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Satellite Drag Estimation Using Retrospective Cost Input Estimation

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Abstract— **Orbit estimation is of increasing interest due to need to avoid close encounters between operational satellites and space debris. This paper uses input estimation to estimate the drag on a satellite, where the drag is modeled as an unknown input forcing. Retrospective cost input estimation is applied to simulated satellite data under three scenarios, namely, 1) indirect estimation of the drag acceleration in the inertial frame (that is, estimation of the total inertial acceleration), 2) direct estimation of the drag acceleration in the inertial frame (that is, estimation of only the inertial drag acceleration), and 3) estimation of the magnitude of the drag acceleration.**

I. Introduction

Orbit estimation is of increasing interest due to the need to avoid collisions between operational satellites and space debris. The number of derelict objects that can threaten satellites numbers in the tens of thousands, and measurements that can be used to track these objects are sparse. There is thus a pressing need for estimation algorithms that can use position and velocity measurements to obtain orbit estimates of the highest possible accuracy.

To address this problem, research has focused on nonlinear estimation techniques. Various classical techniques are applied to this problem in [1]. In [2], the extended Kalman filter (EKF) and the unscented Kalman filter (UKF) were applied to orbit estimation using range data. Optimal transport methods were applied to this problem in [3]. An alternative approach was taken in [4], where optimal control techniques were used to detect the motion of possibly maneuvering objects.

The present paper focuses on the problem of drag estimation, where the goal is to estimate the drag acceleration of the body without assuming knowledge of the nominal orbit of the body. The estimation of satellite drag coefficients has been widely studied [5]–[8]. In the present paper, drag acceleration is estimated by using *input estimation*. As an extension of state estimation, which uses knowledge of the dynamics along with statistical information concerning the process and sensor noise to estimate states, input estimation uses the same information to estimate both states and inputs. Input estimation has been extensively developed, and various approaches are developed in [9]–[23].

The contribution of the present paper is the novel application of input estimation to the problem of estimating drag acceleration. The approach used in the present paper is based on retrospective cost optimization. This technique is a variation of retrospective cost input estimation used in [24], [25]. Related techniques have been applied to adaptive control [26].

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II. Kinematics of a Satellite Orbiting the Earth

The Earth-Centered Inertial (ECI) frame is denoted by $\mathrm{F_E}$. The origin $\mathrm{O_E}$ of $\mathrm{F_E}$ is fixed at the center of the Earth. The axes $\hat{\imath}_\mathrm{E}$ points toward the vernal equinox, $\hat{k}_\mathrm{E}$ points North, and the axis $\hat{\jmath}_\mathrm{E} = \hat{k}_\mathrm{E} \times \hat{\imath}_\mathrm{E}$. Note that, $\hat{\imath}_\mathrm{E}$ and $\hat{\jmath}_\mathrm{E}$ lie in the equatorial plane.

Let p denote a point that is fixed on a satellite orbiting the Earth. The location of p relative to $\mathrm{O_E}$ is denoted by $\vec{r}_{\mathrm{p/O_E}}$ and is resolved in $\mathrm{F_E}$ as

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \triangleq \vec{r}_{\mathrm{p/O_E}}\Big|_\mathrm{E}. \tag{1}$$

The velocity of p relative to $\mathrm{O_E}$ with respect to $\mathrm{F_E}$ is given by

$$\vec{v}_{\mathrm{p/O_E/E}} = \overset{\mathrm{E}\bullet}{\vec{r}}_{\mathrm{p/O_E}}, \tag{2}$$

where E$\bullet$ denotes the derivative with respect to the time taken in ECI frame. The acceleration of p relative to $\mathrm{O_E}$ with respect to $\mathrm{F_E}$ is given by

$$\vec{a}_{\mathrm{p/O_E/E}} = \overset{\mathrm{E}\bullet}{\vec{v}}_{\mathrm{p/O_E/E}} = \overset{\mathrm{E}\bullet\bullet}{\vec{r}}_{\mathrm{p/O_E}}. \tag{3}$$

Define

$$\hat{r} \triangleq \frac{\vec{r}_{\mathrm{p/O_E}}}{|\vec{r}_{\mathrm{p/O_E}}|},\ \hat{v} \triangleq \frac{\vec{v}_{\mathrm{p/O_E/E}}}{|\vec{v}_{\mathrm{p/O_E/E}}|},\ \hat{h} \triangleq \frac{\vec{r}_{\mathrm{p/O_E}} \times \vec{v}_{\mathrm{p/O_E}}}{|\vec{r}_{\mathrm{p/O_E}} \times \vec{v}_{\mathrm{p/O_E}}|}, \tag{4}$$

and $\mathrm{F_P} \triangleq \begin{bmatrix} \hat{\imath}_\mathrm{P} & \hat{\jmath}_\mathrm{P} & \hat{k}_\mathrm{P} \end{bmatrix} = \begin{bmatrix} \hat{v} \times \hat{h} & \hat{v} & \hat{h} \end{bmatrix}$. The frames $\mathrm{F_P}$ and $\mathrm{F_E}$ are related by

$$\mathrm{F_E} = \vec{R}_{\mathrm{E/P}}\, \mathrm{F_P}, \tag{5}$$

where $\vec{R}_{\mathrm{E/P}}$ is a physical rotation matrix. We resolve $\vec{R}_{\mathrm{E/P}}$ in $\mathrm{F_E}$ as

