \bar{\omega}_3/J_{12}$, so that $x_{s1} = 0, x_{s2} = \omega_{s2}$, and $x_{s3} = \omega_{s3}/J_{12}$. In the following development, we assume $x_{s2} > 0$. If $x_{s2} \leq 0$ then minor modifications discussed at the end of this section are required. Rewriting equation (33) as

$$\begin{aligned} \dot{x}_1 &= J_{23}J_{12}(x_2x_3 + x_{s2}x_3 + x_{s3}x_2 + x_{s2}x_{s3}) + u_1 \triangleq v_1, \\ \dot{x}_2 &= J_{31}J_{12}(x_3x_1 + x_{s3}x_1) + u_2 \triangleq v_2, \\ \dot{x}_3 &= x_1x_2 + x_{s2}x_1, \end{aligned} \tag{34}$$

we have, in the notation of (13),

$$f(x)=\begin{bmatrix}0\\0\\x_1x_2+x_{s2}x_1\end{bmatrix},\quad g(x)=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}.$$

For equation (34) we apply Proposition 2.4 to derive an optimal nonlinear feedback control and the corresponding performance functional. By letting

$$y_1 = x_1+\alpha x_3^k, \tag{35}$$

$$y_2 = x_2+\beta x_3^{k+1}, \tag{36}$$

where k is a positive integer, and α,β are arbitrary real numbers, it is easy to check that the system has relative degree $\{1,1\}$. To complete the diffeomorphism, the third coordinate z can be obtained by solving the partial differential equation $L_g z = 0$, and its simplest solution is

$$z = x_3. \tag{37}$$

In the notation of the normal form (25), we have

$$f_0(z) = -\alpha x_{s2}z^k+\alpha\beta z^{2k+1}, \tag{38}$$

and there are two possible choices for $\mathbf{r}(z,y)$, namely,

$$\mathbf{r}(z,y)=[x_{s2}-\beta z^{k+1}+y_2,-\alpha z^k] \tag{39}$$

and

$$\mathbf{r}(z,y)=[x_{s2}-\beta z^{k+1},-\alpha z^k+y_1]. \tag{40}$$

By taking

$$V_0(z)=p_3z^2, \tag{41}$$

where $p_3>0$, we have

$$\dot V_0(z)=\frac{\partial V_0}{\partial z}f_0(z)=-2\alpha p_3x_{s2}z^{k+1}+2\alpha\beta p_3z^{2(k+1)}. \tag{42}$$

To make $\dot V_0(z)<0$, we take $\beta\le 0$, $\alpha>0$ and $k=1,3,5,\ldots$. Hence, by properly choosing α,β,k, the original system is minimum phase. Furthermore, we have

$$L_fh(x)=\begin{bmatrix}k\alpha x_3^{k-1}(x_1x_2+x_{s2}x_1)\\(k+1)\beta x_3^k(x_1x_2+x_{s2}x_1)\end{bmatrix},$$

$$L_gh(x)=\begin{bmatrix}1&0\\0&1\end{bmatrix}.$$

Now we take

$$V(z,y)=V_0(z)+y^{\mathrm T}Py, \tag{43}$$

where

$$P=\begin{bmatrix}p_1&0\\0&p_2\end{bmatrix},\quad p_1>0,p_2>0,$$

and let R have the form

$$R=\begin{bmatrix}r_1&0\\0&r_2\end{bmatrix},\quad r_1>0,r_2>0.$$

The function $L_2(x)$ corresponding to $\mathbf{r}(z,y)$ in (39) and (40) can be computed directly from (27) as

$$L_2^{\mathrm T}(x)=\begin{bmatrix}2r_1k\alpha x_1(x_2+x_{s2})x_3^{k-1}+2r_1(p_3/p_1)(x_2+x_{s2})x_3\\2r_2(k+1)\beta x_1(x_2+x_{s2})x_3^k-2r_2\alpha(p_3/p_2)x_3^{k+1}\end{bmatrix},$$

$$L_2^{\mathrm T}(x)=\begin{bmatrix}2r_1k\alpha x_1(x_2+x_{s2})x_3^{k-1}+2r_1(p_3/p_1)(x_{s2}-\beta x_3^{k+1})x_3\\2r_2(k+1)\beta x_1(x_2+x_{s2})x_3^k+2r_2(p_3/p_2)x_1x_3\end{bmatrix}.$$

