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Deadbeat Input Reconstruction and State Estimation for Discrete-Time Linear Time-Varying Systems

Ahmad Ansari and Dennis S. Bernstein

***Abstract*— The objective of input reconstruction is to use knowledge of the output of a system to estimate the input. An estimate of the input can be used in an observer to provide more accurate state estimates. This paper presents a technique for combined state and input estimation for discrete-time, linear time-varying systems. The algorithm is based on the analysis of the rank of the time-dependent matrix that relates vectors of states and input values to a vector of outputs. The key contribution is a characterization of the time-varying delay under which the state and input can be estimated. The input and state estimate are given in terms of the generalized inverse of a partitioned matrix. The approach is demonstrated on discrete-time examples with linear periodically time-varying dynamics.**

I. Introduction

State estimators are used in practice to estimate states that are not measured [1], [2]. However, in many applications, it may be equally or more important to estimate unknown inputs. These inputs may represent actuator inputs or disturbances. Knowledge of these inputs, which are typically modeled as white noise signals for the purpose of state estimation, can improve the accuracy of the state estimates. Input estimation has been considered over the past several decades for both continuous-time and discrete-time systems in both stochastic and deterministic settings [3]–[26].

The present paper considers input and state estimation within a discrete-time, deterministic setting, where the goal is to exactly estimate the input and state after a finite delay. We refer to this objective as *deadbeat input reconstruction and state estimation*. Variations of this problem are considered in [27], [28] for linear time-invariant plants. The present paper extends the results of [28] to the linear time-varying plants.

The approach of [28] is based on the analysis of the block-Toeplitz matrix that characterizes the relationship between vectors whose components are past values of the state, input, and output. The ability to reconstruct the input and state after a finite delay depends on the difference in the ranks of these coefficients at successive steps. Conditions of this type, which can be viewed as a measure of the relative degree of a multi-input, multi-output system, were first considered in the seminal work on system invertibility [3], [4]. As shown in [28], there exist linear time-invariant plants for which this delay is infinite; in this case, state and input reconstruction are not possible.

For the case of linear time-varying plants, the present paper shows that the achievable delay depends on the time step. This means that the delay in reconstructing the input and state may be different at different steps. For example, we present an example of a periodically time-varying plant with period two, for which input reconstruction at odd steps entails a delay of 3 steps, whereas input reconstruction at even steps entails a delay of 2 steps.

Ahmad Ansari, and Dennis S. Bernstein are with the Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI, USA. {ansahmad, dsbaero}@umich.edu

II. Problem Statement

Consider the discrete-time, linear time-varying dynamics

$$x(k+1) = A_k x(k) + B_k u(k), \tag{1}$$

$$y(k) = C_k x(k) + D_k u(k), \tag{2}$$

where, for all $k \geq 0$, $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$, $C_k \in \mathbb{R}^{p\times n}$, $D_k \in \mathbb{R}^{p\times m}$, $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and $y(k) \in \mathbb{R}^p$. The goal is to use knowledge of $y(k)$ to estimate the unknown state $x(k)$ and the unknown input $u(k)$.

Let r denote a nonnegative integer. For all $k \geq 0$, define

$$\mathcal{Y}_{k,r} \triangleq \begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+r) \end{bmatrix} \in \mathbb{R}^{(r+1)p}, \quad \mathcal{U}_{k,r} \triangleq \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+r) \end{bmatrix} \in \mathbb{R}^{(r+1)m}, \tag{3}$$

$$\Gamma_{k,r} \triangleq \begin{bmatrix} C_k \\ C_{k+1}A_k \\ C_{k+2}A_{k+1}A_k \\ C_{k+3}A_{k+2}A_{k+1}A_k \\ \vdots \\ C_{k+r}A_{k+r-1}A_{k+r-2}\cdots A_k \end{bmatrix} \in \mathbb{R}^{(r+1)p\times n}, \tag{4}$$

$$M_{k,r} \triangleq \begin{bmatrix} D_k & 0 & 0 & 0 & \cdots & 0 \\ C_{k+1}B_k & D_{k+1} & 0 & \cdots & 0 & \\ C_{k+2}A_{k+1}B_k & C_{k+2}B_{k+1} & D_{k+2} & 0 & \cdots & 0 \\ C_{k+3}A_{k+2}A_{k+1}B_k & C_{k+3}A_{k+2}B_{k+1} & C_{k+3}B_{k+2} & D_{k+3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ C_{k+r}A_{k+r-1}\cdots A_{k+1}B_k & C_{k+r}A_{k+r-1}\cdots A_{k+2}B_{k+1} & C_{k+r}A_{k+r-1}\cdots A_{k+3}B_{k+2} & \cdots & C_{k+r}B_{k+r-1} & D_{k+r} \end{bmatrix} \in \mathbb{R}^{(r+1)p\times(r+1)m}. \tag{5}$$

