How Slippery Is Viscous Friction?

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One of the many things that humans and many animals are very good at is picking up objects. Teaching robots to pick up objects is not so easy, however, for at least two reasons. First, the coefficient of friction may be unknown, and therefore, an object may slip through the robot's grasp. Although this uncertainty can be overcome by using a tighter grip, another issue arises, namely, that excessive force may crush the object. Furthermore, the length of the object must also be accounted for since the ultimate goal is to achieve a slip distance that is less than the length of the object. Consequently, reliable gripping is a control problem that requires a careful tradeoff between the possibility of dropping an object and the potential for crushing it.

The literature on slip control for robot grippers is extensive. In [1], the controller uses the object's velocity, acceleration, and detection of incipient slippage to adjust the grasping force. The control law employs empirical rules and an inference mechanism, which are based on fuzzy logic. In [2], a slip-suppression control algorithm is based on a fixed threshold value for the normal force, while [3] presents a grasp controller inspired by the physiology of human grasping. A controller decides on the initial grasping force, detects object slippage, and regulates the grasp force. In [4], a fuzzy sliding mode controller combined with a disturbance observer is designed for contact force control and slip prevention. Using multiple manipulators to grasp objects is considered in [5]. A proportional-derivative shear-force feedback control law and adaptive slip-prevention algorithm are given in [6] and [7]. Assuming a Coulomb friction model, a Lyapunov-based adaptive controller based on a friction estimate is used in [8] for grasping and lifting from zero initial velocity.

As discussed in "Summary," the goal of this article is to investigate the normal force needed to achieve finite slip distance. To do this, a classical control perspective is taken by assuming that the requested normal force is the output of a proportional-integral (PI) controller, where integral control is motivated by the desire to asymptotically reach the setpoint of zero slip velocity. Although many friction models can be assumed, such as viscous, Coulomb, Dahl, and LuGre [9]–[13], we focus on Coulomb and viscous friction.

First, we consider the case of horizontal slip motion, in which gravity plays no role. Since this is a stabilization problem, integral control for disturbance rejection is not needed. Next, we consider the case of vertical slip motion with arbitrary initial vertical velocity, in which case gravity affects the motion of the object. This is a disturbance-rejection problem with constant disturbance, and thus integral control is needed. As long as the normal force multiplies the velocity through the viscous damping coefficient, the closed-loop dynamics are nonlinear. Since the closed-loop dynamics are nonlinear, standard linear techniques cannot be used to analyze the vertical slip

Summary

A robotic gripper is often required to pick up or catch an object with unknown mass and unknown friction. The goal is to reliably hold the object without applying excessive force, which could crush it. The ability of the robot to grasp the object depends on the nature of the friction between the gripper and the object, and the simplest type of friction is viscous. This article shows that, under viscous friction and with bounded normal force, it is impossible to bring the object to rest. Furthermore, under viscous friction and using a proportional-integral control law with unbounded normal force, it is also impossible to bring the object to rest. This result is based on the asymptotic analysis of a second-order nonlinear differential equation and illustrated through numerical simulation. These results show that the ability of a robot (or human) to grasp an object requires static friction. In other words, viscous friction is extremely slippery.

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motion of the object under PI control. We therefore present a detailed analysis of the nonlinear dynamics. Specifically, the main contribution is to demonstrate that, although the vertical slip velocity converges to zero, the slip distance is infinite. In fact, infinite slip distance occurs whether or not the normal force is bounded; in other words, infinite normal force applied by the PI control is not sufficient to bring the object to rest with finite slip distance. This observation is surprising to us and, to the best of our knowledge, has not been discussed in the literature. Some of the background material in this article is based on [14].

GRIPPER DYNAMICS AND FRICTION MODELS

Consider an object with mass m held by a gripper (as shown in Figure 1), where $q(t)$ is the position of the object relative to the zero reference on the gripper, $f_d(t)$ is the disturbance force applied to the object, $f_n(t) \geq 0$ is the normal force applied to the object by the gripper, and $f_f(t)$ is the friction force applied to the object due to $f_n(t)$. Gravity with acceleration g acts along the x-axis. If $\dot{q} = 0$, then the object is fixed relative to the gripper, and thus no slipping is occurring; the object is said to be sticking. If $\dot{q} \neq 0$, then the object is slipping relative to the gripper.

In Figure 1, the object is held vertically. In this case, the force on the object due to gravity is mg, which can be viewed as a constant disturbance. The objective of feedback control is command following with a zero-velocity setpoint and constant disturbance rejection. We can also consider the simpler case where the object is held horizontally. In this case, gravity is absent, and thus, the objective of feedback control is command following with a zero-velocity setpoint, which is equivalent to stabilization.

For the vertical slip case, the equation of motion of the object is given by

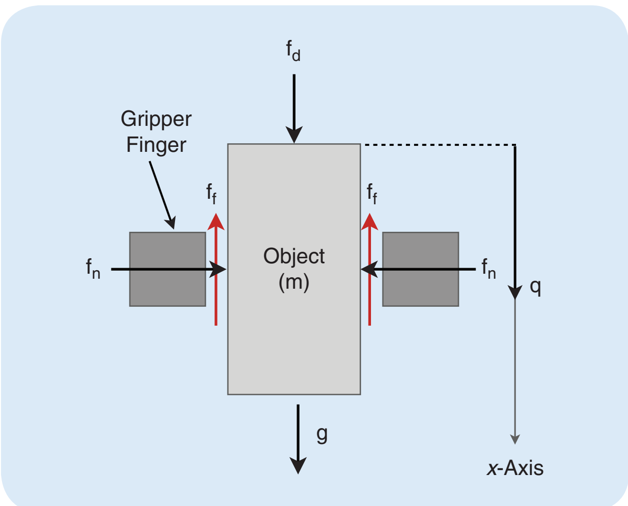

FIGURE 1 The gripper and object. The normal force f_n applied by the gripper is specified by the controller. The case of vertical slip is shown, as indicated by the acceleration due to gravity.

$$m\ddot{q}(t) = mg - f_f(t), \tag{1}$$

where the friction force $f_f(t) \geq 0$ is defined by the friction model. Note the minus sign, which shows that the friction force opposes the downward velocity. In terms of the slip velocity $v(t) \triangleq \dot{q}(t)$, (1) becomes

$$m\dot{v}(t) = mg - f_f(t). \tag{2}$$

Since gravity is in the downward direction, the object can only move downward; therefore, $v(t)$ is nonnegative. Note that f_n is physically constrained to be nonnegative; this sign constraint is enforced for the control signal. For the horizontal slip case, the term mg is omitted.

The *slip distance* $\delta(t)$ is defined as

$$\delta(t) \triangleq q(t) - q(0) = \int_0^t v(\tau)d\tau - q(0) \geq 0, \tag{3}$$

and thus, $\dot{\delta}(t) = \dot{q}(t) = v(t)$. The asymptotic slip distance is given by

$$\delta_\infty \triangleq \lim_{t\to\infty} \delta(t). \tag{4}$$

FRICTION MODELS

We consider two friction models, namely, viscous and Coulomb.