$$\mathcal{O}_{\mathrm{E/P}} \triangleq \vec{R}_{\mathrm{E/P}}\Big|_\mathrm{E} = \begin{bmatrix} \hat{\imath}_\mathrm{E}\cdot\hat{\imath}_\mathrm{P} & \hat{\imath}_\mathrm{E}\cdot\hat{\jmath}_\mathrm{P} & \hat{\imath}_\mathrm{E}\cdot\hat{k}_\mathrm{P} \\ \hat{\jmath}_\mathrm{E}\cdot\hat{\imath}_\mathrm{P} & \hat{\jmath}_\mathrm{E}\cdot\hat{\jmath}_\mathrm{P} & \hat{\jmath}_\mathrm{E}\cdot\hat{k}_\mathrm{P} \\ \hat{k}_\mathrm{E}\cdot\hat{\imath}_\mathrm{P} & \hat{k}_\mathrm{E}\cdot\hat{\jmath}_\mathrm{P} & \hat{k}_\mathrm{E}\cdot\hat{k}_\mathrm{P} \end{bmatrix}. \tag{6}$$

Note that $\mathcal{O}_{\mathrm{P/E}} = \mathcal{O}_{\mathrm{E/P}}^\mathrm{T}$.

We resolve $\vec{v}_{\mathrm{p/O_E/E}}$ and $\vec{a}_{\mathrm{p/O_E/E}}$ in $\mathrm{F_E}$ and $\mathrm{F_P}$ using the notation

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} \triangleq \vec{v}_{\mathrm{p/O_E/E}}\Big|_\mathrm{E}, \quad \begin{bmatrix} v_x \\ v_v \\ v_h \end{bmatrix} \triangleq \vec{v}_{\mathrm{p/O_E/E}}\Big|_\mathrm{P}. \tag{7}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \triangleq \vec{a}_{\mathrm{p/O_E/E}}\Big|_{\mathrm{E}}, \quad \begin{bmatrix} a_x \\ a_v \\ a_h \end{bmatrix} \triangleq \vec{a}_{\mathrm{p/O_E/E}}\Big|_{\mathrm{P}}. \tag{8}$$

Using (6) and (8), $\vec{a}_{\mathrm{p/O_E/E}}$ in $\mathrm{F_E}$ is given by

$$\vec{a}_{\mathrm{p/O_E/E}}\Big|_{\mathrm{E}} = \mathcal{O}_{\mathrm{E/P}}\, \vec{a}_{\mathrm{p/O_E/E}}\Big|_{\mathrm{P}}, \tag{9}$$

and thus,

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \mathcal{O}_{\mathrm{E/P}} \begin{bmatrix} a_x \\ a_v \\ a_h \end{bmatrix}. \tag{10}$$

Note that (5)–(10) are exact kinematic relations that are applicable to an arbitrary point p on the satellite.

III. Dynamics of a Satellite Orbiting the Earth

The dynamics of a satellite moving in the Earth's gravity field is given by

$$\vec{F}_{\mathrm{gravity}} + \vec{F}_{\mathrm{pert}} = m_{\mathrm{sat}}\, \vec{a}_{\mathrm{p/O_E/E}}, \tag{11}$$

where F_{gravity} is the gravitational force acting on the satellite, F_{pert} is the perturbing force acting on the satellite, and m_{sat} is the mass of the satellite.

Gravity Model

We assume that the Earth is homogeneous and spherical. It thus follows from Newton's law of gravitation that

$$\underbrace{\frac{\vec{F}_{\mathrm{gravity}}}{m_{\mathrm{sat}}}}_{\vec{a}_{\mathrm{gravity}}} = -\mu \frac{\vec{r}_{\mathrm{p/O_E}}}{|\vec{r}_{\mathrm{p/O_E}}|^3}, \tag{12}$$

where $\mu = 398600.4405\,\mathrm{km^3/sec^2}$. Substituting (3) and (12) into (11) yields

$$\overset{\mathrm{E}\bullet\bullet}{\vec{r}}_{\mathrm{p/O_E}} = -\mu \frac{\vec{r}_{\mathrm{p/O_E}}}{|\vec{r}_{\mathrm{p/O_E}}|^3} + \frac{F_{\mathrm{pert}}}{m_{\mathrm{sat}}}. \tag{13}$$

A. No Perturbing Force

If $\vec{F}_{\mathrm{pert}} = 0$, then using (1), (13) is given by

$$\ddot{X} = -\mu \frac{X}{(X^2+Y^2+Z^2)^{3/2}}, \tag{14}$$

$$\ddot{Y} = -\mu \frac{Y}{(X^2+Y^2+Z^2)^{3/2}}, \tag{15}$$

$$\ddot{Z} = -\mu \frac{Z}{(X^2+Y^2+Z^2)^{3/2}}. \tag{16}$$

Using (7), (14)–(16) can be written as the following first-order nonlinear ordinary differential equations