It follows from (1), (2) that

$$\mathcal{Y}_{k,r} = \Gamma_{k,r}\, x(k) + M_{k,r}\, \mathcal{U}_{k,r} = \Psi_{k,r} \begin{bmatrix} x(k) \\ \mathcal{U}_{k,r} \end{bmatrix}, \tag{6}$$

where

$$\Psi_{k,r} \triangleq \begin{bmatrix} \Gamma_{k,r} & M_{k,r} \end{bmatrix} \in \mathbb{R}^{(r+1)p \times [n+(r+1)m]}. \tag{7}$$

Next, define

$$P_{k,r} \triangleq \begin{bmatrix} C_{k+1} B_k \\ \vdots \\ C_{k+r} \prod_{i=k+1}^{k+r-1} A_i B_k \end{bmatrix}, \tag{8}$$

$$Q_{k,r} \triangleq \begin{bmatrix} C_{k+r} \prod_{i=k+1}^{k+r-1} A_i B_k & \cdots & C_{k+r} B_{k+r-1} \end{bmatrix}, \tag{9}$$

where $\prod_{i=1}^{2} M_i \triangleq M_2 M_1$. Note that

$$M_{k,r} = \begin{bmatrix} D_k & 0 \\ P_{k,r} & M_{k+1,r-1} \end{bmatrix} = \begin{bmatrix} M_{k,r-1} & 0 \\ Q_{k,r} & D_{k+r} \end{bmatrix}. \tag{10}$$

For all $k \geq 0$, it follows from (10) that

$$\operatorname{rank} M_{k+1,r-1} \leq \operatorname{rank} M_{k,r} \tag{11}$$

$$\leq m + \operatorname{rank} M_{k+1,r-1}. \tag{12}$$

III. State Estimation

For all $k \geq 0$, define

$$\mu_k \triangleq \min\{l \geq 0 : \operatorname{rank} \Psi_{k,l} = n + \operatorname{rank} M_{k,l}\}. \tag{13}$$

If μ_k does not exist, then we set $\mu_k = \infty$. If μ_k is finite, then the following theorem provides state estimates with delay μ_k and without knowledge of u.

Theorem 1. Let $k \geq 0$, assume that μ_k is finite. Then, for all $r \geq \mu_k$,

$$x(k) = \begin{bmatrix} I_n & 0_{n\times(r+1)m} \end{bmatrix} \Psi_{k,r}^{+} \mathcal{Y}_{k,r}. \tag{14}$$

Proof. Let $k \geq 0$. Since μ_k is finite, it follows from (13) that

$$\operatorname{rank} \Psi_{k,\mu_k} = n + \operatorname{rank} M_{k,\mu_k}. \tag{15}$$

Next, noting $\Psi_{k,\mu_k} = [\Gamma_{k,\mu_k}\ \ M_{k,\mu_k}]$ and using Fact 2.11.9 in [29, p. 131] yields

$$\operatorname{rank} \Psi_{k,\mu_k} = \operatorname{rank} \Gamma_{k,\mu_k} + \operatorname{rank} M_{k,\mu_k} - \dim(\mathcal{R}(\Gamma_{k,\mu_k}) \cap \mathcal{R}(M_{k,\mu_k})). \tag{16}$$

Combining (15) with (16) yields

$$0 \leq \dim\left(\mathcal{R}(\Gamma_{k,\mu_k}) \cap \mathcal{R}(M_{k,\mu_k})\right) = \operatorname{rank}\, \Gamma_{k,\mu_k} - n \leq 0,$$

which implies that Γ_{k,μ_k} has full column rank and

$$\mathcal{R}(\Gamma_{k,\mu_k}) \cap \mathcal{R}(M_{k,\mu_k}) = \{0\}. \tag{17}$$

Since Γ_{k,μ_k} has full column rank, it follows from (4) that, for all $r \geq \mu_k$, $\Gamma_{k,r}$ has full column rank.

