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THERMODYNAMIC MODELLING OF INTERCONNECTED SYSTEMS, PART II: DISSIPATIVE COUPLING

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In this paper we present a thermodynamic energy flow model for dissipatively coupled systems. The resulting model shows that there exist two distinct energy flow components through the coupling; namely, an inter-subsystem component and a dissipative component. This energy flow model is shown to be more accurate than models given by prior researchers.

1. INTRODUCTION

Using the conceptual foundation of reference [1] as a starting point, a thermodynamic model for energy flow among coupled systems was developed in Part I [2]. The analysis in reference [2] showed that if the coupling elements neither store nor dissipate energy—that is, the coupling is conservative—then energy flows from higher thermodynamic energy subsystems to lower thermodynamic energy subsystems. Furthermore, under an appropriate condition this property is guaranteed independently of the strength of coupling and the number of subsystems [2].

The purpose of the present paper is to extend the results of reference [2] to the case of dissipative coupling. In prior work, Sun *et al*. [3] and Fahy *et al*. [4] calculated the energy flow between two oscillators connected by a dissipative coupling composed of a spring and damper. Their results show that there exists a term expressing dissipative energy flow in addition to the term proportional to the energy difference between oscillators. Although they obtained an explicit expression for the energy flow model, their model is complicated and difficult to extend to more than two subsystems. In addition, the approach in these papers involves incorporating the damping coefficients of the coupling within the subsystem. In this regard, their energy flow model combines the dissipative ability of the subsystem with that of the coupling, which makes the actual energy flow between each subsystem and the coupling difficult to interpret.

Our contribution to this problem is twofold. First we show that, as in the conservative coupling case [2], there exists an energy flow model for dissipatively coupled systems that predicts energy flow in a systematic manner, independently of the number of subsystems and the strength of the coupling. To obtain such an energy flow model, we define a general dissipatively coupled system (Section 2) and analyze energy flow among subsystems (sections 3–5) in a manner similar to reference [2]. It is shown that there exist two distinct energy flow components through the coupling; namely, an inter-subsystem component and a dissipative component.

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Our second contribution is an examination of the relationship between this energy flow model and those obtained by Sun *et al*. and Fahy *et al*. (section 6). An analysis of three oscillators coupled by dissipative elements illustrates the improved accuracy of the thermodynamic energy flow model (section 7).

The notation is listed in Appendix F of reference [2].

2. DEFINITIONS AND ASSUMPTIONS

As in Part I, we consider r subsystems $z_1(s), \ldots, z_r(s)$ interconnected by an $r \times r$ linear time-invariant coupling $L(s)$. An electrical representation of this interconnection involving scalar impedances $z_i(s)$ is given by Figure 1 of reference [2]. The assumptions and notations concerning the subsystems and disturbances are stated in Part I. In contrast to reference [2], however, where $L(s)$ was assumed to be conservative, we assume now that $L(s)$ is dissipative; that is

$$L(\mathrm{j}\omega) + L^*(\mathrm{j}\omega) \geqslant 0, \qquad \omega \in \mathscr{R}. \tag{1}$$

This condition implies that $\mathrm{Re}\,[L(\mathrm{j}\omega)] + \mathrm{Re}\,[L(\mathrm{j}\omega)]^{\mathrm{T}} \geqslant 0$. Since $Z^{-1}(s)$ is strictly positive real, where $Z(s) = \mathrm{diag}\,(z_1(s), z_2(s), \ldots, z_r(s))$, and $L(s)$ is positive real, it follows that the closed loop system is asymptotically stable.

3. ENERGY FLOW ANALYSIS IN THE FREQUENCY DOMAIN

In this section, we analyze energy flow among the dissipatively coupled systems shown in Figure 1 of reference [2], while an equivalent block diagram is given by Figure 2 of reference [2]. Since the following results are straightforward extensions of results given in reference [2], the proofs are omitted.

Our first result involving the energy flow through the ports of $L(s)$ is due to the dissipative nature of the coupling $L(\mathrm{j}\omega)$.

Lemma 3.1. The coupling energy flow $E_i^c(\omega)$ satisfies

$$\sum_{i=1}^{r} E_i^c(\omega) \leqslant 0, \qquad \omega \in \mathscr{R}. \tag{2}$$

The energy flow per unit bandwidth quantities $E_i^c(\omega)$, $E_i^d(\omega)$ and $E_i^e(\omega)$ satisfy the following relations.

Lemma 3.2. The energy flows per unit bandwidth matrices $E^c(\omega)$, $E^d(\omega)$ and $E^e(\omega)$ satisfy

$$E^c(\omega) + E^d(\omega) + E^e(\omega) = 0, \quad \omega \in \mathscr{R}. \tag{3}$$

Corollary 3.1. The energy flows per unit bandwidth quantities $E_i^c(\omega)$, $E_i^d(\omega)$ and $E_i^e(\omega)$ satisfy

$$E_i^c(\omega) + E_i^d(\omega) + E_i^e(\omega) = 0, \qquad i = 1, \ldots, r, \quad \omega \in \mathscr{R}, \tag{4}$$

and

$$\sum_{i=1}^{r} [E_i^d(\omega) + E_i^e(\omega)] \geqslant 0, \quad \omega \in \mathscr{R}. \tag{5}$$

Lemma 3.1 and Corollary 3.1 describe the properties of the average energy flows per unit bandwidth among the coupled subsystems. That is, equation (5) shows that more external power is generated than is dissipated at each subsystem, while equation (4) indicates that at each subsystem the effect of the external power generated by the disturbance is to change the rate of energy dissipation and energy flow through its port. External power is dissipated by the dissipative coupling $L(j\omega)$, as shown by equation (2).

Corollary 3.2. The steady state energy flow quantities P_i^c, P_i^d and P_i^e satisfy

$$\sum_{i=1}^{r} P_i^c \leqslant 0, \qquad P_i^c + P_i^d + P_i^e = 0, \quad i = 1, \ldots, r, \tag{6, 7}$$

and

$$\sum_{i=1}^{r} (P_i^d + P_i^e) \geqslant 0. \tag{8}$$

As in reference [2], by defining the steady state thermodynamic energy of the ith subsystem as

$$E_i^{th}(\omega) \triangleq S_{w_i w_i}(\omega)/2c_i(\omega), \tag{9}$$

and the steady state thermodynamic cross energy as

$$E_{ij}^{th}(\omega) \triangleq S_{w_i w_j}(\omega)/2\sqrt{c_i(\omega)c_j(\omega)}, \tag{10}$$

we obtain the following result.

Theorem 3.1. For each $i = 1, \ldots, r$, $E_i^c(\omega)$ and $E_i^d(\omega)$ can be expressed as

$$E_i^c(\omega) = E_{Inc,i}^c(\omega) + E_{Coh,i}^c(\omega), \quad E_i^d(\omega) = E_{Inc,i}^d(\omega) + E_{Coh,i}^d(\omega), \qquad \omega \in \mathscr{R}, \tag{11, 12}$$

respectively, where

$$E_{Inc,i}^c(\omega) \triangleq \sum_{\substack{j=1 \\ j \neq i}}^{r} [\delta_{ij}(\omega)E_j^{th}(\omega) - \delta_{ji}(\omega)E_i^{th}(\omega)] - \alpha_i(\omega)E_i^{th}(\omega), \tag{13}$$

$$E_{Coh,i}^c(\omega) \triangleq \sum_{p=1}^{r} \sum_{\substack{q=1 \\ q \neq p}}^{r} \delta_{ipq}(\omega)E_{pq}^{th}(\omega) - \sum_{\substack{q=1 \\ q \neq i}}^{r} \delta_{piq}E_{iq}^{th}(\omega)] - \sum_{\substack{s=1 \\ s \neq i}}^{r} \alpha_{is}(\omega)E_{is}^{th}(\omega), \tag{14}$$

$$E_{Inc,i}^d(\omega) \triangleq -\delta_{ii}(\omega)E_i^{th}(\omega) - \sum_{\substack{j=1 \\ j \neq i}}^{r} \delta_{ij}(\omega)E_j^{th}(\omega), \tag{15}$$

$$E_{Coh,i}^d(\omega) \triangleq -\sum_{p=1}^{r} \sum_{\substack{q=1 \\ q \neq p}}^{r} \delta_{ipq}(\omega)E_{pq}^{th}(\omega), \tag{16}$$

and, for $i, j, p, q, s = 1, \ldots, r$, $\omega \in \mathscr{R}$,

$$\delta_{ipq} \triangleq \frac{c_i(\omega)}{\pi} \sqrt{c_p(\omega)c_q(\omega)} \operatorname{Re} [(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))_{(i,p)}^{-1}(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))_{(q,i)}^{-*}], \tag{17}$$

$$\delta_{ij}(\omega) \triangleq \delta_{ijj}(\omega) = \frac{1}{\pi} c_i(\omega)c_j(\omega)|[(Z(\mathrm{j}\omega) + L(\mathrm{j}\omega))^{-1}]_{(i,j)}|^2, \tag{18}$$

$$\alpha_{is}(\omega) \triangleq \frac{1}{2\pi} \sqrt{c_i(\omega)c_s(\omega)} \operatorname{Re} [(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}(L(\mathrm{j}\omega) + L^*(\mathrm{j}\omega))(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}]_{(i,s)}, \tag{19}$$

$$\alpha_i(\omega) \triangleq \alpha_{ii}(\omega) = \frac{1}{2\pi} c_i(\omega)[(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*} (L(\mathrm{j}\omega) + L^*(\mathrm{j}\omega))(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}]_{(i,i)}. \tag{20}$$