From (29) we obtain a family of optimal nonlinear feedback control laws

$$\phi(x)=\begin{bmatrix}-k\alpha x_1(x_2+x_{s2})x_3^{k-1}-(p_3/p_1)(x_2+x_{s2})x_3\\-(p_1/r_1)(x_1+\alpha x_3^k)\\-(k+1)\beta x_1(x_2+x_{s2})x_3^k+\alpha(p_3/p_2)x_3^{k+1}\\-(p_2/r_2)(x_2+\beta x_3^{k+1})\end{bmatrix}, \tag{44}$$

$$\phi(x)=\begin{bmatrix}-k\alpha x_1(x_2+x_{s2})x_3^{k-1}-(p_3/p_1)(x_{s2}-\beta x_3^{k+2})\\-(p_1/r_1)(x_1+\alpha x_3^k)\\-(k+1)\beta x_1(x_2+x_{s2})x_3^k-(p_3/p_2)x_1x_3\\-(p_2/r_2)(x_2+\beta x_3^{k+1})\end{bmatrix}, \tag{45}$$

corresponding to $\mathbf{r}(z,y)$ given by (39) and (40), respectively.

It is noted that any linear combinations of (44) and (45) reduce to a general expression for optimal nonlinear stabilizing feedback control law [23]. Without loss of generality, we multiply (44) by ρ and (45) by $1-\rho$, where $\rho\in\mathcal{R}$, and add them together to obtain

$$\phi(x)=\begin{bmatrix}-(p_3/p_1)[\rho(x_2+x_{s2})x_3+(1-\rho)(x_{s2}-\beta x_3^{k+2})]\\-k\alpha x_1(x_2+x_{s2})x_3^{k-1}-(p_1/r_1)(x_1+\alpha x_3^k)\\-(k+1)\beta x_1(x_2+x_{s2})x_3^k-(p_2/r_2)(x_2+\beta x_3^{k+1})\\+(p_3/p_2)[\rho\alpha x_3^{k+1}-(1-\rho)x_1x_3]\end{bmatrix}. \tag{46}$$

Equation (46) provides a family of optimal nonlinear feedback control laws for (32), where the corresponding $L_2(x)$ is

$$L_2^{\mathrm T}(x)=\begin{bmatrix}2r_1(p_3/p_1)[(x_{s2}-\beta x_3^{k+1})+\rho(x_2+\beta x_3^{k+1})]x_3\\+2r_1k\alpha x_1(x_2+x_{s2})x_3^{k-1}\\2r_2(p_3/p_2)[x_1-\rho(x_1+\alpha x_3^k)]x_3\\+2r_2(k+1)\beta x_1(x_2+x_{s2})x_3^k\end{bmatrix}. \tag{47}$$

The decay rate of the function $V(x)$ is the same for both cases, namely,

$$\begin{aligned}\dot V(x) &= -2\frac{p_1^2}{r_1}(x_1+\alpha x_3^k)^2-2\frac{p_2^2}{r_2}(x_2+\beta x_3^{k+1})^2\\&\quad+2\alpha\beta p_3x_3^{2(k+1)}-2\alpha p_3x_{s2}x^{k+1}.\end{aligned} \tag{48}$$