Next, note that

$$\mathcal{R}(\Gamma_{k,\mu_k+1}) \cap \mathcal{R}(M_{k,\mu_k+1}) = \mathcal{R}\left(\begin{bmatrix} \Gamma_{k,\mu_k} \\ R_{k,\mu_k+1} \end{bmatrix}\right) \cap \mathcal{R}\left(\left[\begin{array}{c|c} M_{k,\mu_k} & 0 \\ \hline Q_{k,\mu_k+1} & D_{k+\mu_k+1} \end{array}\right]\right), \tag{18}$$

where $R_{k,r} \triangleq C_{k+r} \prod_{i=k}^{k+r-1} A_i$. Since Γ_{k,μ_k} has full column rank and $\mathcal{R}(\Gamma_{k,\mu_k}) \cap \mathcal{R}(M_{k,\mu_k}) = \{0\}$, it follows from (18) and Lemma A that

$$\mathcal{R}(\Gamma_{k,\mu_k+1}) \cap \mathcal{R}(M_{k,\mu_k+1}) = \{0\}.$$

By similar arguments, it follows that, for all $r \geq \mu_k$,

$$\mathcal{R}(\Gamma_{k,r}) \cap \mathcal{R}(M_{k,r}) = \{0\}. \tag{19}$$

Using (19), it follows from Lemma B that, for all $r \geq \mu_k$,

$$\Psi_{k,r}^{+} \Psi_{k,r} = \begin{bmatrix} \Gamma_{k,r}^{+} \Gamma_{k,r} & 0 \\ 0 & M_{k,r}^{+} M_{k,r} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & M_{k,r}^{+} M_{k,r} \end{bmatrix}. \tag{20}$$

Next, multiplying (20) by $[x^{\mathrm{T}}(k)\ \mathcal{U}_{k,r}]^{\mathrm{T}}$ implies that, for all $r \geq \mu_k$,

$$\begin{bmatrix} I_n & 0 \\ 0 & M_{k,r}^{+} M_{k,r} \end{bmatrix} \begin{bmatrix} x(k) \\ \mathcal{U}_{k,r} \end{bmatrix} = \Psi_{k,r}^{+} \mathcal{Y}_{k,r}. \tag{21}$$

Finally, multiplying (21) by $[I_n\ 0_{n\times(r+1)m}]$ implies that, for all $r \geq \mu_k$, (14) holds. □

IV. Input Reconstruction

For all $k \geq 0$, define

$$\eta_k \triangleq \min\{l \geq 0 : \operatorname{rank} M_{k,l} = m + \operatorname{rank} M_{k+1,l-1}\}. \tag{22}$$

If η_k does not exist, then we set $\eta_k = \infty$.

The following result shows that, if for a given $k \geq 0$, η_k is finite and $x(k)$ is known, then the unknown input $u(k)$ can be reconstructed with delay η_k.

Theorem 2. Let $k \geq 0$, and assume that η_k is finite and $x(k)$ is known. Then, for all $r \geq \eta_k$,

$$u(k) = \begin{bmatrix} I_m & 0_{m\times rm} \end{bmatrix} M_{k,r}^{+} (\mathcal{Y}_{k,r} - \Gamma_{k,r} x(k)). \tag{23}$$

Proof. Let $k \geq 0$. Since η_k is finite, it follows from (22) that

$$\operatorname{rank} M_{k,\eta_k} = m + \operatorname{rank} M_{k+1,\eta_k-1}. \tag{24}$$

Next, noting

$$M_{k,\eta_k} = \left[\begin{array}{c|c} D_k & 0 \\ \hline P_{k,\eta_k} & M_{k+1,\eta_k-1} \end{array}\right], \tag{25}$$

and using Fact 2.11.9 in [29, p. 131] yields

$$\operatorname{rank} M_{k,\eta_k} = \operatorname{rank}\left[\begin{array}{c} D_k \\ P_{k,\eta_k}\end{array}\right] + \operatorname{rank} M_{k+1,\eta_k-1} - \dim\left(\mathcal{R}\left(\left[\begin{array}{c} D_k \\ P_{k,\eta_k}\end{array}\right]\right)\cap\mathcal{R}\left(\left[\begin{array}{c} 0 \\ M_{k+1,\eta_k-1}\end{array}\right]\right)\right). \quad (26)$$