From equations (18) and (20), it can be seen that $\delta_{ij}(\omega) \geqslant 0$ and $\sigma_i(\omega) \geqslant 0$, $i, j = 1, \ldots, r$, $\omega \in \mathscr{R}$. Thus it can be seen from equation (13) that the incoherent coupling energy flow $E^c_{Inc,i}(\omega)$ is decomposed into two kinds of energy flows; namely, the inter-subsystem energy flow (the summation in equation (13)) determined by $\delta_{ij}(\omega)$, and the dissipative energy flow (the last term in equation (13)) governed by $\alpha_i(\omega)$. The inter-subsystem energy flow depends on the thermodynamic energy of each subsystem, whereas the dissipative energy flow depends only on the thermodynamic energy of the subsystem into which the energy is flowing.

By introducing one additional assumption on the dissipative coupling $L(s)$, we can guarantee that the inter-subsystem energy flows from higher thermodynamic energy subsystems to lower thermodynamic energy subsystems.

Corollary 3.3. If $L(j\omega)$ is symmetric for all $\omega \in \mathscr{R}$, then

$$\delta_{ij}(\omega) = \delta_{ji}(\omega), \qquad i, j = 1, \ldots, r, \quad \omega \in \mathscr{R}, \tag{21}$$

and, for each $i = 1, \ldots, r$,

$$E^c_{Inc,i}(\omega) = \sum_{\substack{j=1 \\ j \neq i}}^{r} \delta_{ij}(\omega)[E^{th}_j(\omega) - E^{th}_i(\omega)] - \alpha_i(\omega) E^{th}_i(\omega), \qquad \omega \in \mathscr{R}. \tag{22}$$

Remark 3.1. Note that the assumption that $L(j\omega)$ is symmetric implies that $\mathrm{Re}\,[L(j\omega)]$ is symmetric non-negative definite and $\mathrm{Im}\,[L(\mathrm{j}\omega)]$ is symmetric.

Equation (22) can be interpreted thermodynamically as saying that energy flows among subsystems in proportion to energy differences and dissipatively in proportion to subsystem energy. As in reference [2], this equation holds independently of the number of subsystems and the strength of the coupling.

Finally, we rewrite the rate of energy dissipation $E^d_{Inc,i}(\omega)$ in equation (15) so that $E^d_{Inc,i}(\omega)$ can be expressed as a function of $E^{th}_i(\omega)$ only; that is,

$$E^d_{Inc,i}(\omega) = -\delta_i(\omega) E^{th}_i(\omega), \qquad \omega \in \mathscr{R}, \tag{23}$$

where

$$\delta_i(\omega) \triangleq \sum_{j=1}^{r} \delta_{ij}(\omega) \frac{E^{th}_j(\omega)}{E^{th}_i(\omega)}. \tag{24}$$

Obviously $\delta_i(\omega) \geqslant 0$, $i = 1, \ldots, r$, $\omega \in \mathscr{R}$.

Theorem 3.1 is illustrated in Figure 1 for the case $r = 3$ and $\mathrm{Coh}\,[S_{ww}(\omega)] = 0$. In Figure 1 it is shown that there exist inter-subsystem energy flows $\delta_{ji}(\omega)E^{th}_i(\omega)$ and $\delta_{ij}(\omega)E^{th}_j(\omega)$ between the ith and jth subsystems, and that the difference between these energy flows is the net energy flow $\Xi_{ij}(\omega) \triangleq \delta_{ij}(\omega)E^{th}_j(\omega) - \delta_{ji}(\omega)E^{th}_i(\omega)$. Furthermore, there also exists dissipative energy flow $\alpha_i(\omega)E^{th}_i(\omega)$ through the coupling $L(s)$. Then $E^c_{Inc,i}(\omega) = \Sigma^r_{j=1, j \neq i}\, \Xi_{ij}(\omega) - \alpha_i(\omega)E^{th}_i(\omega)$. According to Corollary 3.3, if $\delta_{ji}(\omega) = \delta_{ij}(\omega)$, $i, j = 1, \ldots, r$, there is a net energy flow among subsystems, from higher thermodynamic energy subsystems to lower thermodynamic energy subsystems.

4. ENERGY FLOW MODEL FOR TIME DOMAIN ANALYSIS

In this section, we consider an alternative point of view involving compartmental modelling and time domain analysis. As in reference [2], we now assume that $w_i(t)$ is white noise with intensity matrix $S_{ww} = DD^{\mathrm{T}}$, where D is given in reference [2], and that $c_i = c_i(\omega)$, the real part of $z_i(\omega)$, is constant. Using the notation of reference [2] for C_d, C_1, C_2, $\tilde{A}$, $\tilde{B}$ and B, we have the following results.

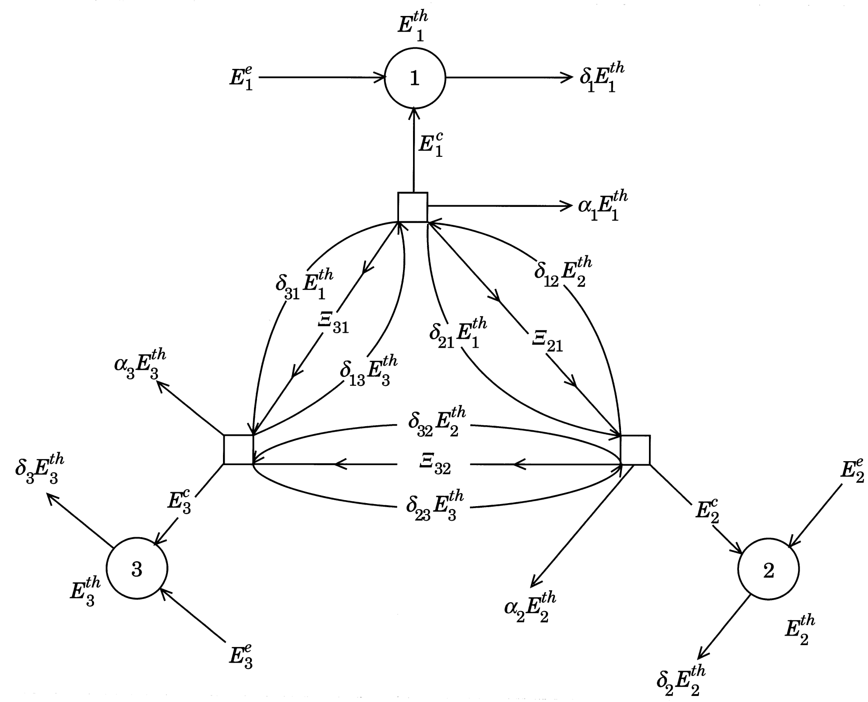

Figure 1. The subsystem energy flow diagram (frequency domain).

Theorem 4.1. The steady state energy flow matrices P^c, P^e and P^d are given by

$$P^c = -C_2\tilde{Q}C_1^{\mathrm{T}}, \qquad P^e = \tfrac{1}{2}D\tilde{B}^{\mathrm{T}}C_1^{\mathrm{T}}, \qquad P^d = (C_2\tilde{Q} - \tfrac{1}{2}D\tilde{B}^{\mathrm{T}})C_1^{\mathrm{T}}, \tag{25–27}$$

where the steady state covariance $\tilde{Q} \triangleq \mathscr{E}[x(t)x^{\mathrm{T}}(t)]$ satisfies the algebraic Lyapunov equation

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{B}\tilde{B}^{\mathrm{T}}. \tag{28}$$

Corollary 4.1. For $i = 1, \ldots, r$, P_i^d is given by

$$P_i^d = -(C_d C_1 \tilde{Q} C_1^{\mathrm{T}})_{(i,i)}, \tag{29}$$

where the steady state covariance $\tilde{Q}$ satisfies the algebraic Lyapunov equation (28).