Viscous Friction

The viscous friction model is given by [9]

$$f_f = (c + c_n f_n)v, \tag{5}$$

where the normal force $f_n(t) \geq 0$ augments the effect of the viscous damping coefficient $c \geq 0$ as a result of the normal force viscous damping coefficient $c_n > 0$. In (5) and henceforth, where no confusion can arise, the time argument t is omitted.

Note that the viscous friction model (5) does not include a static friction term. This means that, at zero velocity, the tangential friction force is zero. In the presence of gravity, the object would begin to accelerate. Consequently, (5) implies that the object cannot be brought to rest and maintained at rest. However, the viscous friction model does not a priori preclude the possibility that an appropriate normal force can be applied by the gripper to attain asymptotic stopping with finite slip distance.

Coulomb Friction

The Coulomb friction model is given by

$$f_f = \begin{cases} f_C \operatorname{sgn}(v), & v \neq 0, \\ \min(|f_d|, f_{st}) \operatorname{sgn}(f_d), & v = 0, \end{cases} \tag{6}$$

where the Coulomb friction force f_C and the static friction force f_{st} are given by

$$f_C = \mu f_n, \tag{7}$$

$$f_{st} = \mu_{st} f_n, \tag{8}$$

where $\mu > 0$ is the sliding-friction coefficient and $\mu_{st} > 0$ is the static friction coefficient. During sticking, the direction of the friction force is defined by the direction of the disturbance force f_d. During slip, the friction force f_f is equal to f_C or $-f_C$, depending on the sliding direction. In this article, all sliding is in a single direction.

Comparison of the Free Response

The parameters of the viscous and Coulomb friction models are chosen so that the time constants of the corresponding free responses are approximately the same. In particular, for viscous friction, let $c = 0.4$ N-s/m and $c_n = 1$ s/m; for Coulomb friction, let $\mu_{st} = 1$ N-m and $\mu = 0.7$ N-m. For the open-loop responses, consider an object with mass $m = 1$ kg. Let the initial slip velocity of the object be $v_0 = 1$ m/s, and consider the constant normal force $f_n = 1$ N. Figure 2 shows the slip distance, slip velocity, and friction force with both viscous and Coulomb friction in the case of horizontal slip, while Figure 3 shows the corresponding response for vertical slip. For horizontal slip, viscous and Coulomb friction lead to finite slip distance, whereas, for vertical slip, both friction models yield infinite slip distance. In Figure 4, the normal force is increased by a factor of 15 compared to Figure 3. With this increase, it can be seen that, for Coulomb friction, the slip distance is finite; however, for viscous friction, the slip distance remains infinite.

PROPORTIONAL INTEGRAL CONTROL WITH VISCOUS FRICTION

The objective of feedback control is to adjust the normal force applied by the gripper to drive the slip velocity to zero with finite asymptotic slip distance. Note that asymptotic convergence of the slip velocity to zero does not imply that the asymptotic slip distance is finite. However, a sufficient condition for achieving finite slip distance is for the

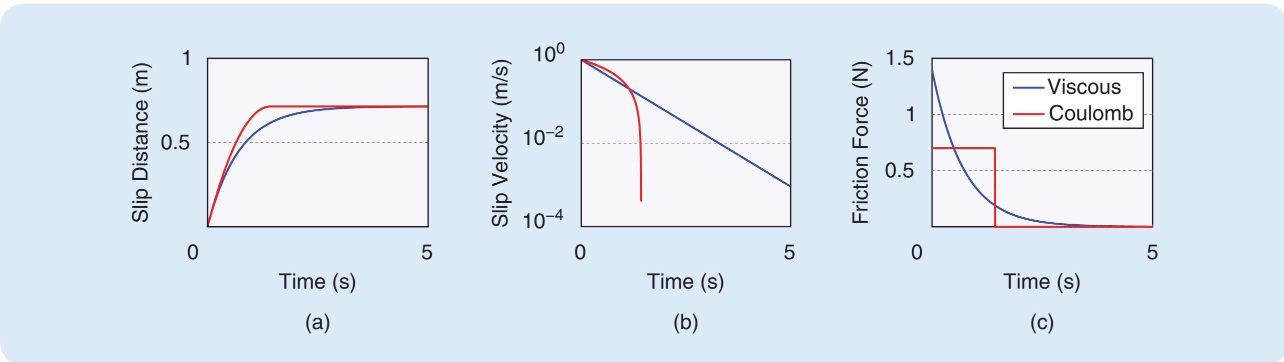

FIGURE 2 Open-loop response of the gripper model (2) for horizontal slip with the viscous and Coulomb friction models. The parameters of both friction models are chosen so that the time constants and slip distances of the free responses are approximately the same. For both simulations, the initial slip velocity is $v_0 = 1$ m/s, and the normal force is $f_n = 1$ N. (a) Both friction models yield finite slip distance. (b) Coulomb friction provides finite-time convergence but viscous friction does not. (c) Friction forces versus time.

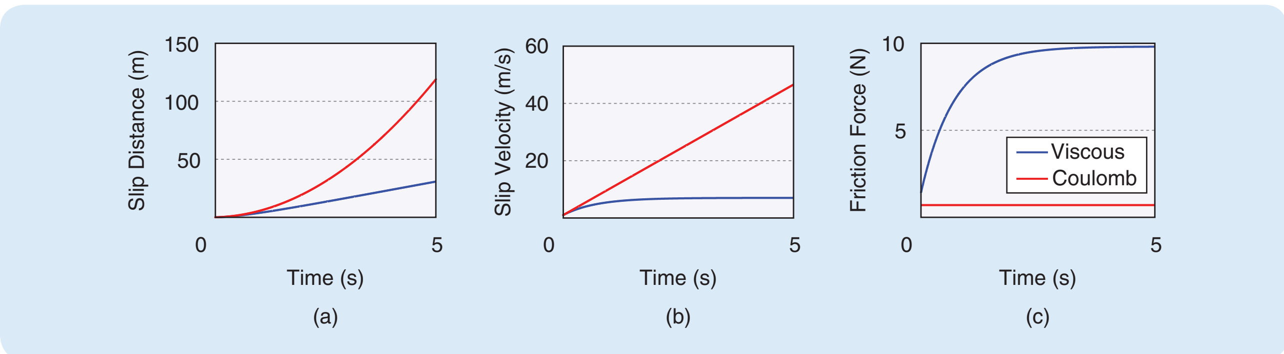

FIGURE 3 Open-loop response of the gripper model (2) for vertical slip with the viscous and Coulomb friction models. For both simulations, the initial slip velocity is $v_0 = 1$ m/s, and the normal force is $f_n = 1$ N. For viscous friction, (a) the slip distance increases linearly, (b) the slip velocity reaches a terminal velocity, and (c) the asymptotic friction force is approximately equal to the force of gravity. For Coulomb friction, the slip distance in (a) increases quadratically corresponding to the linear increase in slip velocity shown in (b). Note that, for both types of friction, the object does not come to rest.

slip velocity to converge exponentially to zero. A stronger condition is for the slip velocity to converge to zero in finite time. To avoid damaging the object, it is necessary to achieve finite asymptotic slip distance without applying excessive normal force.