Clearly, if $\beta\le 0$, then by taking $\alpha>0$ and $k=1,3,5,\ldots$, the Lyapunov derivative is negative. Hence, by properly choosing the parameters, the time derivative of the Lyapunov function is negative for all nonzero $x\in\mathcal{R}^3$. Thus, the control law $\phi(x)$ in (46) globally asymptotically stabilizes (34), and $V(x)$ is a Lyapunov function for the closed-loop system. It should be noted that because of the normalization in taking the linear combinations, the coefficient ρ does not appear in $\dot V(x)$. Finally, $L_1(x)$ can be calculated directly from (21), so that the performance integrand from (14) is

$$\begin{aligned}L(x,u) &= [u+\frac12R^{-1}L_2^{\mathrm T}(x)]^{\mathrm T}R[u+\frac12R^{-1}L_2^{\mathrm T}(x)]\\&\quad+\frac{p_1^2}{r_1}(x_1+\alpha x_3^k)^2+\frac{p_2^2}{r_2}(x_2+\beta x_3^{k+1})^2\\&\quad-2\alpha\beta p_3x_3^{2(k+1)}+2\alpha p_3x_{s2}x^{k+1},\end{aligned} \tag{49}$$

where R and $L_2^{\rm T}(x)$ are as defined previously. Hence $L(x,u)$ is nonnegative definite for all x and u.

The optimal nonlinear feedback control laws (46) are a direct generalization of the results of [22] to the case in which $x_{s1} = 0, x_{s2} \neq 0, x_{s3} \neq 0$. If $x_{s2} = 0, x_{s3} = 0$, then (46) specialize to the results obtained in [22]. Note that, in (46) $\beta = 0$ is allowed and for global stabilization k is restricted to be a positive odd integer.

The control laws developed here correspond to the case in which $x_{s2} > 0$. If $x_{s2} = 0$, then $\alpha\beta < 0$ is required to guarantee stabilization and α is not necessarily positive. If, on the other hand, $x_{s2} < 0$ then we can redefine the coordinate to have positive x_{s2} in the new coordinate, or we can restrict α to be negative. In the latter case, β should be chosen to be nonnegative.

Finally, we can write the control law (46) in the angular velocity coordinate (30) as $u(\omega) = (u_1(\omega), u_2(\omega))^{\rm T}$, where

$$\begin{aligned} u_1(\omega) &= -J_{23}\omega_2\omega_3 - k\alpha\omega_1\omega_2(\omega_3-\omega_{s3})^{k-1}/J_{12}^{k-1} \\ &\quad -(p_1/r_1)(\omega_1 + \alpha(\omega_3-\omega_{s3})^k/J_{12}^k) \\ &\quad -(p_3/p_1)[\rho\omega_2(\omega_3-\omega_{s3})/J_{12} \\ &\quad +(1-\rho)(\omega_{s2} - \beta(\omega_3-\omega_{s3})^{k+2}/J_{12}^{k+2})], \end{aligned} \tag{50}$$

$$\begin{aligned} u_2(\omega) &= -J_{31}\omega_3\omega_1 - (k+1)\beta\omega_1\omega_2(\omega_3-\omega_{s3})^k/J_{12}^k \\ &\quad -(p_2/r_2)[(\omega_2-\omega_{s2}) + \beta(\omega_3-\omega_{s3})^{k+1}/J_{12}^{k+1}] \\ &\quad +(p_3/p_2)[\rho\alpha(\omega_3-\omega_{s3})^{k+1}/J_{12}^{k+1} \\ &\quad -(1-\rho)\omega_1(\omega_3-\omega_{s3})/J_{12}]. \end{aligned} \tag{51}$$

4 Simulation Results

For illustration, we consider rotational stabilization for an idealized spacecraft with $J_1 = 4, J_2 = 3$, and $J_3 = 2$. Suppose we want to stabilize this spacecraft to $\omega_s = (0,1,1)^{\rm T}$ from an arbitrary initial angular velocity, say $(-1,-2,-3)^{\rm T}$. Choosing $\rho = 1, k = 1, p_1 = p_2 = r_1 = r_2 = \frac{1}{2}, p_3 = 1, \alpha = 1$, and $\beta = 0$, the globally asymptotically stabilizing control law (50) (51) becomes

$$u_1(\omega) = -\omega_2\omega_3/4 - \omega_1\omega_2 - 4\omega_2(\omega_3-\omega_{s3}) - \omega_1 - 2(\omega_3-\omega_{s3}), \tag{52}$$

$$u_2(\omega) = 2\omega_3\omega_1/3 + 8(\omega_3-\omega_{s3})^2 - (\omega_2-\omega_{s2}). \tag{53}$$