$$\dot{X} = V_x, \quad \dot{Y} = V_y, \quad \dot{Z} = V_z, \tag{17}$$

$$\dot{V}_x = -\mu \frac{X}{(X^2+Y^2+Z^2)^{3/2}}, \tag{18}$$

$$\dot{V}_y = -\mu \frac{Y}{(X^2+Y^2+Z^2)^{3/2}}, \tag{19}$$

$$\dot{V}_z = -\mu \frac{Z}{(X^2+Y^2+Z^2)^{3/2}}. \tag{20}$$

B. Drag as a Perturbing Force

Let the drag acting on the satellite be given by

$$\underbrace{\frac{\vec{F}_{\mathrm{pert}}}{m_{\mathrm{sat}}}}_{\vec{a}_{\mathrm{drag}}} = -\alpha \frac{\vec{v}_{\mathrm{p/O_E/E}}}{|\vec{v}_{\mathrm{p/O_E/E}}|}, \tag{21}$$

where $\alpha \in \mathbb{R}\,(\mathrm{kN/kg})$ is the magnitude of the acceleration due to drag. Using (1), (7), (13), and (21), the satellite dynamics are given by

$$\dot{X} = V_x, \quad \dot{Y} = V_y, \quad \dot{Z} = V_z, \tag{22}$$

$$\dot{V}_x = \underbrace{-\mu \frac{X}{(X^2+Y^2+Z^2)^{3/2}}}_{A_{x,\mathrm{gravity}}} \underbrace{-\alpha \frac{V_x}{(V_x^2+V_y^2+V_z^2)^{1/2}}}_{A_{x,\mathrm{drag}}}, \tag{23}$$

$$\dot{V}_y = \underbrace{-\mu \frac{Y}{(X^2+Y^2+Z^2)^{3/2}}}_{A_{y,\mathrm{gravity}}} \underbrace{-\alpha \frac{V_y}{(V_x^2+V_y^2+V_z^2)^{1/2}}}_{A_{y,\mathrm{drag}}}, \tag{24}$$

$$\dot{V}_z = \underbrace{-\mu \frac{Z}{(X^2+Y^2+Z^2)^{3/2}}}_{A_{z,\mathrm{gravity}}} \underbrace{-\alpha \frac{V_z}{(V_x^2+V_y^2+V_z^2)^{1/2}}}_{A_{z,\mathrm{drag}}}. \tag{25}$$

Note from (11), (12), and (21) that

$$\vec{a}_{\mathrm{drag}} = \vec{a}_{\mathrm{p/O_E/E}} - \vec{a}_{\mathrm{gravity}}. \tag{26}$$

Furthermore note that $\vec{a}_{\mathrm{drag}}|_{\mathrm{P}} = \begin{bmatrix} 0 & -\alpha & 0 \end{bmatrix}^{\mathrm{T}}$.

IV. Model for Input Estimation

A continuous-time state-space model for input estimation can be formulated as

$$\dot{x}(t) = A_{\mathrm{c}}(t)x(t) + B_{\mathrm{c}}(t)u(t) + G_{\mathrm{c}}(t)d(t) + \bar{D}_1(t)w(t), \tag{27}$$

$$y(t) = C(t)x(t) + D_2(t)v(t), \tag{28}$$

where $x \in \mathbb{R}^{l_x}$ is the unknown state, $u \in \mathbb{R}^{l_u}$ is the known input, $d \in \mathbb{R}^{l_d}$ is the unknown input, $\bar{D}_1 w \in \mathbb{R}^{l_x}$ is the process noise with known covariance $\bar{V}_1 \triangleq \bar{D}_1\bar{D}_1^{\mathrm{T}} \in \mathbb{R}^{l_x \times l_x}$, $y \in \mathbb{R}^{l_y}$ is the measured output, and $D_2 \in \mathbb{R}^{l_v}$ is the measurement noise with known covariance $V_2 \triangleq D_2 D_2^{\mathrm{T}} \in \mathbb{R}^{l_y \times l_y}$. It is shown below that estimating acceleration is equivalent to estimating the unknown input d. We consider three scenarios for estimating the drag acceleration $\vec{a}_{\mathrm{drag}}$ by estimating the unknown input d.

A. Indirect Estimation of Drag Acceleration in F_{E}

For indirect estimation of the drag acceleration, we first estimate $\vec{a}_{\mathrm{p/O_E/E}}$ resolved in $\mathrm{F_E}$. In doing so, we use (2), (7), (8), and write (3) in state space form

$$\dot{x} = A_{\mathrm{c}}x + G_{\mathrm{c}}d, \tag{29}$$

where

$$A_{\rm c} = \begin{bmatrix} 0_{3\times3} & I_3 \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}, \quad G_{\rm c} = \begin{bmatrix} 0_{3\times3} \\ I_3 \end{bmatrix}, \tag{30}$$

$$x = \begin{bmatrix} X & Y & Z & V_x & V_y & V_z \end{bmatrix}^{\rm T}, d = \begin{bmatrix} A_x & A_y & A_z \end{bmatrix}^{\rm T}. \tag{31}$$

Note that (29) is an exact kinematic equation, and thus it does not include process noise. Next, using (26) and the knowledge of gravity (12), the acceleration due to drag $\vec{a}_{\rm drag}$ resolved in $F_{\rm E}$ is given by

$$\vec{a}_{\rm drag}\Big|_{\rm E} = \vec{a}_{\rm p/O_E/E}\Big|_{\rm E} - \vec{a}_{\rm gravity}\Big|_{\rm E}, \tag{32}$$

$$\begin{bmatrix} A_{x,\rm drag} \\ A_{y,\rm drag} \\ A_{z,\rm drag} \end{bmatrix} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} + \frac{\mu}{(X^2+Y^2+Z^2)^{3/2}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}. \tag{33}$$

Note that (33) gives an indirect estimate of the drag acceleration. A direct estimate of drag acceleration is presented in the subsection below.