A block diagram of the control system is shown in Figure 5, where the friction model block represents either viscous or Coulomb friction. The gripper-object dynamics are given by (2). The slip-velocity error e is defined by $e \triangleq v - v_{\mathrm{ref}}$. For slip suppression, the reference slip velocity is $v_{\mathrm{ref}} = 0$.

Consider the PI controller

$$\dot{h} = v, \tag{9}$$

$$f_{\mathrm{n}} = K_I h + K_{\mathrm{P}} v, \tag{10}$$

where the constants $K_{\mathrm{P}} \geq 0$ and $K_{\mathrm{I}} \geq 0$ are the proportional and integral gains, respectively. The purpose of the proportional term $K_{\mathrm{P}} v$ is to bring the slip velocity to zero for horizontal slip, while the purpose of the integral term $K_{\mathrm{I}} h$ is to asymptotically reject the effect of gravity for vertical slip.

With the viscous friction model (5), the horizontal slip velocity satisfies

$$m\dot{v} + (c + c_{\mathrm{n}} f_{\mathrm{n}})v = mg. \tag{11}$$

Combining (9)–(11) and using $v = \dot{\delta}$ yields

$$m\ddot{\delta} + (c + c_{\mathrm{n}} K_{\mathrm{P}} \dot{\delta})\dot{\delta} + c_{\mathrm{n}} K_{\mathrm{I}} \dot{\delta}\delta = mg \tag{12}$$

with the initial conditions

$$\delta(0) = 0, \quad \dot{\delta}(0) = v_0 \geq 0. \tag{13}$$

PROPORTIONAL CONTROL OF HORIZONTAL SLIP

We first consider horizontal slip. Considering that no disturbance is present, this is a stabilization problem, and only proportional control is used. Using (5), the closed-loop dynamics for the viscous friction model are given by

$$m\dot{v} = -cv - c_{\mathrm{n}} K_{\mathrm{P}} v^2. \tag{14}$$

Solving (14) yields

$$v(t) = \frac{c v_0 e^{-(c/m)t}}{c + c_{\mathrm{n}} K_{\mathrm{P}} v_0 (1 - e^{-(c/m)t})}. \tag{15}$$

Note that, since the open-loop velocity is $v(t) = v_0 e^{-(c/m)t}$, it follows from (15) that the ratio of the controlled slip velocity to the open-loop slip velocity decreases as K_{P} increases. In addition, for each value of $K_{\mathrm{P}} \geq 0$, the terminal slip velocity is given by

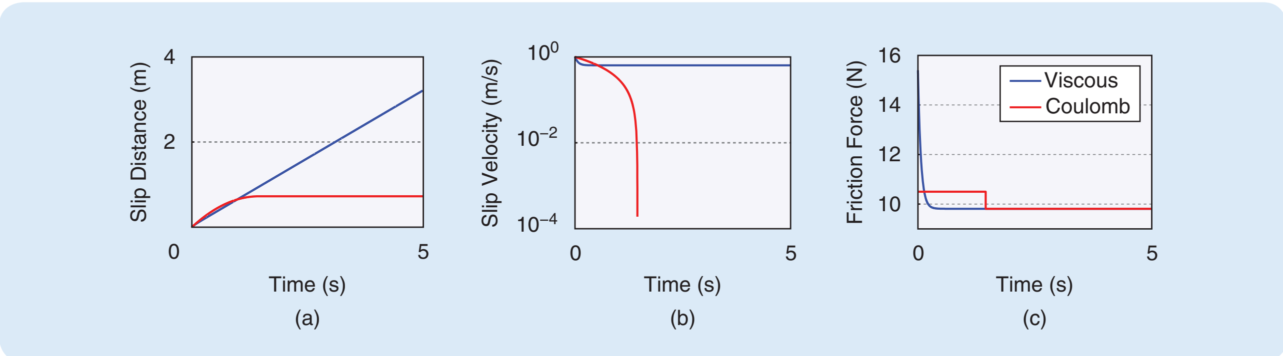

FIGURE 4 Open-loop response of the gripper (2) model for vertical slip with the viscous and Coulomb friction models. For both simulations, the initial slip velocity is $v_0 = 1$ m/s, and the normal force is $f_{\mathrm{n}} = 15$ N. For viscous friction, (a) the slip distance increases linearly, (b) the slip velocity reaches a terminal velocity, and (c) the friction force is approximately equal to the force of gravity. For Coulomb friction, the slip distance in (a) reaches a terminal value as the slip velocity converges to zero, as shown in (b). (c) During sticking, $f_{\mathrm{f}} = \min(|f_{\mathrm{d}}|, f_{\mathrm{st}})\,\mathrm{sgn}(f_{\mathrm{d}}) = mg$.

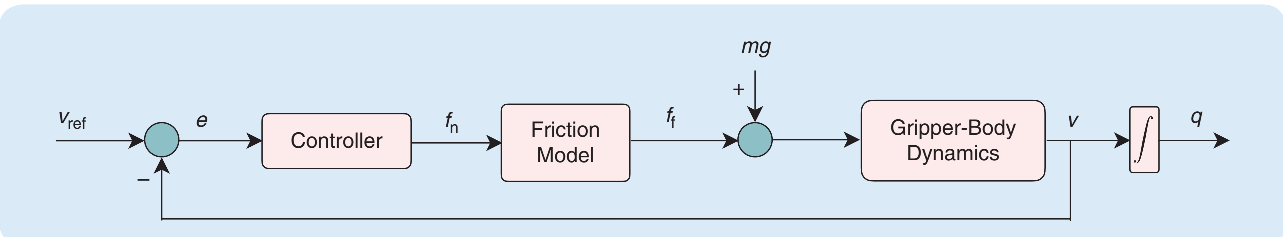

FIGURE 5 Feedback control of the gripper. The controller specifies the normal force f_{n}.

$$v_\infty \triangleq \lim_{t\to\infty} v(t) = 0. \tag{16}$$

Furthermore, integrating (15) yields the slip distance

$$\delta(t) = \frac{mv_0}{c_n K_P}\log\left[1 + \frac{c_n K_P}{c}\left(1 - e^{-(c/m)t}\right)\right], \tag{17}$$

and thus

$$\delta_\infty = \frac{mv_0}{c_n K_P}\log\left(1 + \frac{c_n K_P}{c}\right), \tag{18}$$

which is a decreasing function of K_P. The velocity converges exponentially to zero, and thus the asymptotic slip distance is finite. Finally, the normal force f_n is decreasing and converges to zero. Figure 6 illustrates proportional control of horizontal slip for viscous and Coulomb friction.