Proposition 4.1. For $i = 1, \ldots, r$, P_i^c, P_i^d and P_i^c can be decomposed as

$$P_i^c = P_{Inc,i}^c + P_{Coh,i}^c, \qquad P_i^d = P_{Inc,i}^d + P_{Coh,i}^d, \qquad P_i^e = P_{Inc,i}^e + P_{Coh,i}^e, \tag{30–32}$$

where

$$P_{Inc,i}^c \triangleq -(C_2\tilde{Q}_{Inc}C_1^{\mathrm{T}})_{(i,i)}, \qquad P_{Coh,i}^c \triangleq -(C_2\tilde{Q}_{Coh}C_1^{\mathrm{T}})_{(i,i)},$$
$$P_{Inc,i}^d \triangleq -(C_d C_1\tilde{Q}_{Inc}C_1^{\mathrm{T}})_{(i,i)}, \qquad P_{Coh,i}^d \triangleq -(C_d C_1\tilde{Q}_{Coh}C_1^{\mathrm{T}})_{(i,i)},$$
$$P_{Inc,i}^e \triangleq \tfrac{1}{2}(\mathrm{Inc}\,[S_{ww}]B^{\mathrm{T}}C_1^{\mathrm{T}})_{(i,i)}, \qquad P_{Coh,i}^e \triangleq \tfrac{1}{2}(\mathrm{Coh}\,[S_{ww}]B^{\mathrm{T}}C_1^{\mathrm{T}})_{(i,i)},$$

and $\tilde{Q}_{Inc}$ and $\tilde{Q}_{Coh}$ satisfy

$$0 = \tilde{A}\tilde{Q}_{Inc} + \tilde{Q}_{Inc}\tilde{A}^{\mathrm{T}} + B\,\mathrm{Inc}\,[S_{ww}]B^{\mathrm{T}}, \tag{33}$$

$$0 = \tilde{A}\tilde{Q}_{Coh} + \tilde{Q}_{Coh}\tilde{A}^{\mathrm{T}} + B\,\mathrm{Coh}\,[S_{ww}]B^{\mathrm{T}}. \tag{34}$$

Proposition 4.2. For $i = 1, \ldots, r$,

$$P^c_{Inc,i} + P^d_{Inc,i} + P^e_{Inc,i} = 0, \qquad P^c_{Coh,i} + P^d_{Coh,i} + P^e_{Coh,i} = 0. \tag{35, 36}$$

Now, as in reference [2] we consider a compartmental model expression for the coupled system.

Theorem 4.2. Define

$$\sigma_{ij} \triangleq \int_{-\infty}^{\infty} \delta_{ij}(\omega)\, \mathrm{d}\omega, \qquad i \neq j, \quad i, j = 1, \ldots, r, \tag{37}$$

$$\sigma_i \triangleq \int_{-\infty}^{\infty} \delta_i(\omega)\, \mathrm{d}\omega, \quad \bar{\alpha}_i \triangleq \int_{-\infty}^{\infty} \alpha_i(\omega)\, \mathrm{d}\omega, \qquad i = 1, \ldots, r, \tag{38, 39}$$

$$\eta_i \triangleq \sigma_i + \bar{\alpha}_i, \quad i = 1, \ldots, r; \qquad \Pi_{ij} \triangleq \sigma_{ij} E^{th}_j - \sigma_{ji} E^{th}_i, \quad i, j = 1, \ldots, r, \tag{40, 41}$$

where $\delta_{ij}(\omega)$, $\alpha_i(\omega)$ and $\delta_i(\omega)$ are given by equations (18), (20) and (24), respectively. Then, energy flow in the coupled system satisfies

$$-\eta_i E^{th}_i + \sum_{\substack{j=1 \\ j \neq i}}^{r} \Pi_{ij} + P^e_{Inc,i} = 0, \qquad i = 1, \ldots, r. \tag{42}$$

Proof. From the definition of $P^c_{Inc,i}$ it follows that

$$\begin{aligned} P^c_{Inc,i} &= \int_{-\infty}^{\infty} E^c_{Inc,i}(\omega)\, \mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} \sum_{\substack{j=1 \\ j \neq i}}^{r} [\delta_{ij}(\omega)\, \mathrm{d}\omega\, E^{th}_j - \delta_{ji}(\omega)\, \mathrm{d}\omega\, E^{th}_i] - \int_{-\infty}^{\infty} \alpha_i(\omega)\, \mathrm{d}\omega\, E^{th}_i \\ &= \sum_{\substack{j=1 \\ j \neq i}}^{r} \left[\int_{-\infty}^{\infty} \delta_{ij}(\omega)\, \mathrm{d}\omega\, E^{th}_j - \int_{-\infty}^{\infty} \delta_{ji}(\omega)\, \mathrm{d}\omega\, E^{th}_i \right] - \int_{-\infty}^{\infty} \alpha_i(\omega)\, \mathrm{d}\omega\, E^{th}_i \\ &= \sum_{\substack{j=1 \\ j \neq i}}^{r} [\sigma_{ij} E^{th}_j - \sigma_{ji} E^{th}_i] - \bar{\alpha}_i E^{th}_i = \sum_{\substack{j=1 \\ j \neq i}}^{r} \Pi_{ij} - \bar{\alpha}_i E^{th}_i. \end{aligned}$$

In a similar manner,

$$P^d_{Inc,i} = \int_{-\infty}^{\infty} E^d_{Inc,i}(\omega)\, \mathrm{d}\omega = -\int_{-\infty}^{\infty} \delta_i(\omega)\, \mathrm{d}\omega\, E^{th}_i = -\sigma_i E^{th}_i.$$

Finally, by using equation (35) in Proposition 4.2, equation (42) can be obtained. □

Obviously, $\eta_i \geqslant 0$, $\sigma_{ij} \geqslant 0$ and $P^e_{Inc,i} \geqslant 0$, so that equations (41) and (42) represent a form of a compartmental model [5].

We also obtain the following result corresponding to Corollary 3.3.

Corollary 4.2. If $L(\mathrm{j}\omega)$ is symmetric, then

$$\sigma_{ij} = \sigma_{ji}, \qquad i, j = 1, \ldots, r, \tag{43}$$

and

$$P^c_{Inc,i} = \sum_{\substack{j=1 \\ j \neq 1}}^{r} \sigma_{ij}(E^{th}_j - E^{th}_i) - \bar{\alpha}_i E^{th}_i. \tag{44}$$

Proof. This result follows immediately from Corollary 3.3. □

Although Theorem 4.2 provides closed form expressions for σ_{ij} and $\bar{\alpha}_i$, the integration may be difficult, especially for $r \geqslant 3$. Next we provide explicit expressions for integrals in terms of algebraic Lyapunov equations.

Proposition 4.3. The coefficients σ_{ij} and σ_i, $i, j = 1, \ldots, r$, defined by equations (37) and (38), respectively, are given by

$$\sigma_{ij} = 2c_i c_j (C_1 \tilde{Q}_j C_1^{\mathrm{T}})_{(i,i)}, \quad i, j = 1, \ldots, r, \qquad \sigma_i = \sum_{j=1}^{r} \sigma_{ij} \frac{E^{th}_j}{E^{th}_i}, \quad i = 1, \ldots, r, \tag{45, 46}$$

where $\tilde{Q}_j$, $j = 1, \ldots, r$, satisfies the algebraic Lyapunov equation

$$0 = \tilde{A}\tilde{Q}_j + \tilde{Q}_j \tilde{A}^{\mathrm{T}} + B e_j e_j^{\mathrm{T}} B^{\mathrm{T}}, \tag{47}$$

and e_j denotes the jth column of the $r \times r$ identity matrix. Furthermore, suppose that $\overline{T} \triangleq L(\mathrm{j}\omega) + L^*(\mathrm{j}\omega)$ is constant. Then $\bar{\alpha}_i$ defined by equation (39) is given by

$$\bar{\alpha}_i = c_i (B^{\mathrm{T}} \hat{Q} B)_{(i,i)}, \tag{48}$$

where $\hat{Q}$ satisfies the algebraic Lyapunov equation

$$0 = \tilde{A}^{\mathrm{T}} \hat{Q} + \hat{Q} \tilde{A} + C_1^{\mathrm{T}} \overline{T} C_1. \tag{49}$$

If, in addition, $L(\mathrm{j}\omega)$ is symmetric, then $\bar{\alpha}_i$ is also given by

$$\bar{\alpha}_i = c_i (C_1 \overline{Q} C_1^{\mathrm{T}})_{(i,i)}, \tag{50}$$

where $\overline{Q}$ satisfies the algebraic Lyapunov equation

$$0 = \tilde{A}\overline{Q} + \overline{Q}\tilde{A}^{\mathrm{T}} + B\overline{T}B^{\mathrm{T}}. \tag{51}$$

Proof. See Appendix A. □

By using Proposition 4.3, we can obtain coefficients for the time-domain energy flow model, which allows us to calculate energy flow among an arbitrary number of coupled subsystems.