Combining (24) with (26) yields

$$0 \le \dim\left(\mathcal{R}\left(\left[\begin{array}{c} D_k \\ P_{k,\eta_k}\end{array}\right]\right)\cap\mathcal{R}\left(\left[\begin{array}{c} 0 \\ M_{k+1,\eta_k-1}\end{array}\right]\right)\right) = \operatorname{rank}\left[\begin{array}{c} D_k \\ P_{k,\eta_k}\end{array}\right] - m \le 0,$$

which implies that $\left[\begin{array}{c} D_k \\ P_{k,\eta_k}\end{array}\right]$ has full column rank and

$$\mathcal{R}\left(\left[\begin{array}{c} D_k \\ P_{k,\eta_k}\end{array}\right]\right)\cap\mathcal{R}\left(\left[\begin{array}{c} 0 \\ M_{k+1,\eta_k-1}\end{array}\right]\right) = \{0\}. \quad (27)$$

Since $\left[\begin{array}{c} D_k \\ P_{k,\eta_k}\end{array}\right]$ has full column rank, it follows from (8) that, for all $r \ge \eta_k$, $\left[\begin{array}{c} D_k \\ P_{k,r}\end{array}\right]$ has full column rank.

Next, note that

$$\mathcal{R}\left(\left[\begin{array}{c} D_k \\ P_{k,\eta_k+1}\end{array}\right]\right)\cap\mathcal{R}\left(\left[\begin{array}{c} 0 \\ M_{k+1,\eta_k}\end{array}\right]\right) = \mathcal{R}\left(\left[\begin{array}{c} D_k \\ P_{k,\eta_k} \\ \hline C_{k+\eta_k+1}\prod_{i=k+1}^{k+\eta_k} A_i B_k\end{array}\right]\right) \cap \mathcal{R}\left(\left[\begin{array}{c|c} 0 & 0 \\ M_{k+1,\eta_k-1} & 0 \\ \hline S_{k,\eta_k+1} & D_{k+\eta_k+1}\end{array}\right]\right), \quad (28)$$

where $S_{k,r} \triangleq \left[\begin{array}{ccc} C_{k+r}\prod_{i=k+2}^{k+r-1} A_i B_{k+1} & \cdots & C_{k+r}B_{k+r-1}\end{array}\right]$. Since $\left[\begin{array}{c} D_k \\ P_{k,\eta_k}\end{array}\right]$ has full column rank and $\mathcal{R}\left(\left[\begin{array}{c} D_k \\ P_{k,\eta_k}\end{array}\right]\right)\cap\mathcal{R}\left(\left[\begin{array}{c} 0 \\ M_{k+1,\eta_k-1}\end{array}\right]\right) = \{0\}$, it follows from (28) and Lemma A that

$$\mathcal{R}\left(\left[\begin{array}{c} D_k \\ P_{k,\eta_k+1}\end{array}\right]\right)\cap\mathcal{R}\left(\left[\begin{array}{c} 0 \\ M_{k+1,\eta_k}\end{array}\right]\right) = \{0\}.$$

By similar arguments, it follows that, for all $r \ge \eta_k$,

$$\mathcal{R}\left(\left[\begin{array}{c} D_k \\ P_{k,r}\end{array}\right]\right)\cap\mathcal{R}\left(\left[\begin{array}{c} 0 \\ M_{k+1,r-1}\end{array}\right]\right) = \{0\}. \quad (29)$$

Using (29), it follows from Lemma B that, for all $r \ge \eta_k$,

$$M_{k,r}^+ M_{k,r} = \left[\begin{array}{cc} \left[\begin{array}{c} D_k \\ P_{k,r}\end{array}\right]^+ \left[\begin{array}{c} D_k \\ P_{k,r}\end{array}\right] & 0 \\ 0 & \left[\begin{array}{c} 0 \\ M_{k+1,r-1}\end{array}\right]^+ \left[\begin{array}{c} 0 \\ M_{k+1,r-1}\end{array}\right]\end{array}\right] = \begin{bmatrix} I_m & 0 \\ 0 & M_{k+1,r-1}^+ M_{k+1,r-1}\end{bmatrix}. \quad (30)$$