ASYMPTOTIC VERTICAL SLIP UNDER BOUNDED CONTROL

To avoid unbounded f_n, assume, as is the case in practice, that the normal force is bounded; that is, there exists $f_{n,\max} > 0$ such that, for all $t \geq 0$, $f_n(t) \leq f_{n,\max}$. This constraint can be enforced by replacing (9) and (10) with

$$\dot{h} = v, \tag{19}$$

$$f_n = \mathrm{sat}(K_I h + K_P v), \tag{20}$$

where the saturation function is defined as

$$\mathrm{sat}(z) = \begin{cases} z, & |z| < f_{n,\max}, \\ \mathrm{sign}(z) f_{n,\max}, & |z| \geq f_{n,\max}. \end{cases} \tag{21}$$

For all $t \geq \tau \geq 0$, defining the state transition matrix

$$\Phi(t,\tau) \triangleq e^{-(c/m)(t-\tau)} e^{-(c_n/m)\int_\tau^t f_n(s)ds}, \tag{22}$$

it follows from (11) that

$$v(t) = \Phi(t,0)v_0 + \int_0^t \Phi(t,\tau)\,d\tau g. \tag{23}$$

It thus follows from (23) that

$$\begin{aligned} v(t) &\geq \Phi(t,0)\,v_0 + \int_0^t e^{-(c/m)(t-\tau)} e^{-(c_n/m)\int_\tau^t f_{n,\max} ds}\,d\tau g \\ &\geq \frac{mg}{c + c_n f_{n,\max}}\left(1 - e^{-mt/(c + c_n f_{n,\max})}\right). \end{aligned} \tag{24}$$

Therefore,

$$\liminf_{t\to\infty} v(t) \geq \frac{mg}{c + c_n f_{n,\max}}. \tag{25}$$

Consequently, the slip velocity $v(t)$ is asymptotically bounded from below, and thus the slip distance is infinite. Therefore, finite slip distance cannot be achieved under bounded control by any feedback control law. Figure 7 illustrates bounded PI control of vertical slip for viscous and Coulomb friction.

The lower bound in (25) suggests that, if unbounded normal force could be applied, then the slip velocity may converge to zero and, perhaps, the asymptotic slip distance would be finite. We examine this conjecture in the next section.

ASYMPTOTIC VERTICAL SLIP UNDER UNBOUNDED PROPORTIONAL INTEGRAL CONTROL

The goal is to analyze the asymptotic properties of the asymptotic vertical slip δ satisfying (12) as well as the corresponding normal force f_n. Unlike in the previous section, where the normal force was assumed to be bounded, the normal force in this section is allowed to be unbounded.

Theorem 1

Let δ satisfy (12) with the initial conditions (13). Then the following statements hold:

1) δ is defined and C^2 on $[0,\infty)$.
2) If

$$mg < cv_0 + c_n K_P v_0^2, \tag{26}$$

then $v = \dot{\delta}$ is decreasing on $(0,\infty)$. If

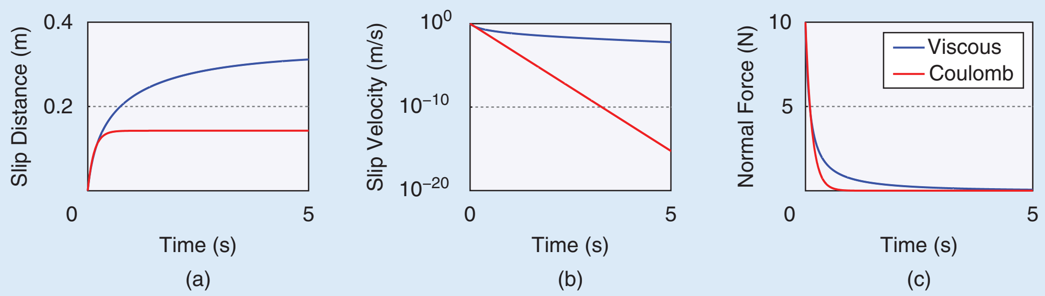

FIGURE 6 Proportional control of the gripper model (2) for horizontal slip with the viscous and Coulomb friction models and $K_P = 10$ N-s/m. The parameters of both friction models and the initial velocities are chosen as in Figure 2. (a) The slip distance, (b) slip velocity, and (c) normal force are shown. For both friction models, the asymptotic slip distance is finite.

$$mg \geq cv_0 + c_n K_P v_0^2, \tag{27}$$

then there exists $\sigma > 0$ such that $v = \dot{\delta}$ is increasing on $(0,\sigma)$ and decreasing on (σ,∞).

3) For all $t \geq 0$, $v(t) \geq 0$.
4) $\lim_{t\to\infty} v(t) = 0$.
5) As $t \to \infty$,

$$\delta(t) \sim \sqrt{\frac{2mg}{c_n K_I} t}. \tag{28}$$

Hence, δ and f_n are unbounded.

Proof

Note that, for positive constants p, q, and r, the function

$$\varphi(t) \triangleq p\delta(qt) + r \tag{29}$$

satisfies

$$\ddot{\varphi} + \frac{c_n K_P}{pm}\left[\left(\frac{cp}{c_n K_I} - r\right)\frac{qK_I}{K_P} + \dot{\varphi} + \frac{qK_I}{K_P}\varphi\right]\dot{\varphi} = pq^2 g. \tag{30}$$

Defining

$$p \triangleq \frac{c_n K_P}{m}, \quad q \triangleq \frac{2K_P}{K_I}, \quad r \triangleq \frac{cp}{c_n K_I} = \frac{cK_P}{mK_I}, \tag{31}$$

it follows from (30) that φ satisfies

$$\ddot{\varphi} + \dot{\varphi}^2 + 2\dot{\varphi}\varphi = \mu \tag{32}$$

with the initial conditions

$$\varphi(0) = \alpha, \quad \dot{\varphi}(0) = \beta, \tag{33}$$

where

$$\alpha \triangleq \frac{cK_P}{mK_I}, \quad \beta \triangleq \frac{2c_n K_P^2}{mK_I} v_0, \quad \mu \triangleq \frac{4gc_n K_P^3}{mK_I^2}. \tag{34}$$

Note that $\alpha \geq 0$, $\beta \geq 0$, and $\mu > 0$.

Considering that (32) is an autonomous differential equation with a Lipschitz-continuous vector field, it follows from [15, Ch. 1] that there exists a unique, maximal solution φ of (32) satisfying the initial conditions (33). The following result describes the asymptotic properties of the maximal solution of (32) and yields statements 2–5 of Theorem 1.