Finally, the energy flow model obtained in this section is illustrated in Figure 2 for the case $r = 3$ and $\mathrm{Coh}\,[S_{ww}] = 0$. Figure 2 has the same interpretation as Figure 1.

5. EQUIPARTITION OF ENERGY

In this section we show, as in reference [2], that equipartition of energy is equivalent to zero net energy flow into or out of each subsystem.

Theorem 5.1. Assume that $L(\mathrm{j}\omega)$ is symmetric. If

$$E^{th}_i(\omega) = E^{th}_j(\omega), \qquad i, j = 1, \ldots, r, \tag{52}$$

then

$$\sum_{\substack{j=1 \\ j \neq i}}^{r} \Xi_{ij}(\omega) = 0, \qquad i = 1, \ldots, r, \quad \omega \in \mathscr{R}. \tag{53}$$

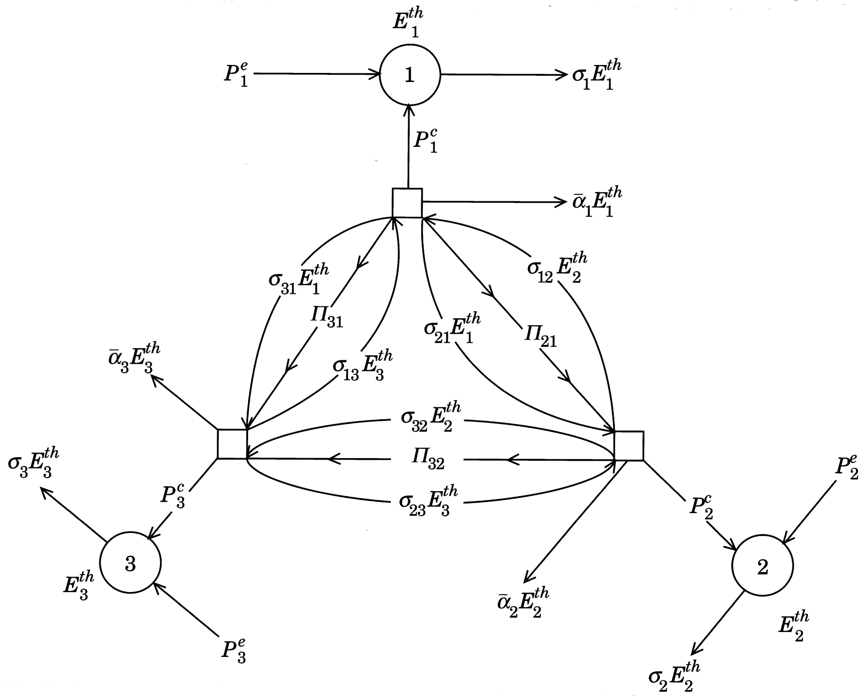

Figure 2. The subsystem energy flow diagram (time domain).

Corollary 5.1. Assume that $S_{w_i w_i}$ and c_i are constant for $i = 1, \ldots, r$ and that $L(\mathrm{j}\omega)$ is symmetric. If

$$E_i^{th} = E_j^{th}, \qquad i, j = 1, \ldots, r, \tag{54}$$

then

$$\sum_{\substack{j=1 \\ j \neq i}}^{r} \Pi_{ij} = 0, \qquad i = 1, \ldots, r. \tag{55}$$

Theorem 5.1 and Corollary 5.1 show that there is no net energy flow among subsystems when each subsystem has the same thermodynamical energy; that is, when equipartition of energy holds.

The converse of Theorem 5.1 and Corollary 5.1 can be obtained by defining the $r \times r$ matrices $\mathscr{H}(\omega)$ and $\mathscr{S}$ by

$$\mathscr{H}(\omega)_{(i,j)} \triangleq \delta_{ij}(\omega), \quad \mathscr{H}(\omega)_{(i,i)} \triangleq -\sum_{\substack{j=1 \\ j \neq i}}^{r} \delta_{ji}(\omega), \qquad i, j = 1, \ldots, r,$$

$$\mathscr{S}_{(i,j)} \triangleq \sigma_{ij}, \quad \mathscr{S}_{(i,i)} \triangleq -\sum_{\substack{j=1 \\ j \neq i}}^{r} \sigma_{ji}, \qquad i, j = 1, \ldots, r,$$

respectively.

Theorem 5.2. Assume that $L(j\omega)$ is symmetric and that rank $\mathscr{H}(\omega) = r - 1$ for all $\omega \in \mathscr{R}$. If

$$\sum_{\substack{j=1 \\ j \neq i}}^{r} \Xi_{ij}(\omega) = 0, \qquad i = 1, \ldots, r, \quad \omega \in \mathscr{R}, \tag{56}$$

then

$$E_i^{th}(\omega) = E_j^{th}(\omega), \qquad i, j = 1, \ldots, r, \quad \omega \in \mathscr{R}. \tag{57}$$

Corollary 5.2. Assume that $L(j\omega)$ is symmetric and that rank $\mathscr{S} = r - 1$. If

$$\sum_{\substack{j=1 \\ j \neq i}}^{r} \Pi_{ij} = 0, \qquad i = 1, \ldots, r, \tag{58}$$

then

$$E_i^{th} = E_j^{th}, \qquad i, j = 1, \ldots, r. \tag{59}$$

6. INCREASED DISSIPATION FEEDBACK SYSTEM

As mentioned in section 1, Sun *et al*. [3] and Fahy *et al*. [4] obtained an energy flow model for two dissipatively coupled oscillators, as shown in Figure 3. In their framework, the thermodynamical energy of each subsystem is effectively increased according to

$$\tilde{E}_i^{th} = S_{w_i w_i}/2(c_i + C), \qquad i = 1, 2, \tag{60}$$

where C is the damping coefficient of the coupling. In reference [4] $\tilde{E}_i^{th}$ is called the time-averaged total energy of the ith uncoupled oscillator. The difference between their energy flow model and the energy flow model obtained in the previous sections results from the signals to be used for calculating the energy flow. To clarify this point, we formulate the system interconnection by using feedback in Figure 2 of reference [2]. Because of equation (60), the energy flow model of references [3, 4] can be viewed as an increased dissipation feedback system. For simplicity, we assume that Coh $[S_{ww}(\omega)] = 0$.

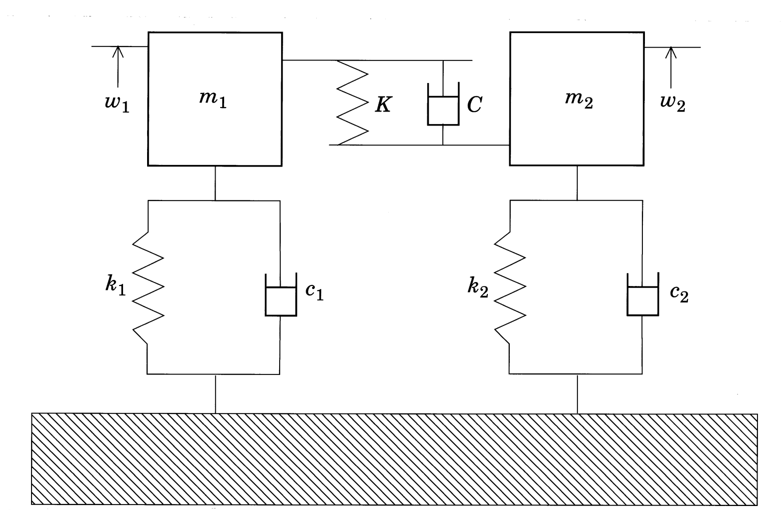

Figure 3. The two coupled oscillator system with dissipative coupling.