The simulation results are shown in Figure 1 and Figure 2. For $\beta = -1$ and the remaining parameters as above, the resulting control law is

$$u_1(\omega) = -\omega_2\omega_3/4 - \omega_1\omega_2 - 4\omega_2(\omega_3-\omega_{s3}) - \omega_1 - 2(\omega_3-\omega_{s3}), \tag{54}$$

$$u_2(\omega) = 2\omega_3\omega_1/3 + 4\omega_1\omega_2(\omega_3-\omega_{s3}) + 12(\omega_3-\omega_{s3})^2 - (\omega_2-\omega_{s2}). \tag{55}$$

The simulation results are shown in Figure 3 and Figure 4.

5 Conclusions

It was shown that the Euler's equation can be asymptotically stabilized to a nonzero state using only two torque inputs along two principal axes if and only if the nonzero state has only two nonzero components and the zero component is in one of the two controlled-principal-axis. The Hamilton-Jacobi-Bellman theorem was used to synthesize smooth control laws that globally asymptotically stabilized the Euler's equation to the prescribed state.

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FIGURE 1.

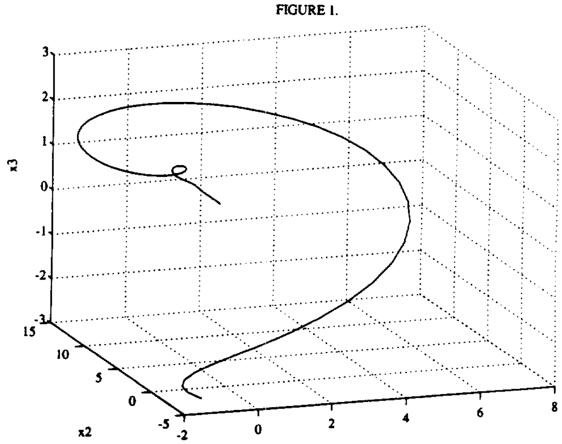

FIGURE 2.

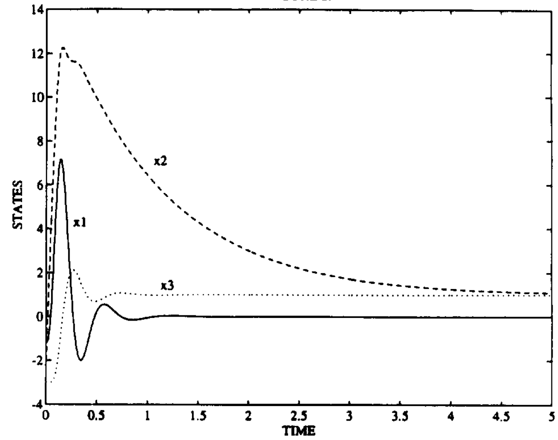

FIGURE 3.

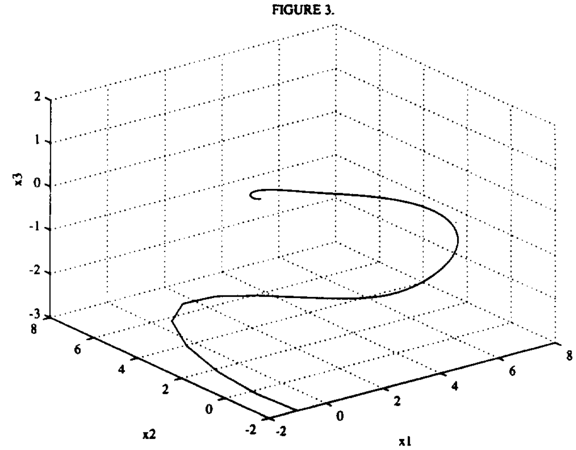

FIGURE 4.