Proposition 1

Let $\varphi: \mathcal{I} \triangleq [0, t_{\max}) \to \mathbb{R}$ denote the maximal solution satisfying (32) with the initial conditions (33). In addition, consider the conditions

$$\mu < \beta^2 + 2\alpha\beta, \tag{35}$$

$$\mu \geq \beta^2 + 2\alpha\beta. \tag{36}$$

Then the following statements hold:

1) There exists $t_0 \in \mathcal{I}$ such that $\ddot{\varphi}(t_0) < 0$.
2) If $t_0 \in \mathcal{I}$ and $\ddot{\varphi}(t_0) < 0$, then, for all $t \in [t_0, t_{\max})$, $\ddot{\varphi}(t) < 0$.
3) If (35) is satisfied, then $\ddot{\varphi}(t) < 0$ for all $t \in (0, t_{\max})$. If (36) is satisfied, then there exists a unique $\sigma \in (0, t_{\max})$ such that $\ddot{\varphi}(t) > 0$ for all $t \in (0,\sigma)$ and $\ddot{\varphi}(t) < 0$ for all $t \in (\sigma, t_{\max})$.
4) If the first case in 3) holds, then, for all $t \in (0, t_{\max})$, $\dot{\varphi}(t) > 0$. If the second case in 3) holds, then, for all $t \in (\sigma, t_{\max})$, $\dot{\varphi}(t) > 0$.
5) For all $t \in \mathcal{I}$, $\dot{\varphi}(t) \geq 0$ and $\varphi(t) \geq 0$.
6) $\lim_{t \uparrow t_{\max}} \dot{\varphi}(t)$ exists and is nonnegative.
7) $t_{\max} = \infty$.
8) For all $t \geq 0$, $\varphi(t) \leq \sqrt{\mu t + \alpha^2} + \beta$.
9) $\lim_{t\to\infty} \dot{\varphi}(t) = 0$.
10) $\lim_{t\to\infty} (\varphi(t)/\sqrt{\mu t}) = 1$. Hence, φ is unbounded.

Proof of Proposition 1

To prove 1), note that, in the case where $\ddot{\varphi}(0) = \mu - \beta^2 - 2\alpha\beta < 0$, it suffices to let $t_0 = 0$. Hence, we consider the

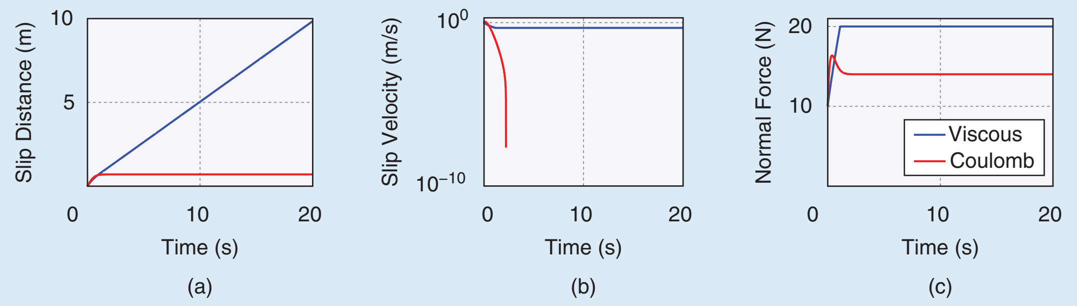

FIGURE 7 Bounded proportional-integral (PI) control of the gripper model (2) for vertical slip with the viscous and Coulomb friction models using $K_p = 10$ N-s/m, $K_i = 20$ N/m, and $f_{n,\max} = 20$ N. The parameters of both friction models and the initial velocities are chosen as in Figure 2. (a) The Coulomb friction yields finite slip distance, whereas, for viscous friction, the slip distance is infinite. (b) For Coulomb friction, zero slip velocity is reached in finite time. For viscous friction, however, bounded PI control does not yield finite slip distance. (c) For viscous friction, the corresponding normal force reaches a saturation limit equal to $f_{n,\max}$. For Coulomb friction, saturation of the normal force does not occur.

case where $\ddot{\varphi}(0) = \mu - \beta^2 - 2\alpha\beta \geq 0$, and thus, $\beta \leq \sqrt{\mu}$. Now, suppose that $\ddot{\varphi}(t) \geq 0$ for all $t \in \mathcal{I}$, and thus, $\dot{\varphi}$ is nondecreasing on $\mathcal{I}$. As $\dot{\varphi}(0) \geq 0$, it follows that $\dot{\varphi}$ is nonnegative on $\mathcal{I}$. Therefore, φ is nondecreasing on $\mathcal{I}$, and since $\varphi(0) \geq 0$, it follows that φ is nonnegative on $\mathcal{I}$.

Next, since all terms in (32) are nonnegative, it follows that $\dot{\varphi}(t) \leq \sqrt{\mu}$ for all $t \in \mathcal{I}$. Therefore, since $\dot{\varphi}$ is nondecreasing on $\mathcal{I}$, it follows that $\ell \triangleq \lim_{t\uparrow t_{\max}} \dot{\varphi}(t) \leq \sqrt{\mu}$ exists and is nonnegative. Now, suppose that $\ell = 0$. Since $\dot{\varphi}$ is nonnegative and nondecreasing on $\mathcal{I}$, it follows that $\dot{\varphi} \equiv 0$ on $\mathcal{I}$, and, thus, (32) implies that $\mu = 0$, which is a contradiction. Hence, $\ell \in (0, \sqrt{\mu})$. Since $\dot{\varphi}$ is continuous and nondecreasing on $\mathcal{I}$, there exists $t_1 \in (0, t_{\max})$ such that, for all $t \in (t_1, t_{\max})$, $\dot{\varphi}(t) \geq \ell/2$. Furthermore, since φ is nondecreasing and satisfies $\varphi(t) \leq \mu/(2\dot{\varphi}(t))$ for all $t \in (t_1, t_{\max})$, it follows that $k \triangleq \lim_{t\uparrow t_{\max}} \varphi(t) \leq \lim_{t\uparrow t_{\max}} \mu/(2\dot{\varphi}(t)) = \mu/(2\ell)$ exists. Now suppose that $t_{\max}$ is finite. Since $\lim_{t\uparrow t_{\max}} \varphi(t)$ and $\lim_{t\uparrow t_{\max}} \dot{\varphi}(t)$ exist, it follows that φ can be continued to the right. Therefore, $t_{\max} = \infty$. Finally, since $\dot{\varphi}(t) \geq \ell/2$ for all $t \in (t_1, \infty)$, it follows that $k = \lim_{t\to\infty} \varphi(t) = \infty$, which contradicts $k \leq \mu/(2\ell)$. Consequently, there exists $t_0 \in \mathcal{I}$ such that $\ddot{\varphi}(t_0) < 0$, which proves 1).