B. *Direct Estimation of the Drag Acceleration in* $F_{\rm E}$

For a direct estimation of drag acceleration $\vec{a}_{\rm drag}$ resolved in $\rm F_E$, we use (2), (7), (8), (32), and write (3) in state space form

$$\dot{x} = A_{\rm c}x + B_{\rm c}u + G_{\rm c}d + \bar{D}_1 w, \tag{34}$$

where

$$A_{\rm c} = \begin{bmatrix} 0_{3\times3} & I_3 \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}, \quad B_{\rm c} = G_{\rm c} = \begin{bmatrix} 0_{3\times3} \\ I_3 \end{bmatrix}, \tag{35}$$

$$x = \begin{bmatrix} X & Y & Z & V_x & V_y & V_z \end{bmatrix}^{\rm T}, \tag{36}$$

$$u = \begin{bmatrix} A_{x,\rm gravity} & A_{y,\rm gravity} & A_{z,\rm gravity} \end{bmatrix}^{\rm T}, \tag{37}$$

$$d = \begin{bmatrix} A_{x,\rm drag} & A_{y,\rm drag} & A_{z,\rm drag} \end{bmatrix}^{\rm T}. \tag{38}$$

Likewise (29), (34) is an exact kinematic equation, but process noise is included to account for errors due to uncertainty in u.

C. *Estimation of Drag Acceleration in* $F_{\rm P}$

For estimating the drag acceleration $\vec{a}_{\rm drag}$ resolved in $\rm F_P$, we use (2), (6), (7), (8), (32), (10), and write (3) in state space form

$$\dot{x} = A_{\rm c}x + B_{\rm c}u + G_{\rm c}d + \bar{D}_1 w, \tag{39}$$

where

$$A_{\rm c} = \begin{bmatrix} 0_{3\times3} & I_3 \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}, B_{\rm c} = \begin{bmatrix} 0_{3\times3} \\ I_3 \end{bmatrix}, G_{\rm c} = \begin{bmatrix} 0_{3\times1} \\ \hat{v}|_{\rm E} \end{bmatrix}, \tag{40}$$

$$x = \begin{bmatrix} X & Y & Z & V_x & V_y & V_z \end{bmatrix}^{\rm T}, \tag{41}$$

$$u = \begin{bmatrix} A_{x,\rm gravity} & A_{y,\rm gravity} & A_{z,\rm gravity} \end{bmatrix}^{\rm T}, \tag{42}$$

$$d = -\alpha. \tag{43}$$

Note that (39) is an exact kinematic equation, but process noise is now included to account for errors in the measurements of $\hat{v}|_{\rm E}$ appearing in (40). Finally, notice that, due to $\hat{v}|_{\rm E}$, (39) is a continuous-time linear, time-varying system, and therefore its discretization is linear, time-varying.

V. Input and State Estimation

Consider the linear discrete-time system

$$x(k) = A(k-1)x(k-1) + B(k-1)u(k-1) + G(k-1)d(k-1) + D_1(k-1)w(k-1), \tag{44}$$

$$y(k) = C(k)x(k) + D_2(k)v(k). \tag{45}$$

This model represents a sampled-data version of the continuous-time plant [(27), (28)] with sample time $T_{\rm s}$, in which case $x(k)$ denotes the state at time $t = kT_{\rm s}$. The goal is to estimate the unknown input $d(k)$ and the unknown state $x(k)$.

A. *Retrospective Cost Input Estimation (RCIE)*

In order to estimate the unknown input $d(k)$, we consider the Kalman filter forecast step

$$x_{\rm fc}(k) = A(k-1)x_{\rm da}(k-1) + B(k-1)u(k-1) + G(k-1)\hat{d}(k-1), \tag{46}$$

$$y_{\rm fc}(k) = C(k)x_{\rm fc}(k), \tag{47}$$

$$z(k) = y_{\rm fc}(k) - y(k), \tag{48}$$

where $\hat{d}(k) \in \mathbb{R}^{l_d}$ is the input estimate, $x_{\rm da}(k) \in \mathbb{R}^{l_x}$ is the data-assimilation state, $x_{\rm fc}(k) \in \mathbb{R}^{l_x}$ is the forecast state, and $z(k) \in \mathbb{R}^{l_y}$ is the innovations. The goal is to develop an input estimator that minimizes $z(k)$ by estimating $d(k)$.