Next, multiplying (6) by $M_{k,r}^+$ and rearranging terms yields, for all $r \ge \eta_k \ge 0$,

$$M_{k,r}^+ M_{k,r}\mathcal{U}_{k,r} = M_{k,r}^+\left(\mathcal{Y}_{k,r} - \Gamma_{k,r}\, x(k)\right). \quad (31)$$

Substituting (30) into (31) yields, for all $r \ge \eta_k$,

$$\begin{bmatrix} I_m & 0 \\ 0 & M_{k+1,r-1}^+ M_{k+1,r-1}\end{bmatrix}\mathcal{U}_{k,r} = M_{k,r}^+\left(\mathcal{Y}_{k,r} - \Gamma_{k,r}\, x(k)\right). \quad (32)$$

Finally, multiplying (32) by $[I_m\ 0_{m\times rm}]$ implies that, for all $r \ge \eta_k$, (23) holds. ☐

V. Simultaneous Input Reconstruction and State Estimation

The following theorem combines Theorem 1 and Theorem 2 to provide simultaneous input reconstruction and state estimation.

Theorem 3. Let $k \ge 0$, and assume that η_k and μ_k are finite. Then, for all $r \ge \max\{\eta_k, \mu_k\}$,

$$\begin{bmatrix} x(k) \\ u(k)\end{bmatrix} = \left[\begin{array}{cc} I_{n+m} & 0_{(n+m)\times rm}\end{array}\right]\Psi_{k,r}^+\mathcal{Y}_{k,r}. \quad (33)$$

VI. Numerical Examples

Example 1. Consider the linear periodically time-varying system, where, for all $k \ge 0$,

$$A_k = \begin{bmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 1 \\ 0.5 & 0 & 0\end{bmatrix}, \quad (34)$$

$$B_k = \left[\begin{array}{ccc} 0 & 0 & 1\end{array}\right]^{\mathrm{T}},\ D_k = \left[\begin{array}{cc} 0 & 0\end{array}\right]^{\mathrm{T}}, \quad (35)$$

and

$$C_k = \begin{cases} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0\end{array}\right], & k = 0, 2, 4, \ldots, \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right], & k = 1, 3, 5, \ldots. \end{cases} \quad (36)$$

To apply Theorem 3 at each time step k using (33), we choose $r = \gamma_k \triangleq \max\{\eta_k, \mu_k\}$. It thus follows from (33) that

$$\begin{bmatrix} x(k) \\ u(k)\end{bmatrix} = \left[\begin{array}{cc} I_{n+m} & 0_{(n+m)\times \gamma_k m}\end{array}\right]\Psi_{k,\gamma_k}^+\mathcal{Y}_{k,\gamma_k}. \quad (37)$$

Therefore, for all $0 \le k \le 18$, $(\eta_k, \mu_k, \gamma_k)$ are given by

$$(\eta_k, \mu_k, \gamma_k) = \begin{cases} (1, 2, 2), & k = 0, 2, 4, \ldots, 18, \\ (3, 1, 3), & k = 1, 3, 5, \ldots, 17, \end{cases} \quad (38)$$

which implies that the delay for simultaneous input reconstruction and state estimation is 2 steps and 3 steps at even and odd time steps, respectively.

Let the unknown initial condition be $x(0) = [4\ 6\ 10]^{\mathrm{T}}$, and let the unknown input be $u(k) = 1 + w(k) + \sin(kT_{\mathrm{s}})$, where w is zero-mean Gaussian white noise with variance 0.1. Furthermore, let the available measurement be $[y^{\mathrm{T}}(0)\ y^{\mathrm{T}}(1)\ \cdots\ y^{\mathrm{T}}(20)]^{\mathrm{T}}$. For all $0 \le k \le 20 - \gamma_{18} = 18$, Figures 1 and 2 show that the estimated state is equal to the actual state, and the reconstructed input is equal to the actual input, respectively. Therefore, (37) is satisfied, which confirms Theorem 3. $\Diamond$