To prove 2), let $t_0 \in \mathcal{I}$ be such that $\ddot{\varphi}(t_0) < 0$, and suppose there exists $t_2 \in (t_0, t_{\max})$ such that $\ddot{\varphi}(t_2) \geq 0$. Therefore, $\{t \in (t_0, t_{\max}) : \ddot{\varphi}(t) \geq 0\}$ is not empty, and $t_1 \triangleq \inf\{t \geq t_0 : \ddot{\varphi}(t) \geq 0\} \geq t_0$. Since $\ddot{\varphi}$ is continuous on $\mathcal{I}$, it follows that $\ddot{\varphi}(t_1) = 0$ and, for all $t \in [t_0, t_1)$, $\ddot{\varphi}(t) < 0$. Therefore, for all sufficiently small $\varepsilon > 0$, $\ddot{\varphi}$ is negative in $(t_1 - \varepsilon, t_1)$ and positive in $(t_1, t_1 - \varepsilon)$. Furthermore, since φ is C^3 and zero at t_1, it follows that $\dddot{\varphi}(t_1) \geq 0$. Now, differentiating

$$\ddot{\varphi} + \dot{\varphi}^2 + 2\dot{\varphi}\varphi = \mu$$

implies that

$$\dddot{\varphi} + 2\ddot{\varphi}\dot{\varphi} + 2\dot{\varphi}^2 + 2\ddot{\varphi}\varphi = 0.$$

In particular, setting $t = t_1$ implies

$$\dddot{\varphi}(t_1) + 2(\dot{\varphi}(t_1))^2 = 0,$$

which is the sum of two nonnegative terms. Hence, $\dot{\varphi}(t_1) = 0$. Setting $t = t_1$ in (32) implies $\mu = 0$, which is a contradiction. Therefore, for all $t \in [t_0, t_{\max})$, $\ddot{\varphi}(t) < 0$, which proves 2). To prove 3), note that 1) and 2) imply that $\{t_0 \in [0, t_{\max}):$ for all $t \in (t_0, t_{\max}), \ddot{\varphi}(t) < 0\}$ is not empty. Hence, define $\sigma \triangleq \inf\{t_0 \in [0, t_{\max}):$ for all $t \in (t_0, t_{\max}), \ddot{\varphi}(t) < 0\}$.

In the case where $\sigma = 0$, which holds if and only if (35) is satisfied, it follows that, for all $t \in (0, t_{\max})$, $\ddot{\varphi}(t) < 0$. Now consider the case where $\sigma > 0$, which holds if and only if (36) is satisfied, and suppose there exists $t_0 \in (0, \sigma)$ such that $\ddot{\varphi}(t_0) < 0$. Then *2)* implies that, for all $t \in (t_0, t_{\max})$, $\ddot{\varphi}(t) < 0$, which, since $t_0 < \sigma$, contradicts the definition of σ. Therefore, $\ddot{\varphi}(t) \geq 0$ for all $t \in (0, \sigma)$. Next, suppose there exists $t_1 \in (0, \sigma)$ such that $\ddot{\varphi}(t_1) = 0$. As a result of $\ddot{\varphi}(t) \geq 0$ for all $t \in (0, \sigma)$, it follows that t_1 is a local minimizer of $\ddot{\varphi}$, and thus $\dddot{\varphi}(t_1) = 0$. However, as in the proof of 2),

$$\dddot{\varphi}(t_1) + 2(\dot{\varphi}(t_1))^2 = 0,$$

which implies that $\dot{\varphi}(t_1) = 0$. Now, setting $t = t_1$ in (32) yields $\mu = 0$, which is a contradiction. Therefore, $\ddot{\varphi}(t) > 0$ for all $t \in (0, \sigma)$ and $\ddot{\varphi}(t) < 0$ for all $t \in (\sigma, t_{\max})$, which proves 3).

To prove 4), suppose that the first case in 3) holds, that is, for all $t \in (0, t_{\max})$, $\ddot{\varphi}(t) < 0$. Therefore, for all $t \in [0, t_{\max})$,

$$\dot{\varphi}^2(t) + 2\dot{\varphi}(t)\varphi(t) = \mu - \ddot{\varphi}(t) > \mu > 0,$$

and thus $\dot{\varphi}(t) \neq 0$ for all $t \in (0, t_{\max})$. Since $\dot{\varphi}$ is continuous, it follows that either $\dot{\varphi}(t) > 0$ for all $t \in (0, t_{\max})$ or $\dot{\varphi}(t) < 0$ for all $t \in (0, t_{\max})$. Suppose that, for all $t \in (0, t_{\max})$, $\dot{\varphi}(t) < 0$. Therefore, $\beta = \dot{\varphi}(0) \leq 0$, and thus, $\beta = 0$. Now, setting $t = 0$ in (32) yields $\mu \leq 0$, which is a contradiction. Therefore, for all $t \in (0, t_{\max})$, $\dot{\varphi}(t) > 0$.

Now, suppose that the second case in 3) holds, and let $\sigma > 0$ given by 3). Since, for all $t \in (\sigma, t_{\max})$,

$$\dot{\varphi}^2(t) + 2\dot{\varphi}(t)\,\varphi(t) = \mu - \ddot{\varphi}(t) > \mu > 0,$$

it follows that $\dot{\varphi}(t) \neq 0$ for all $t \in (\sigma, t_{\max})$. Since $\dot{\varphi}$ is continuous, it follows that either $\dot{\varphi}(t) > 0$ for all $t \in (\sigma, t_{\max})$ or $\dot{\varphi}(t) < 0$ for all $t \in (\sigma, t_{\max})$. Suppose that, for all $t \in (\sigma, t_{\max})$, $\dot{\varphi}(t) < 0$. Then, $\dot{\varphi}(\sigma) \leq 0$. Since, for all $t \in (0, \sigma)$, $\ddot{\varphi}(t) > 0$, it follows that $0 \geq \dot{\varphi}(\sigma) > \dot{\varphi}(0) = \beta \geq 0$, which is a contradiction. Therefore, $\dot{\varphi}(t) > 0$ for all $t \in (\sigma, t_{\max})$, which proves 4).

To prove 5), consider the second case in 4), and suppose there exists $t_2 \in (0, \sigma)$ such that $\dot{\varphi}(t_2) < 0$. Since $\varphi(0) \geq 0$, it follows that there exists $t_1 \in (0, t_2)$ such that $\ddot{\varphi}(t_1) < 0$. However, this contradicts the fact that, for all $t \in (0, \sigma)$, $\ddot{\varphi}(t) > 0$. Therefore, for all $t \in \mathcal{I}$, $\dot{\varphi}(t) \geq 0$. Finally, since $\varphi(0) \geq 0$ and φ is nondecreasing on $\mathcal{I}$, it follows that φ is nonnegative on $\mathcal{I}$, which proves 5).