We obtain the input estimate $\hat{d}(k)$ as the output of the *input-estimation subsystem* of order $n_{\rm c}$ given by

$$\hat{d}(k) = \sum_{i=1}^{n_{\rm c}} P_i(k)\hat{d}(k-i) + \sum_{i=0}^{n_{\rm c}} Q_i(k)z(k-i), \tag{49}$$

where $P_i(k) \in \mathbb{R}^{l_d\times l_d}$ and $Q_i(k) \in \mathbb{R}^{l_d\times l_y}$. Note that (49) represents an exactly proper transfer function with direct feedthrough from the innovations $z(k)$ to the estimate $\hat{d}(k)$ of $d(k)$. RCIE minimizes $z(k)$ by updating $P_i(k)$ and $Q_i(k)$. The subsystem (49) can be reformulated as

$$\hat{d}(k) = \Phi(k)\theta(k), \tag{50}$$

where the regressor matrix $\Phi(k)$ is defined by

$$\Phi(k) \triangleq \begin{bmatrix} \hat{d}(k-1) \\ \vdots \\ \hat{d}(k-n_{\rm c}) \\ z(k) \\ \vdots \\ z(k-n_{\rm c}) \end{bmatrix}^{\rm T} \otimes I_{l_d} \in \mathbb{R}^{l_d\times l_\theta}$$

and

$$\theta(k) \triangleq {\rm vec} \begin{bmatrix} P_1(k)\cdots P_{n_{\rm c}}(k) \ Q_0(k)\cdots Q_{n_{\rm c}}(k) \end{bmatrix} \in \mathbb{R}^{l_\theta},$$

where $l_\theta \triangleq l_d^2 n_{\rm c} + l_d l_y(n_{\rm c}+1)$, "$\otimes$" is the Kronecker product, and "vec" is the column-stacking operator.

Define the $l_y \times l_d$ filter $G_{{\rm f},k}(\mathbf{q})$ as

$$G_{{\rm f},k}(\mathbf{q}) \triangleq \sum_{i=1}^{n_{\rm f}} H_i(k)\frac{1}{\mathbf{q}^i}, \tag{51}$$

where $\mathbf{q}$ is the forward shift operator, $n_{\mathrm{f}} \geq 1$ is the order of G_{f}, and, for all $i \geq 1$,

$$H_i(k) \triangleq \begin{cases} C(k)G(k-1), & i=1, \\ C(k)\left(\prod_{j=1}^{i-1} \bar{A}(k-j)\right)G(k-i), & i \geq 2. \end{cases} \tag{52}$$

Next, for all $k \geq 0$, we define the *retrospective input*

$$d_{\mathrm{rc}}(\hat{\theta}, k) \triangleq \Phi(k)\hat{\theta} \tag{53}$$

and the corresponding *retrospective performance variable*

$$z_{\mathrm{rc}}(\hat{\theta}, k) \triangleq z(k) + G_{\mathrm{f},k}(\mathbf{q})[d_{\mathrm{rc}}(\hat{\theta}, k) - \hat{d}(k)], \tag{54}$$

where the coefficient vector $\hat{\theta} \in \mathbb{R}^{l_\theta}$ is determined by optimization below. Defining

$$\Phi_{\mathrm{f}}(k) \triangleq G_{\mathrm{f},k}(\mathbf{q})\Phi(k) \in \mathbb{R}^{l_y \times l_\theta}, \tag{55}$$

$$\hat{d}_{\mathrm{f}}(k) \triangleq G_{\mathrm{f},k}(\mathbf{q})\hat{d}(k) \in \mathbb{R}^{l_y}, \tag{56}$$

it follows that $z_{\mathrm{rc}}(\hat{\theta}, k)$ can be written as

$$z_{\mathrm{rc}}(\hat{\theta}, k) = z(k) + \Phi_{\mathrm{f}}(k)\hat{\theta} - \hat{d}_{\mathrm{f}}(k). \tag{57}$$

For $k \geq 1$, we define the retrospective cost function

$$J(\hat{\theta}, k) \triangleq \sum_{i=0}^{k} \lambda^{k-i} \Big(z_{\mathrm{rc}}(\hat{\theta}, i)^{\mathrm{T}} R_z z_{\mathrm{rc}}(\hat{\theta}, i) + [\Phi(i)\hat{\theta}]^{\mathrm{T}} \cdot R_d \Phi(i)\hat{\theta}\Big) + \lambda^k [\hat{\theta} - \theta(0)]^{\mathrm{T}} R_\theta [\hat{\theta} - \theta(0)], \tag{58}$$

where $R_z \in \mathbb{R}^{l_y \times l_y}$, $R_d \in \mathbb{R}^{l_d \times l_d}$, and $R_\theta \in \mathbb{R}^{l_\theta \times l_\theta}$ are positive definite, and $\lambda \in (0, 1]$ is the forgetting factor. Let $P(0) = R_\theta^{-1}$ and $\theta(0) = \theta_0$. Then, for all $k \geq 1$, the cumulative cost function (58) has the unique global minimizer $\theta(k)$ given by the RLS update

$$\theta(k) = \theta(k-1) - P(k-1)\tilde{\Phi}(k)^{\mathrm{T}}\Gamma(k)[\tilde{\Phi}(k)\theta(k-1) + \tilde{z}(k)], \tag{59}$$

$$P(k) = \frac{1}{\lambda}[P(k-1) - P(k-1)\tilde{\Phi}(k)^{\mathrm{T}}\Gamma(k)\tilde{\Phi}(k)P(k-1)], \tag{60}$$

where

$$\tilde{\Phi}(k) \triangleq \begin{bmatrix} \Phi_{\mathrm{f}}(k) \\ \Phi(k) \end{bmatrix} \in \mathbb{R}^{(l_y+l_d) \times l_\theta}, \tag{61}$$

$$\tilde{R}(k) \triangleq \begin{bmatrix} R_z(k) & 0 \\ 0 & R_d(k) \end{bmatrix} \in \mathbb{R}^{(l_y+l_d) \times (l_y+l_d)}, \tag{62}$$

$$\tilde{z}(k) \triangleq \begin{bmatrix} z(k) - \hat{d}_{\mathrm{f}}(k) \\ 0 \end{bmatrix} \in \mathbb{R}^{l_y+l_d}, \tag{63}$$

$$\Gamma(k) \triangleq [\lambda \tilde{R}(k)^{-1} + \tilde{\Phi}(k)P(k-1)\tilde{\Phi}(k)^{\mathrm{T}}]^{-1}. \tag{64}$$

Note that [24] provides the pseudo code of RCIE.