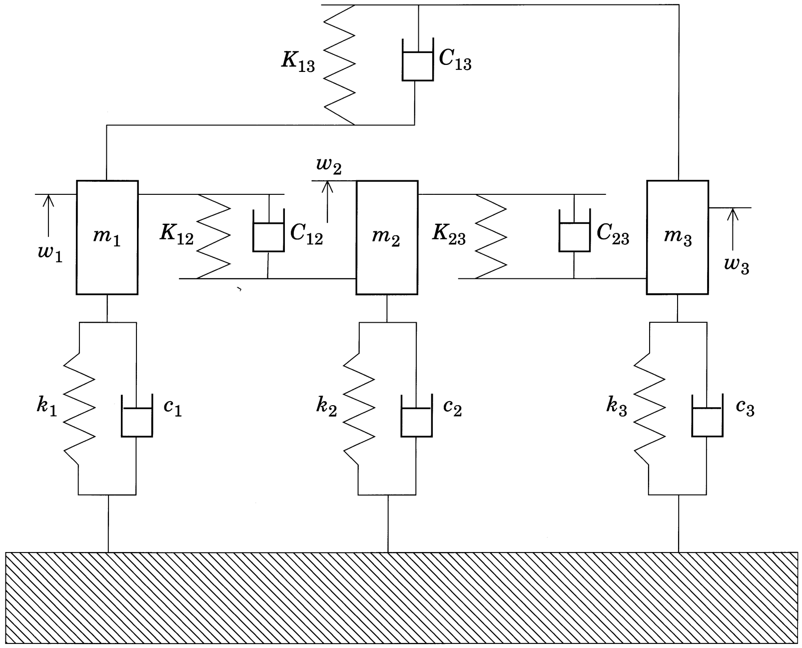

Figure 4. The three coupled oscillator system with dissipative coupling.

First we extend the energy flow model of references [3, 4] to more than two coupled oscillators. For such systems the coupling $L(s)$ has the form

$$L(s) = D_L + C_L/s, \tag{61}$$

where $D_L \in \mathscr{R}^{r\times r}$ is symmetric and satisfies

$$D_{L(i,j)} = D_{L(j,i)} \triangleq -d_{ij} \leqslant 0, \quad i \neq j, \; i, j = 1, \ldots, r; \tag{62}$$

$$D_{L(i,i)} = \sum_{\substack{j=1 \\ j\neq i}}^{r} d_{ij}, \quad i = 1, \ldots, r, \tag{63}$$

and $C_L \in \mathscr{R}^{r\times r}$ is symmetric. For example, the coupling $L(\mathrm{j}\omega)$ of the three coupled oscillator system in Figure 4 is given by

$$L(\mathrm{j}\omega) = D_L + C_L/s,$$

where

$$D_L = \begin{bmatrix} C_{12}+C_{13} & -C_{12} & -C_{13} \\ -C_{12} & C_{12}+C_{23} & -C_{23} \\ -C_{13} & -C_{23} & C_{13}+C_{23} \end{bmatrix},$$

and

$$C_L = \begin{bmatrix} K_{12}+K_{13} & -K_{12} & -K_{13} \\ -K_{12} & K_{12}+K_{23} & -K_{23} \\ -K_{13} & -K_{23} & K_{13}+K_{23} \end{bmatrix}.$$

Obviously, this coupling $L(\mathrm{j}\omega)$ satisfies equation (62), equation (63), and C_L is symmetric.

To transform the feedback system, we define

$$\{D_L\} \triangleq \operatorname{diag}(D_{L(1,1)}, D_{L(2,2)}, \ldots, D_{L(r,r)}), \qquad \tilde{Z}(j\omega) \triangleq Z(j\omega) + \{D_L\}, \qquad (64, 65)$$

and

$$\tilde{L}(j\omega) \triangleq D_L - \{D_L\} + C_L/s. \qquad (66)$$

Then from Figure 2 of reference [2] it follows that

$$y = Z^{-1}(s)(w - v) = Z^{-1}(s)(w - L(s)y) = Z^{-1}(s)(w - \{D_L\}y - \tilde{L}(s)y). \qquad (67)$$

Since $Z(s)$ is square and invertible, and $I + Z^{-1}(s)\{D_L\}$ is also square and invertible, it follows from equation (67) that

$$y = (I + Z^{-1}(s)\{D_L\})^{-1}Z^{-1}(s)(w - \tilde{L}(s)y) = (Z(s) + \{D_L\})^{-1}(w - \tilde{L}(s)y)$$
$$= \tilde{Z}^{-1}(s)(w - \tilde{L}(s)y). \qquad (68)$$

Thus, the feedback system in Figure 2 of reference [2] is transformed to a system with increased dissipation as shown in Figure 5, where $v_1, v_2, u_2 \in \mathscr{R}^r$ are given by

$$v_1 = \{D_L\}(\tilde{L}(s) + \tilde{Z}(s))^{-1}w, \quad v_2 = \tilde{L}(s)(\tilde{L}(s)+\tilde{Z}(s))^{-1}w, \quad u_2 = \tilde{Z}(s)(\tilde{L}(s) + \tilde{Z}(s))^{-1}w.$$

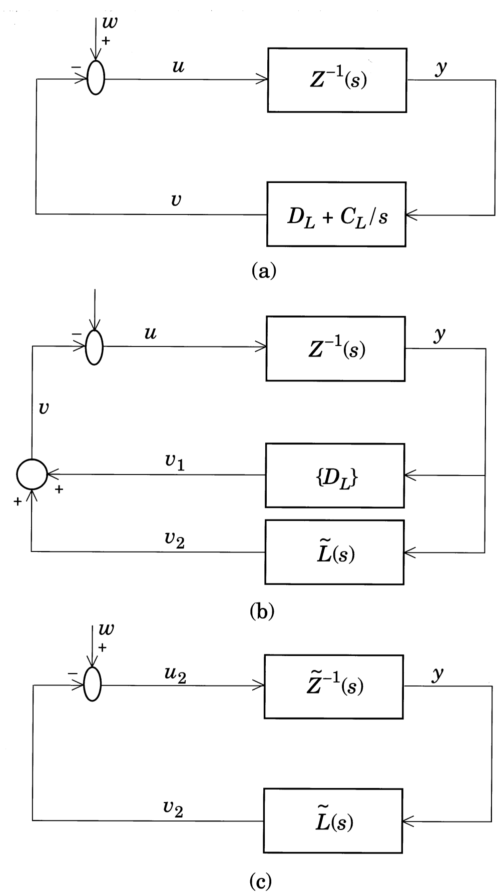

Figure 5. Transformation of the feedback system: (a) a feedback system equivalent to Figure 2 of reference [2]; (b) reformulation of (a); (c) increased dissipation feedback system.

Now

$$\mathrm{Re}\,[\tilde{Z}(\mathrm{j}\omega)]_{(i,i)} = c_i + \sum_{\substack{j=1\\ j\neq i}}^{r} d_{ij} \geqslant 0, \quad i = 1, \ldots, r; \tag{69}$$

that is, the dissipative ability of each subsystem $\tilde{z}_i(\mathrm{j}\omega) \triangleq [\tilde{Z}(\mathrm{j}\omega)]_{(i,i)}$ is increased compared with that of $z_i(\mathrm{j}\omega)$. Thus the thermodynamical energy of each subsystem $\tilde{z}_i(\mathrm{j}\omega)$ can be defined by

$$\tilde{E}_i^{th}(\omega) \triangleq \frac{S_{w_i w_i}(\omega)}{2\left(c_i + \sum_{\substack{j=1\\ j\neq i}}^{r} d_{ij}\right)}, \tag{70}$$

and the coupling and dissipation energy flow matrices are now calculated according to y, v_2 and u_2 (instead of y, v and u) as

$$\tilde{P}^c \triangleq -\mathscr{E}[v_2(t)y^{\mathrm{T}}(t)], \qquad \tilde{p}^d \triangleq -\mathscr{E}[u_2(t)y^{\mathrm{T}}(t)].$$

The energy flow model of references [3, 4], which was calculated without using the feedback representation, is based on this increased dissipation system. As shown later, the energy flow predicted by the approach of references [3, 4] is completely different from the energy flow obtained by means of the model developed in the previous sections.