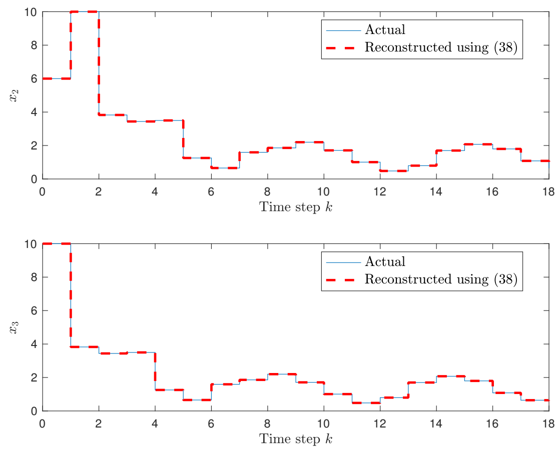

Fig. 1: Application of Theorem 3 to Example 1. For all $0 \le k \le 20-\gamma_{18} = 18$, the estimates of states x_2 and x_3 are equal to the actual states.

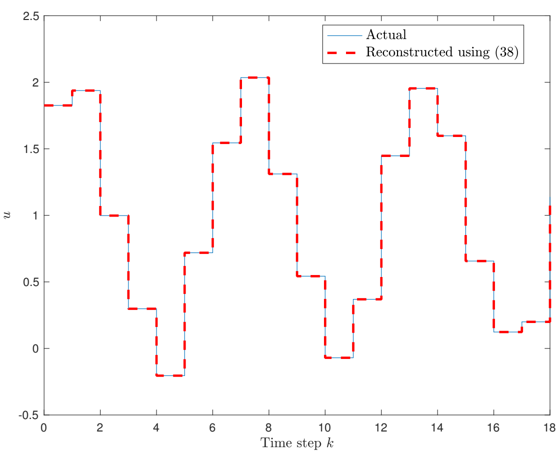

Fig. 2: Application of Theorem 3 to Example 1. For all $0 \le k \le 20-\gamma_{18} = 18$, the reconstructed input is equal to the actual input. Together with the state estimates in Figure 1, these numerical results satisfy (37), and thus confirm Theorem 3.

Example 2. Consider the Mathieu equation

$$\ddot{q} + (\alpha + \beta\cos(\omega t))q = bu, \tag{39}$$

where $\omega > 0$ is the stiffness frequency, $\alpha \in \mathbb{R}$, and $\beta \in \mathbb{R}$. A continuous-time state-space representation of (39) with state vector $x \triangleq [\ q \quad \dot{q}\ ]^{\mathrm{T}}$ is given by

$$\dot{x} = A_{\mathrm{c}}x + B_{\mathrm{c}}u, \tag{40}$$

where

$$A_{\mathrm{c}}(t) \triangleq \begin{bmatrix} 0 & 1 \\ -(\alpha+\beta\cos(\omega t)) & 0 \end{bmatrix}, \ B_{\mathrm{c}}(t) \triangleq \begin{bmatrix} 0 \\ b \end{bmatrix}. \tag{41}$$

Let $\alpha = 1$, $\beta = 1$, $\omega = 1$, and $b = 1$. For all $k \ge 0$, we discretize (40) as

$$A_k \triangleq I_n + T_{\mathrm{s}}A_{\mathrm{c}}(kT_{\mathrm{s}}), \quad B_k \triangleq T_{\mathrm{s}}B_{\mathrm{c}}(kT_{\mathrm{s}}), \tag{42}$$

$$C_k \triangleq [\ 1 \quad 0\ ], \quad D_k \triangleq 0, \tag{43}$$

where $T_{\mathrm{s}} = 0.01\,\mathrm{sec}$ is the sample time. This discretized model is taken as the truth model for demonstrating input and state estimation. Let the unknown initial condition be $x(0) = [1 \ -1]^{\mathrm{T}}$, and let the unknown input be $u(t) = 1 + \sin(2t)$. Furthermore, let the available measurement from the continuous-time system (40) be $[y(0)\ y(0.01)\ \cdots\ y(10)]^{\mathrm{T}}$.