To prove 6), consider the first case in 4). Since $\ddot{\varphi}(t) < 0$ for all $t \in (0, t_{\max})$, it follows that $\dot{\varphi}$ is decreasing and nonnegative on $(0, t_{\max})$. Therefore, $\lim_{t\uparrow t_{\max}} \dot{\varphi}(t)$ exists and is nonnegative. The same conclusion holds in the second case in 4), which proves 6).

To prove 7), suppose that $t_{\max}$ is finite. Consider the first case in 4). It follows from 3) that, for all $t \in (0, t_{\max})$, $\ddot{\varphi}(t) < 0$, and thus, $\dot{\varphi}$ is decreasing on $(0, t_{\max})$. In addition, it follows from 4) that $\dot{\varphi}$ is positive on $(0, t_{\max})$. Hence, for all $t \in (0, t_{\max})$, $0 < \dot{\varphi}(t) < \dot{\varphi}(0) = \beta$. Therefore, φ is increasing on $\in (0, t_{\max})$ and bounded by $\beta t_{\max}$. It thus follows that $\lim_{t\uparrow t_{\max}} \varphi(t)$ exists. However, since $t_{\max}$ is finite, it follows that $\mathcal{I}$ cannot be the maximal interval of existence of φ. Hence, $t_{\max} = \infty$, and φ is defined on $[0, \infty)$. A similar argument applies in the second case in 4), which proves 7).

To prove 8), note that integrating

$$\ddot{\varphi} + 2\dot{\varphi}\varphi \leq \ddot{\varphi} + 2\dot{\varphi}\varphi + \dot{\varphi}^2 = \mu$$

implies that, for all $t \geq 0$,

$$\varphi^2(t) \leq \dot\varphi(t) + \varphi^2(t) \leq \mu t + \alpha^2 + \beta,$$

which proves 8).

To prove 9), consider the first case in 4). It follows from 3) that, for all $t \in (0,\infty)$, $\ddot\varphi(t) < 0$. Therefore, $\dot\varphi(t)$ is decreasing and, by 5), is nonnegative. Thus, $L \triangleq \lim_{t\to\infty}\dot\varphi(t)$ exists and is nonnegative. Therefore, for all $t \in (0,\infty)$,

$$\varphi(t) = \int_0^t \dot\varphi(\tau)\mathrm{d}\tau > Lt.$$

It therefore follows from 8) that, for all $t \in (0,\infty)$,

$$Lt < \sqrt{\mu t + \alpha^2 + \beta},$$

and thus

$$L < \sqrt{\mu t + \alpha^2 + \beta}\,/t.$$

Hence,

$$0 \leq L \leq \lim_{t\to\infty}\sqrt{\mu t + \alpha^2 + \beta}\,/t = 0,$$

which implies that $L = 0$. A similar argument yields the same result for the second case in 4), which proves 9).

To prove 10), consider the first case in 4), so that, for all $t > 0$, $\ddot\varphi(t) < 0$. It thus follows from (32) that

$$[\dot\varphi(t) + \varphi(t)]^2 \geq \mu + \varphi^2(t),$$

which implies that

$$\dot\varphi(t) \geq \sqrt{\mu + \varphi^2(t)} - \varphi(t) = \frac{\mu}{\sqrt{\mu + \varphi^2(t)} + \varphi(t)} \geq \frac{\mu}{2\varphi(t) + \sqrt{\mu}}.$$

Therefore,

$$(2\varphi(t) + \sqrt{\mu})\dot\varphi(t) \geq \mu.$$

Integrating yields

$$\left(\varphi(t) + \frac{\sqrt{\mu}}{2}\right)^2 \geq \mu t + \left(\varphi(0) + \frac{\sqrt{\mu}}{2}\right)^2 \geq \mu t + \frac{\mu}{4},$$

which implies

$$\varphi(t) \geq \sqrt{\mu t + \frac{\mu}{4}} - \frac{\sqrt{\mu}}{2}.$$

Combining this with 9) yields

$$\lim_{t\to\infty}\frac{\varphi(t)}{\sqrt{\mu t}} = 1.$$

A similar argument yields the same result for the second case in 4), which proves 10). □

Figure 8 illustrates unbounded PI control of vertical slip for viscous and Coulomb friction. As shown in Theorem 1, the slip distance is infinite despite the fact that the normal force is unbounded. Finally, Figure 9 illustrates both cases in 2) of Theorem 1.

CONCLUSIONS AND OPEN QUESTIONS

This article showed that, in the presence of viscous friction, finite slip distance is not achievable under bounded normal force by any control law. Additionally, allowing unbounded normal force, finite slip distance is not achievable under PI control. In both cases, the control law is unable to reject the force due to gravity, which constitutes a step disturbance. Although extension to proportional-integral-derivative control was not considered, numerical experiments (not shown) suggest that the addition of derivative action provides no additional benefit in terms of slip distance.

These properties are a consequence of the gripper model, where the normal force multiplies the velocity analogously with the viscous friction coefficient. Since standard linear techniques are not applicable, these results were obtained by analyzing the nonlinear closed-loop dynamics.

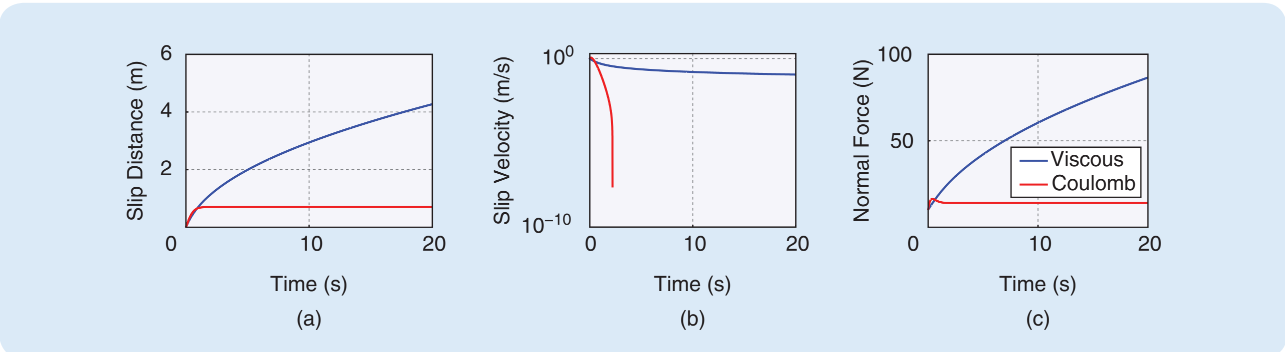

FIGURE 8 Unbounded proportional-integral control of the gripper model (2) for vertical slip with the viscous and Coulomb friction models using $K_p = 10$ N-s/m and $K_i = 20$ N/m. The parameters of both friction models and the initial velocities are chosen as in Figure 2. (a) The Coulomb friction yields finite slip distance, whereas, for viscous friction, the slip distance is infinite, despite the fact that the normal force is unbounded. (b) For Coulomb friction, finite slip velocity is reached in finite time. (c) For viscous friction, however, the slip velocity converges to zero but not sufficiently fast to achieve finite slip distance, despite the unbounded normal force.