B. State Estimation

In order to estimate the state $x(k)$, we use $x_{\mathrm{fc}}(k)$ given by (46) to obtain the estimate $x_{\mathrm{da}}(k)$ of $x(k)$ given by the Kalman filter data-assimilation step

$$x_{\mathrm{da}}(k) = x_{\mathrm{fc}}(k) + K_{\mathrm{da}}(k)z(k), \tag{65}$$

where the state estimator gain $K_{\mathrm{da}}(k) \in \mathbb{R}^{l_x \times l_y}$ is given by

$$K_{\mathrm{da}}(k) = -P_{\mathrm{f}}(k)C(k)^{\mathrm{T}}[C(k)P_{\mathrm{f}}(k)C(k)^{\mathrm{T}} + V_2(k)]^{-1}, \tag{66}$$

and the forecast error covariance $P_{\mathrm{f}}(k) \in \mathbb{R}^{l_x \times l_x}$ and the data-assimilation error covariance $P_{\mathrm{da}}(k) \in \mathbb{R}^{l_x \times l_x}$ are given by

$$P_{\mathrm{f}}(k) = A(k-1)P_{\mathrm{da}}(k-1)A(k-1)^{\mathrm{T}} + V_1(k-1) + V_{\hat{d}}(k-1), \tag{67}$$

$$P_{\mathrm{da}}(k) = [I + K_{\mathrm{da}}(k)C(k)]P_{\mathrm{f}}(k), \tag{68}$$

where $V_{\hat{d}}(k)$ is the covariance of $\hat{d}(k)$. Note that, if $\hat{d}(k) = d(k)$ for all $k \geq 0$, then, for all $k \geq 0$, $V_{\hat{d}}(k) = 0$ and the state estimate x_{da} given by (65) is the standard Kalman filter estimate.

VI. Numerical Results

A. Simulation Setup

Using retrospective cost input estimation (RCIE), we estimate the drag acceleration $\vec{a}_{\mathrm{drag}}$ of a satellite in $\mathrm{F_E}$ and $\mathrm{F_P}$ using [(29),(34)] and (39), respectively, and with $C = I_6$. We choose $\alpha = 10^{-5}\,\mathrm{kN/kg}$ in (21), which is unknown to RCIE. The position $\vec{r}_{\mathrm{p/O_E}}\Big|_{\mathrm{E}}$ and velocity $\vec{v}_{\mathrm{p/O_E}}\Big|_{\mathrm{E}}$ of the satellite are obtained by integrating (17)–(20) using the Matlab function ode45 with a numerical tolerance of 10^{-12}. The initial position and velocity are chosen such that the satellite orbit is circular with inclination $51.6\,\mathrm{deg}$ and radius $6731\,\mathrm{km}$. The duration of the simulation is set for $1\,\mathrm{hr}$, and the noise-free position and velocity data are recorded using the sampling-time $T_{\mathrm{s}} = 0.1\,\mathrm{sec}$. Since RCIE is a discrete-time algorithm, we discretize (29), (34), and (39) using the Matlab function c2d, which uses zero-order hold on the input for discretization. To assess the accuracy of the RCIE estimate, we define the relative error $e_d = |\frac{d-\hat{d}}{d}|$, where d is the actual input and $\hat{d}$ is the RCIE estimated input.

B. Indirect Estimation of the Drag Acceleration in F_{E}

Let the RCIE parameters be $n_{\mathrm{c}} = 2$, $n_{\mathrm{f}} = 24$, $\lambda = 1$, $R_\theta = 10^{-12}I_{l_\theta}$, $R_d = 10^{-12}I_3$, $R_z = I_6$, $V_{\hat{d}} + V_1 = 10^{-8}I_6$, and $V_2 = 10^{-8}I_6$.

For indirect estimation of drag acceleration, we first use (29) to estimate the total inertial acceleration (A_x, A_y, A_z) of the satellite. Figure 1 shows that the RCIE estimates follow the actual acceleration (A_x, A_y, A_z). After the initial transient, the maximum relative errors in the directions x, y, z of F_{E} are $1.07, 3.52, 3.52$, respectively, whereas the minimum relative errors in the directions x, y, z of F_{E} are $1.05 \times 10^{-9}, 3.11 \times 10^{-9}, 3.11 \times 10^{-9}$, respectively.