Remark 6.1. It is important to note that $\tilde{L}(\mathrm{j}\omega)$ defined by equation (66) for the increased dissipation feedback system does not necessarily satisfy $\tilde{L}(\mathrm{j}\omega) + \tilde{L}^*(\mathrm{j}\omega) \geqslant 0$. Thus, some of the results obtained in the previous sections cannot be used to analyze this system. Specifically, Lemma 3.1, equation (5) in Corollary 3.1, equation (6), equation (8) in Corollary 3.2, equations (48) and (50) in Proposition 4.3 and the non-negativity of $\alpha_i(\omega)$ and $\bar{\alpha}_i$ do not apply. With these exceptions, the remaining results can be used to analyze energy flow in the increased dissipation feedback representation.

Remark 6.2. Note that since equation (6) in Corollary 3.2 does not necessarily hold, the increased dissipation feedback system may imply that energy is increasing at the coupling. On the other hand, since Corollary 3.3 still holds for symmetric $\tilde{L}(\mathrm{j}\omega)$, the inter-subsystem energy flows according to $\tilde{E}_i^{th}$ in the increased dissipation feedback representation. Thus, if the ordering of $\tilde{E}_1^{th}, \ldots, \tilde{E}_r^{th}$ is different from the ordering of $E_1^{th}, \ldots, E_r^{th}$, then these energy flow models may predict different directions for the inter-subsystem energy flows.

7. EXAMPLES

In this section we numerically illustrate an energy flow model for the dissipatively coupled systems shown in Figures 3 and 4. For simplicity, we assume that $\mathrm{Coh}\,[S_{ww}(\omega)] = \mathrm{Coh}\,[S_{ww}] = 0$. Now the coupling $L(\mathrm{j}\omega)$ is symmetric so that the net inter-subsystem energy flow is from higher thermodynamic energy subsystems to lower thermodynamic energy subsystems according to Corollaries 3.3 and 4.2. From Theorem 5.1 and Corollary 5.1, all the inter-subsystem energy flows are zero when each subsystem has the same thermodynamical energy.

7.1. EXAMPLE 1

Consider the system consisting of two coupled oscillators shown in Figure 3 and define

$$\Delta_{1u} \triangleq c_1/m_1, \qquad \Delta_{2u} \triangleq c_2/m_2, \qquad \omega_1^2 \triangleq (K + k_1)/m_1, \qquad \omega_2^2 \triangleq (K + k_2)/m_2,$$
$$\Delta_1 \triangleq (C + c_1)/m_1, \qquad \Delta_2 \triangleq (C + c_2)/m_2,$$
$$\omega_{1u}^2 \triangleq k_1/m_1, \qquad \omega_{2u}^2 \triangleq k_2/m_2, \qquad \Delta_{1c} \triangleq C/m_1, \qquad \Delta_{2c} \triangleq C/m_2,$$
$$\mu \triangleq C/\sqrt{m_1 m_2}, \qquad v \triangleq K/\sqrt{m_1 m_2}.$$

First we obtain the energy flow coefficients for the actual feedback system in Figure 2 of reference [2]. From equations (18) and (20), $\delta_{12}(\omega)$, $\alpha_1(\omega)$ and $\alpha_2(\omega)$ are given by

$$\delta_{12}(\omega) = \frac{\Delta_{1u}\,\Delta_{2u}\omega^2(\mu^2\omega^2 + v^2)}{\pi\Gamma(\omega)}, \tag{71}$$

$$\alpha_1(\omega) = \frac{\Delta_{1u}\,\Delta_{1c}\omega^2(\Delta_{2u}^2 + (\omega^2 - \omega_{2u}^2)^2]}{\pi\Gamma(\omega)}, \qquad \alpha_2(\omega) = \frac{\Delta_{2u}\,\Delta_{2c}\omega^2[\Delta_{1u}^2 + (\omega^2 - \omega_{1u}^2)^2]}{\pi\Gamma(\omega)}, \tag{72, 73}$$

where

$$\Gamma(\omega) \triangleq [\omega^4 - (\omega_1^2 + \omega_2^2)\omega^2 + \omega_1^2\omega_2^2 - v^2]^2 + \omega^2[(\Delta_1 + \Delta_2)\omega^2 - (\Delta_1\omega_2^2 + \Delta_2\omega_1^2 - 2\mu v)]^2. \tag{74}$$

By using the integral formula [6], we obtain

$$\sigma_{12} = \Delta_{1u}\Delta_{2u}[(\Delta_1 + \Delta_2)v^2 + (\Delta_1\omega_2^2 + \Delta_2\omega_1^2 - 2\mu v)\mu^2]/\Lambda, \tag{75}$$

$$\bar{\alpha}_1 = \Delta_{1u}\Delta_{1c}A_1/\Lambda, \qquad \bar{\alpha}_2 = \Delta_{2u}\Delta_{2c}A_2/\Lambda, \tag{76, 77}$$

where

$$\begin{aligned}\Lambda \triangleq \Delta_1\,\Delta_2[(\omega_1^2 - \omega_2^2)^2 + (\Delta_1 + \Delta_2)(\Delta_1\omega_2^2 + \Delta_2\omega_1^2 - 2\mu v)] \\ + v^2(\Delta_1 + \Delta_2)^2 + 2\mu v[(\Delta_1 - \Delta_2)(\omega_1^2 - \omega_2^2) - 2\mu v] \\ + \mu^2(\Delta_1 + \Delta_2)(2\mu v - \Delta_1\omega_2^2 - \Delta_2\omega_1^2),\end{aligned} \tag{78}$$

$$\begin{aligned}A_1 \triangleq (\Delta_1 + \Delta_2)(v^2 - \omega_1^2\omega_2^2) + (\omega_1^2 + \omega_2^2 + \Delta_1\Delta_2 - \mu^2)(\Delta_1\omega_2^2 + \Delta_2\omega_1^2 - 2\mu v) \\ + (\Delta_1 + \Delta_2)\omega_{2u}^4 + (\Delta_1\omega_2^2 + \Delta_2\omega_1^2 - 2\mu v)(\Delta_{2u}^2 - 2\omega_{2u}^2),\end{aligned} \tag{79}$$

$$\begin{aligned}A_2 \triangleq (\Delta_1 + \Delta_2)(v^2 - \omega_1^2\omega_2^2) + (\omega_1^2 + \omega_2^2 + \Delta_1\Delta_2 - \mu^2)(\Delta_1\omega_2^2 + \Delta_2\omega_1^2 - 2\mu v) \\ + (\Delta_1 + \Delta_2)\omega_{1u}^4 + (\Delta_1\omega_2^2 + \Delta_2\omega_1^2 - 2\mu v)(\Delta_{1u}^2 - 2\omega_{1u}^2).\end{aligned} \tag{80}$$

On the other hand, the coefficients for the increased dissipation feedback system are obtained in a similar manner, as

$$\tilde{\delta}_{12}(\omega) = \Delta_1\Delta_2\omega^2(\mu^2\omega^2 + v^2)/\pi\Gamma(\omega), \tag{81}$$

$$\tilde{\alpha}_1(\omega) = 2[\mu\Delta_1(v - \mu\Delta_2)\omega^2 - \mu v\omega_2^2\Delta_1]\omega^2/\pi\Gamma(\omega), \tag{82}$$

$$\tilde{\alpha}_2(\omega) = 2[\mu\Delta_2(v - \mu\Delta_1)\omega^2 - \mu v\omega_1^2\Delta_2]\omega^2/\pi\Gamma(\omega), \tag{83}$$

$$\tilde{\sigma}_{12} = \Delta_1\Delta_2(\Delta_1 + \Delta_2)v^2 + (\Delta_1\omega_2^2 + \Delta_2\omega_1^2 - 2\mu v)\mu^2]/\Lambda, \tag{84}$$

$$\tilde{\bar{\alpha}}_1 = 2[(\Delta_1\omega_2^2 + \Delta_2\omega_1^2 - 2\mu v)(v - \mu\Delta_2)\mu\Delta_1 - (\Delta_1 + \Delta_2)\mu v\omega_2^2\Delta_1]/\Lambda, \tag{85}$$

$$\tilde{\bar{\alpha}}_2 = 2[(\Delta_1\omega_2^2 + \Delta_2\omega_1^2 - 2\mu v)(v - \mu\Delta_1)\mu\Delta_2 - (\Delta_1 + \Delta_2)\mu v\omega_1^2\Delta_2]/\Lambda. \tag{86}$$

These results show that the energy flow coefficients (71)–(73) and (75)–(77) obtained by the actual feedback system are different from equations (81)–(86) obtained by the increased dissipation feedback system. Furthermore, while in the actual feedback system the positivity of $\delta_{12}(\omega)$, $\alpha_1(\omega)$ and $\alpha_2(\omega)$ is guaranteed from equations (71), (72) and (73), in the increased dissipation feedback system $\tilde{\alpha}_1(\omega)$ and $\tilde{\alpha}_2(\omega)$ given in equations (82) and (83) are not necessarily non-negative. Thus the energy flow model based on the actual feedback system reflects the fact that energy is dissipated through the dissipative element (damper) of the coupling $L(s)$. On the other hand, the energy flow model based on the increased dissipation feedback system may predict energy flow increased through the coupling for some combinations of parameters, which is shown more clearly in the next example.