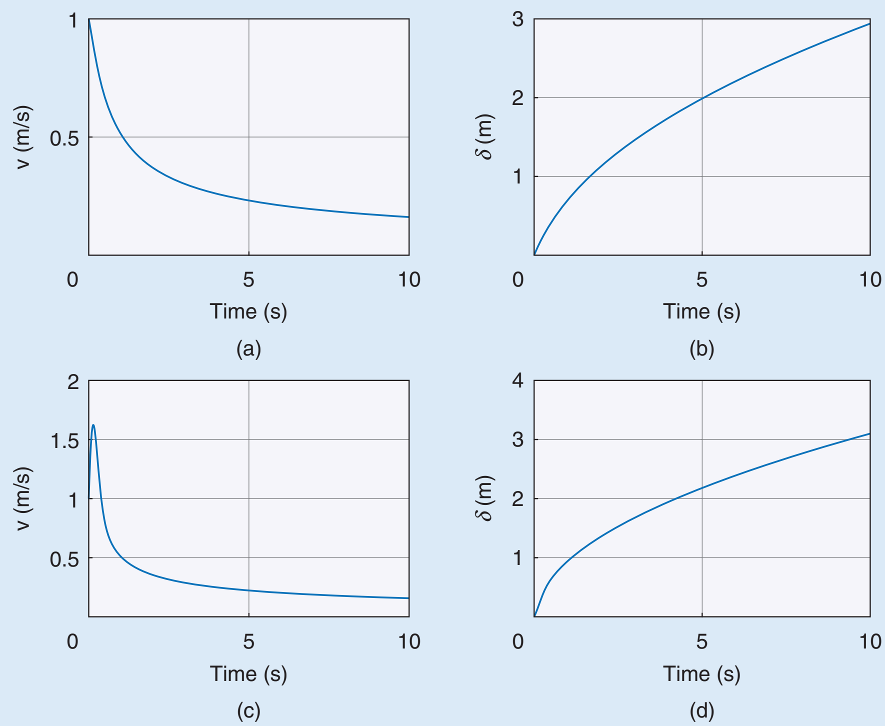

FIGURE 9 Proportional-integral control of the gripper model (2) for vertical slip with the viscous friction model and $K_i = 20$ N/m. (a) With $K_p = 10$ N-s/m, which satisfies $mg < cv_0 + c_n K_P v_0^2$, the slip velocity v is decreasing on $(0, \infty)$. (b) The corresponding slip distance is shown. (c) With $K_p = 1$ N-s/m, which satisfies $mg \geq cv_0 + c_n K_P v_0^2$, the slip velocity v is increasing for all $t \in (0, 0.143)$ and decreasing for all $t \in (0.143, \infty)$. (d) The corresponding slip distance is shown. These two cases illustrate statement 2 of Theorem 1.

The inability to achieve finite slip distance with viscous friction under PI control with unbounded normal force does not imply that it is impossible to achieve finite slip distance under unbounded control. In fact, there may exist nonlinear control laws (such as sliding mode controllers [5]) that meet this objective. However, such control laws would not be useful in practice because they would require infinite normal force. Nevertheless, the form of such a nonlinear control may suggest control laws with improved performance under bounded control for objects that do not have purely viscous friction. The development of such nonlinear control laws is left for future research.

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Recursive Least Squares for Real-Time Implementation

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Many estimation and control problems involve a process of the form

$$y_k = \phi_k \theta, \tag{1}$$

where $k = 0, 1, 2, \ldots$ is the discrete-time step corresponding to the continuous-time step size T_s, the scalar or vector $y_k \in \mathbb{R}^p$ is the measurement at step k, the matrix $\phi_k \in \mathbb{R}^{p\times n}$ is the regressor at step k whose entries consist of current and past data, and $\theta \in \mathbb{R}^n$ is a column vector of n unknown parameters. The objective is to use y_k and ϕ_k to estimate the components of θ. In applications, y_k and ϕ_k are corrupted by noise, and thus (1) does not hold exactly. This motivates the need for the least squares estimates of θ given below.

The measurements y_k and the data in ϕ_k are typically obtained from a continuous-time process and, as such, are available at the sample times kT_s, where T_s is the sample interval. The batch approach to this problem is to collect a large amount of data and then apply least squares optimization to the collected data to compute an estimate of θ. In particular, collecting data over the time window $i = 0, \ldots, k$, it follows from (1) that

$$\Upsilon = \Phi\theta, \tag{2}$$

where

$$\Upsilon \triangleq \begin{bmatrix} y_0 \\ \vdots \\ y_k \end{bmatrix}, \quad \Phi \triangleq \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_k \end{bmatrix}. \tag{3}$$

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Note that (2) has the form $Ax = b$, where A denotes Φ, x denotes θ, and b denotes Υ.

In the presence of noise corrupting the data Υ and Φ, (2) may not have a solution. In this case, it is useful to replace (2) by a least squares optimization problem of the form

$$\begin{aligned} J_k(\hat\theta) &\triangleq \sum_{i=0}^{k} (y_i - \phi_i\hat\theta)^{\mathrm T}(y_i - \phi_i\hat\theta) + (\hat\theta - \theta_0)^{\mathrm T} R(\hat\theta - \theta_0) \\ &= (\Upsilon - \Phi\hat\theta)^{\mathrm T}(\Upsilon - \Phi\hat\theta) + (\hat\theta - \theta_0)^{\mathrm T} R(\hat\theta - \theta_0), \end{aligned} \tag{4}$$

where R is a positive semidefinite (and thus, by definition, symmetric) matrix, and θ_0 is an initial estimate of θ. Assuming that R is chosen such that the inverse in (5) exists, the regularization term $(\hat\theta - \theta_0)^{\mathrm T} R(\hat\theta - \theta_0)$ weights the initial estimate and ensures that J_k has a unique global minimizer. In particular, the *batch least squares* (*BLS*) minimizer of (4) is given by

$$\theta_{\mathrm{opt},R} = (\Phi^{\mathrm T}\Phi + R)^{-1}(\Phi^{\mathrm T}\Upsilon + R\theta_0). \tag{5}$$

Note that the inverse required to compute (5) is of size $n \times n$, and thus the computational requirement of the inverse is of order n^3. In addition to the inverse, three matrix multiplications are needed. Note also that the memory needed to store Φ grows with k. Furthermore, if Φ has full column rank, then R can be set to zero, and thus (5) becomes

$$\theta_{\mathrm{opt},0} = (\Phi^{\mathrm T}\Phi)^{-1}\Phi^{\mathrm T}\Upsilon. \tag{6}$$

In the case where (2) has a solution and Φ has full column rank, (6) is the unique solution of (2).

In many applications, computational speed and memory are limited. One way to alleviate these requirements is

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