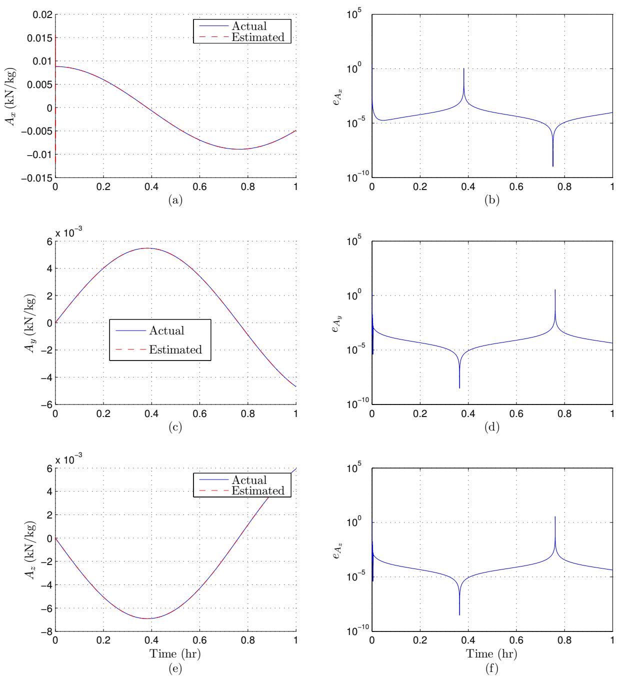

Fig. 1: Estimation of the inertial acceleration (A_x, A_y, A_z) of the satellite using position and velocity measurements with $T_\mathrm{s} = 0.1$ sec. The RCIE estimates (dashed line) follow the actual acceleration (A_x, A_y, A_z) (solid line). After the initial transient, the maximum relative errors in the directions x, y, z of F_E are $1.07, 3.52, 3.52$, respectively, whereas the minimum relative errors in the directions x, y, z of F_E are $1.05 \times 10^{-9}, 3.11 \times 10^{-9}, 3.11 \times 10^{-9}$, respectively.

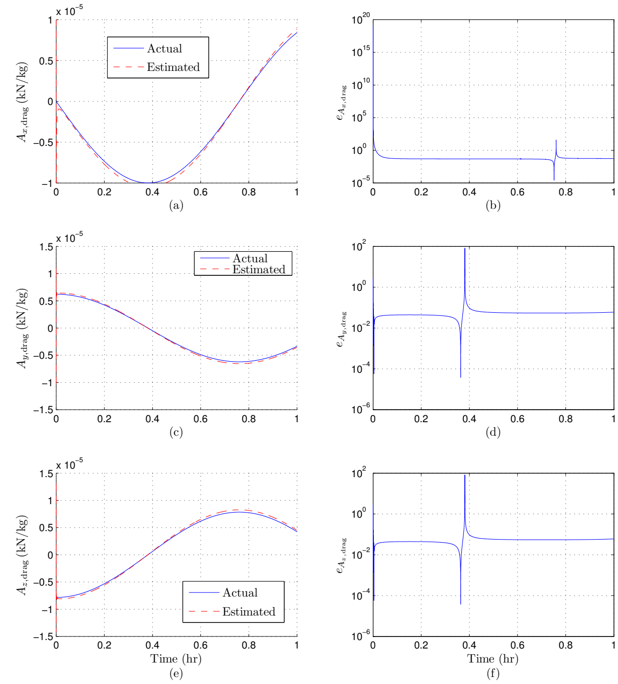

Fig. 2: Indirect estimation of drag acceleration $(A_{x,\mathrm{drag}}, A_{y,\mathrm{drag}}, A_{z,\mathrm{drag}})$ of the satellite using the RCIE estimates of (A_x, A_y, A_z) shown in Figure 1. The drag acceleration estimates (dashed line) follow the actual acceleration $(A_{x,\mathrm{drag}}, A_{y,\mathrm{drag}}, A_{z,\mathrm{drag}})$ (solid line). After the initial transient, the maximum relative errors in the directions x, y, z of F_E are $38.4, 81.2, 81.2$, respectively, whereas the minimum relative errors in the directions x, y, z of F_E are $2.6 \times 10^{-5}, 3.8 \times 10^{-5}, 3.8 \times 10^{-5}$, respectively.

Next, we use (33) to estimate the drag acceleration $(A_{x,\mathrm{drag}}, A_{y,\mathrm{drag}}, A_{z,\mathrm{drag}})$ acting on the satellite. Figure 2 shows that the drag acceleration estimates follow the actual acceleration $(A_{x,\mathrm{drag}}, A_{y,\mathrm{drag}}, A_{z,\mathrm{drag}})$. After the initial transient, the maximum relative errors in the directions x, y, z of F_E are $38.4, 81.2, 81.2$, respectively, whereas the minimum relative errors in the directions x, y, z of F_E are $2.6 \times 10^{-5}, 3.8 \times 10^{-5}, 3.8 \times 10^{-5}$, respectively.

C. Direct Estimation of Drag Acceleration in F_E

Using the same RCIE parameters as in Section VI-B, we use (34) to obtain a direct estimation of the drag acceleration $(A_{x,\mathrm{drag}}, A_{y,\mathrm{drag}}, A_{z,\mathrm{drag}})$. Figure 3 shows that the drag acceleration estimates follow the actual acceleration $(A_{x,\mathrm{drag}}, A_{y,\mathrm{drag}}, A_{z,\mathrm{drag}})$. After the initial transient, the maximum relative errors in the directions x, y, z of F_E are $88.1, 6.5, 6.5$, respectively, whereas the minimum relative errors in the directions x, y, z of F_E are $6.1 \times 10^{-6}, 2.6 \times 10^{-4}, 2.6 \times 10^{-4}$, respectively.