7.2. EXAMPLE 2

Next we analyze the three coupled oscillator system shown in Figure 4, where $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, $K_{12} = 0{\cdot}05$, $K_{13} = 0{\cdot}07$, $K_{23} = 0{\cdot}1$ and other parameters are varied for analysis. Furthermore, let the disturbances $w_i(t)$, $i = 1, 2, 3$, be white noise with unit intensity; that is, $D = I$. In Figure 6 it is shown that at each subsystem, $P_i^c + P_i^d + P_i^e = 0$ for $i = 1, 2, 3$, as stated in Corollary 3.2, and that there exist inter-subsystem energy flow and dissipative energy flow at the coupling. In Figures 7 and 8 are shown the energy flows for the increased dissipation feedback system and the feedback system in Figure 2 of reference [1], respectively. Although the energy flow is calculated for the same system, the results shown in Figures 7 and 8 are completely different, because the signals used to calculate energy flow are different. In particular, the energy flow model for the increased dissipation feedback system, Figure 7, shows that

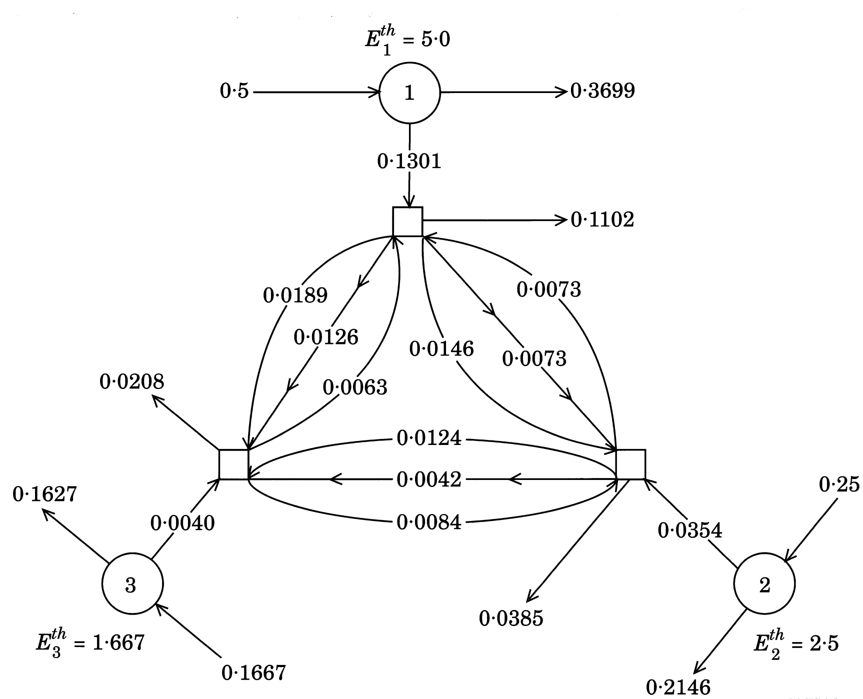

Figure 6. Energy flow among three coupled oscillators with dissipative coupling. $c_1 = 0{\cdot}1$, $c_2 = 0{\cdot}2$, $c_3 = 0{\cdot}3$, $C_{12} = 0{\cdot}01$, $C_{13} = 0{\cdot}02$, $C_{23} = 0{\cdot}03$.

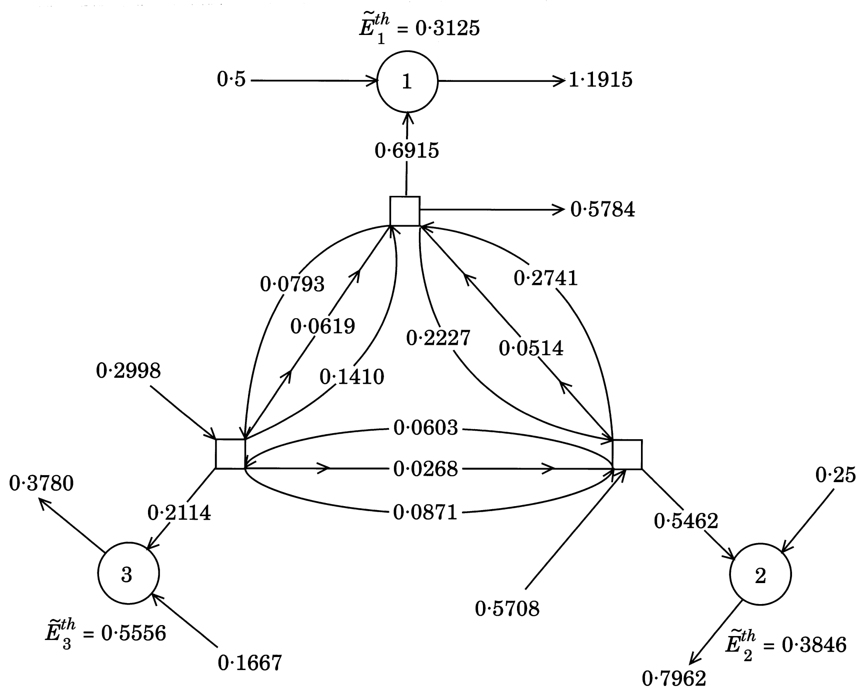

Figure 7. An energy flow model based on the increased dissipation feedback system. $c_1 = 0{\cdot}1$, $c_2 = 0{\cdot}2$, $c_3 = 0{\cdot}3$, $C_{12} = 1{\cdot}0$, $C_{13} = 0{\cdot}5$, $C_{23} = 0{\cdot}1$.

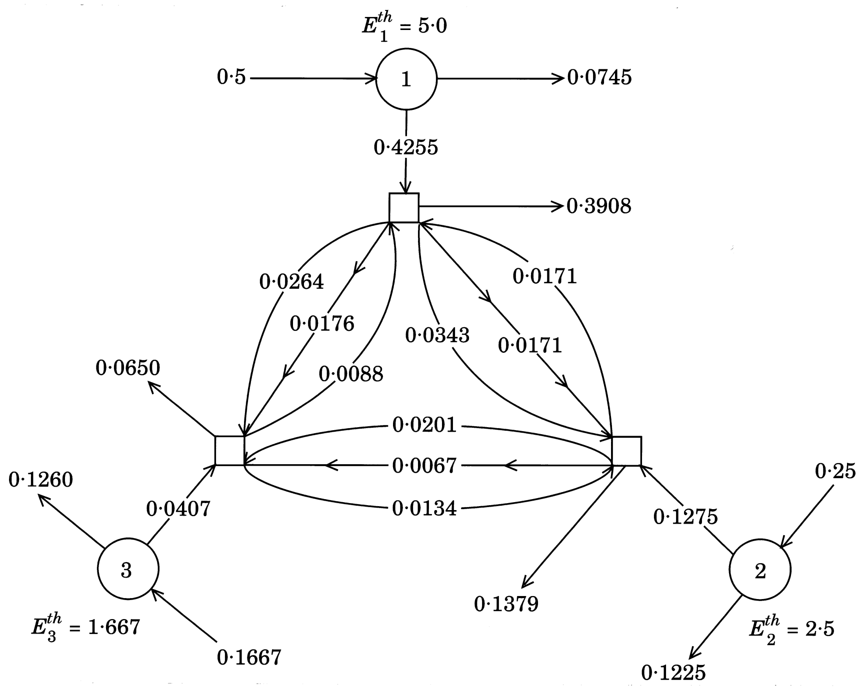

Figure 8. An energy flow model based on the actual feedback representation. $c_1 = 0{\cdot}1$, $c_2 = 0{\cdot}2$, $c_3 = 0{\cdot}3$, $C_{12} = 1{\cdot}0$, $C_{13} = 0{\cdot}5$, $C_{23} = 0{\cdot}1$.