Next, for all $0 \le k \le 998$, it can be shown that $\eta_k = 2$ and $\mu_k = 1$. To apply Theorem 3 at each time step k using (33), we choose $r = \max\{\eta_k, \mu_k\} = 2$. For all $0 \le t \le 9.98\,\mathrm{sec}$, Figures 3 and 4 show that the estimated state is close to the actual state, and the reconstructed input is close to the actual input, respectively. $\Diamond$

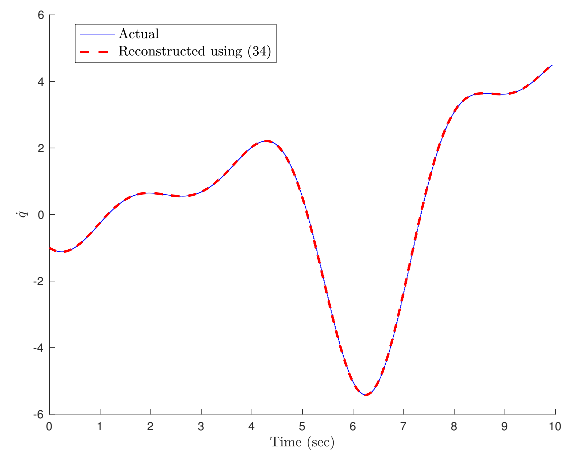

Fig. 3: Application of Theorem 3 to Example 2. For all $0 \le t \le 9.98\,\mathrm{sec}$, the estimates of the state $\dot{q}$ approximate the actual state $\dot{q}$.

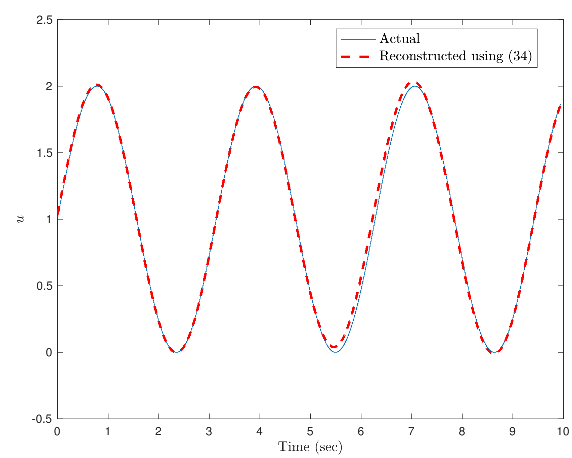

Fig. 4: Application of Theorem 3 to Example 2. For all $0 \le t \le 9.98\,\mathrm{sec}$, the reconstructed input approximates the actual input.

VII. Conclusions

This paper presented a technique for combined state and input estimation for discrete-time, linear time-varying systems based on the analysis of the rank of the time-dependent matrix that relates vectors of states and input values to a vector of outputs. The time-varying delays under which the state and input can be estimated were defined in terms of the rank of a time-dependent matrix, and the state and input estimates were given in terms of the generalized inverse of a partitioned matrix. The approach was demonstrated on discrete-time examples with linear periodically time-varying dynamics.

The development focused on characterizing the estimation delay within a noise-free setting. Future research will include the effects of process and sensor noise in order to obtain optimal input and state estimates.

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Appendix: Partitioned Matrices

The following two lemmas are used in the proofs of Theorem 1 and Theorem 2.

Lemma A. Let $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{l\times p}$, and $E \in \mathbb{R}^{l\times q}$. Assume that A has full column rank, and $\mathcal{R}(A) \cap \mathcal{R}(C) = \{0\}$. Then

$$\mathcal{R}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) \cap \left(\begin{bmatrix} C & 0 \\ D & E \end{bmatrix}\right) = \{0\}. \tag{44}$$

Proof. Let

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{R}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) \cap \left(\begin{bmatrix} C & 0 \\ D & E \end{bmatrix}\right).$$

Therefore, $x \in \mathcal{R}(A) \cap \mathcal{R}(C) = \{0\}$, and thus $x = 0$. Furthermore, there exists $z \in \mathbb{R}^m$ such that $\left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] z$, and thus $Az = 0$ and $y = Bz$. Since A has full column rank, it follows that $z = 0$, and thus $y = 0$. $\square$

Lemma B. Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{n\times l}$, define $C \triangleq (I - AA^+)B$ and $D \triangleq (I - BB^+)A$, and assume that $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$. Then, $C^+A = 0$, $D^+B = 0$, $C^+B = B^+B$, $D^+A = A^+A$,

$$[A\ B]^+ = \begin{bmatrix} D^+ \\ C^+ \end{bmatrix}, \quad [A\ B]^+[A\ B] = \begin{bmatrix} A^+A & 0 \\ 0 & B^+B \end{bmatrix}. \tag{45}$$

Proof. The result follows from Theorem 1, line 6 on page 21, and line 7 on page 22 of [30]. $\square$