D. Estimation of Drag Acceleration in F_P

Let $n_\mathrm{c} = 4$, $n_\mathrm{f} = 4$, $\lambda = 1$, $R_\theta = 10^{-12} I_{l_\theta}$, $R_d = 10^{-8}$, $R_z = I_6$, $V_{\hat{d}} + V_1 = 10^{-12} I_6$, and $V_2 = 10^{-8} I_6$. To estimate the magnitude of drag acceleration α, we use (39). Figure 4 shows that the drag acceleration estimate follows the actual acceleration α. After the initial transient, the maximum relative error is 0.1, whereas the minimum relative error is 6.3×10^{-6}.

VII. Conclusions

Retrospective cost input estimation was used to estimate satellite drag. Three problem formulations were considered, namely, indirect estimation of the drag acceleration by estimating the total acceleration; direct estimation of the drag acceleration; and estimation of the drag magnitude. These results, along with [24], show that input estimation can provide a viable approach to estimating acceleration modeled as an unknown input. Future research will focus on the application of this approach to UAVs and sensor fault detection.

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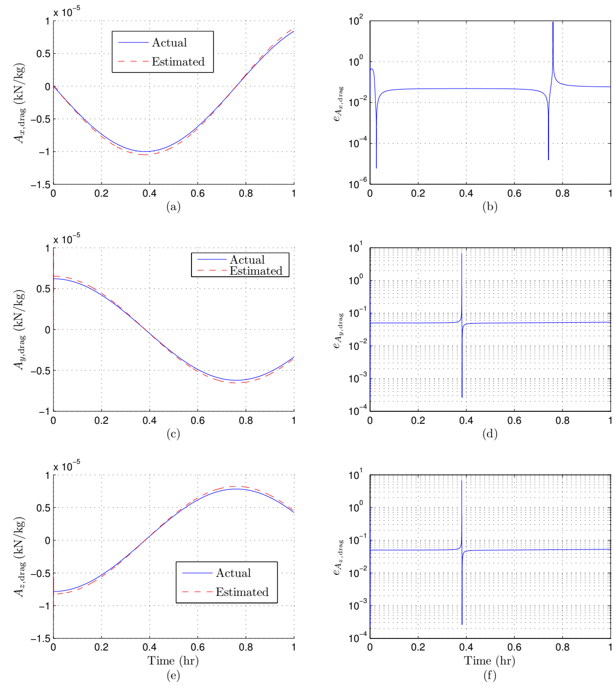

Fig. 3: Direct estimation of drag acceleration $(A_{x,\mathrm{drag}}, A_{y,\mathrm{drag}}, A_{z,\mathrm{drag}})$ of the satellite using gravity, position and velocity measurements with $T_\mathrm{s} = 0.1\,\mathrm{sec}$. The drag acceleration estimates (dashed line) follow the actual acceleration $(A_{x,\mathrm{drag}}, A_{y,\mathrm{drag}}, A_{z,\mathrm{drag}})$ (solid line). After the initial transient, the maximum relative errors in the directions x, y, z of F_E are $88.1, 6.5, 6.5$, respectively, whereas the minimum relative errors in the directions x, y, z of F_E are $6.1 \times 10^{-6}, 2.6 \times 10^{-4}, 2.6 \times 10^{-4}$, respectively.

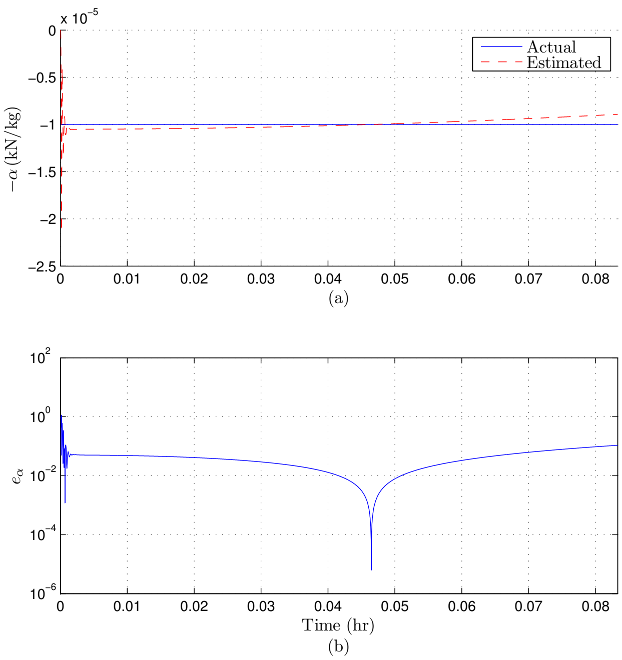

Fig. 4: Estimation of the drag acceleration of the satellite in F_P using position and velocity measurements with $T_\mathrm{s} = 0.1\,\mathrm{sec}$. The drag acceleration estimate (dashed line) follows the actual acceleration α (solid line). After the initial transient, the maximum relative error is 0.1, whereas the minimum relative error is 6.3×10^{-6}.

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