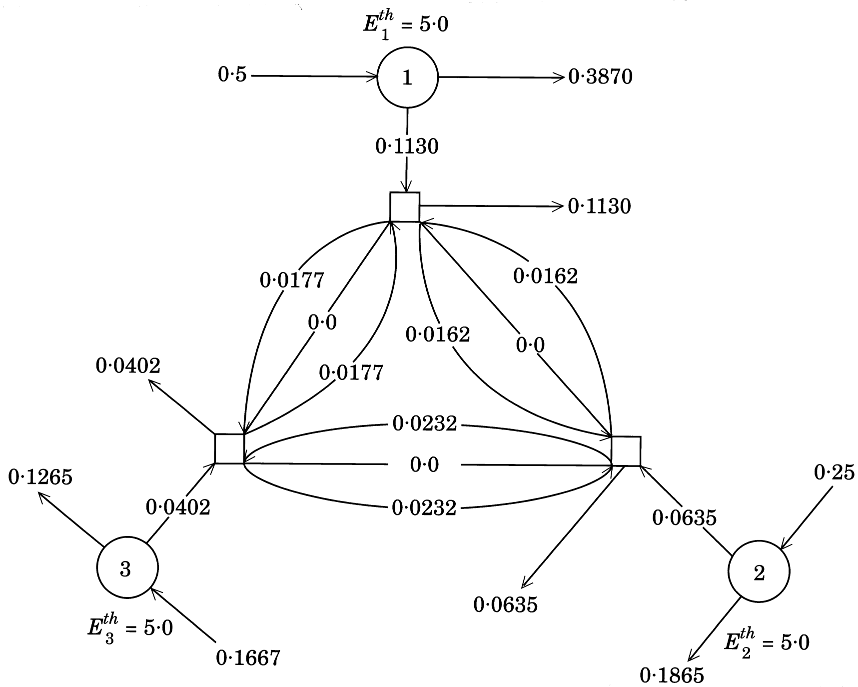

Figure 9. Inter-subsystem energy flow with equipartition of thermodynamic energy. $c_1 = 0{\cdot}1$, $c_2 = 0{\cdot}1$, $c_3 = 0{\cdot}1$, $C_{12} = 0{\cdot}01$, $C_{13} = 0{\cdot}02$, $C_{23} = 0{\cdot}03$.

the dissipation at each subsystem is increased while, at the coupling, energy is not dissipated but *increased*, even though the coupling is dissipative. Furthermore, the predicted inter-subsystem energy flows between all pairs of subsystems are completely reversed in Figures 7 and 8. These results show that the clear assignment of the calculated signals for the energy flow calculation is important and that for this purpose the feedback framework is effective. Finally, in Figure 9 it is shown that there is no net energy flows among subsystems when each subsystem has the same thermodynamic energy.

8. CONCLUSIONS

In this paper we have derived an energy flow model for dissipatively coupled systems. It has been shown that there exist two kinds of energy flow; namely, the inter-subsystem energy flow, which depends on the thermodynamic energy of each subsystem, and the dissipative energy flow, which depends only on the thermodynamic energy of the subsystem into which the energy is flowing. Furthermore, if the coupling matrix $L(s)$ is symmetric, then it has been shown that energy flows between subsystems in proportion to energy differences and dissipatively in proportion to subsystem energy. As in reference [2], this equation holds independently of the number of subsystems and the strength of the coupling. Additionally, it was shown that a compartmental model expression is valid for the dissipatively coupled system. This energy flow model was compared to the energy flow models obtained by Sun *et al*. [3] and Fahy *et al*. [4]. It was shown that the feedback representation of the coupled system provides a rigorous framework for predicting energy flow with dissipative coupling.

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APPENDIX A: PROOF OF PROPOSITION 4.3

Equations (45) and (46) are proved in Proposition 4.4 in reference [2]. By using Parseval's theorem with equations (39) and using the fact that $\bar{T}$ is a non-negative definite constant matrix, we obtain

$$\bar{\alpha}_i = \frac{1}{2\pi} c_i \int_{-\infty}^{\infty} [(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-*}\, \bar{T}(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}]_{(i,i)}\, \mathrm{d}\omega$$

$$= \frac{1}{2\pi} c_i \int_{-\infty}^{\infty} [(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-*}\, \bar{T}^{1/2}(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}]_{(i,i)}\, \mathrm{d}\omega$$

$$= c_i \frac{1}{2\pi} \int_{-\infty}^{\infty} ([\bar{T}^{1/2}C_1(\mathrm{j}\omega I-\tilde{A})^{-1}B]^*[\bar{T}^{1/2}C_1(\mathrm{j}\omega I - \tilde{A})^{-1}B])_{(i,i)}\, \mathrm{d}\omega$$

$$= c_i \frac{1}{2\pi} \int_{-\infty}^{\infty} ([B^{\mathrm{T}}(-\mathrm{j}\omega I - \tilde{A}^{\mathrm{T}})^{-1}(\bar{T}^{1/2}C_1)^{\mathrm{T}}][B^{\mathrm{T}}(-\mathrm{j}\omega I - \tilde{A}^{\mathrm{T}})^{-1}(\bar{T}^{1/2}C_1)^{\mathrm{T}}]^*)_{(i,i)}\, \mathrm{d}\omega$$

$$= c_i \frac{1}{2\pi} \int_{\infty}^{-\infty} ([B^{\mathrm{T}}(\mathrm{j}\omega' I-\tilde{A}^{\mathrm{T}})^{-1}(\tilde{T}^{1/2}C_1)^{\mathrm{T}}][B^{\mathrm{T}}(\mathrm{j}\omega' I - \tilde{A}^{\mathrm{T}})^{-1}(\bar{T}^{1/2}C_1)^{\mathrm{T}}]^*)_{(i,i)}(-\mathrm{d}\omega')$$

$$= c_i \frac{1}{2\pi} \int_{-\infty}^{\infty} ([B^{\mathrm{T}}(\mathrm{j}\omega' I-\tilde{A}^{\mathrm{T}})^{-1}(\bar{T}^{1/2}C_1)^{\mathrm{T}}][B^{\mathrm{T}}(\mathrm{j}\omega' I - \tilde{A}^{\mathrm{T}})^{-1}(\bar{T}^{1/2}C_1)^{\mathrm{T}}]^*)_{(i,i)}(\mathrm{d}\omega')$$

$$= c_i(B^{\mathrm{T}}\hat{Q}B)_{(i,i)},$$

where $\omega' \triangleq -\omega$ and $\hat{Q}$ satisfies equation (49).

Additionally, if $L(\mathrm{j}\omega)$ is symmetric, then $L(\mathrm{j}\omega) + Z(\mathrm{j}\omega)$ is symmetric. Thus

$$\begin{aligned}
\bar{\alpha}_i &= \frac{1}{2\pi} c_i \int_{-\infty}^{\infty} [(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-*}\, \overline{T}(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}]_{(i,i)}\, \mathrm{d}\omega \\
&= \frac{1}{2\pi} c_i \int_{-\infty}^{\infty} \mathrm{tr}\, [e_i^{\mathrm{T}}(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-*}\, \overline{T}(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1} e_i]\, \mathrm{d}\omega \\
&= \frac{1}{2\pi} c_i \int_{-\infty}^{\infty} \mathrm{tr}\, [e_i e_i^{\mathrm{T}}(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-*}\, \overline{T}(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-1}]\, \mathrm{d}\omega \\
&= \frac{1}{2\pi} c_i \int_{-\infty}^{\infty} [(L(\mathrm{j}\omega)+Z(\mathrm{j}\omega))^{-1}\overline{T}(L(\mathrm{j}\omega) + Z(\mathrm{j}\omega))^{-*}\,]_{(i,i)}\, \mathrm{d}\omega \\
&= \frac{1}{2\pi} c_i \int_{-\infty}^{\infty} ([C_1(\mathrm{j}\omega I - \tilde{A})^{-1}B\overline{T}^{1/2}][C_1(\mathrm{j}\omega I - \tilde{A})^{-1}B\overline{T}^{1/2}]^*)_{(i,i)}\, \mathrm{d}\omega \\
&= c_i(C_1\overline{Q}C_1^{\mathrm{T}})_{(i,i)},
\end{aligned}$$

where $\overline{Q}$ satisfies equation (51). □