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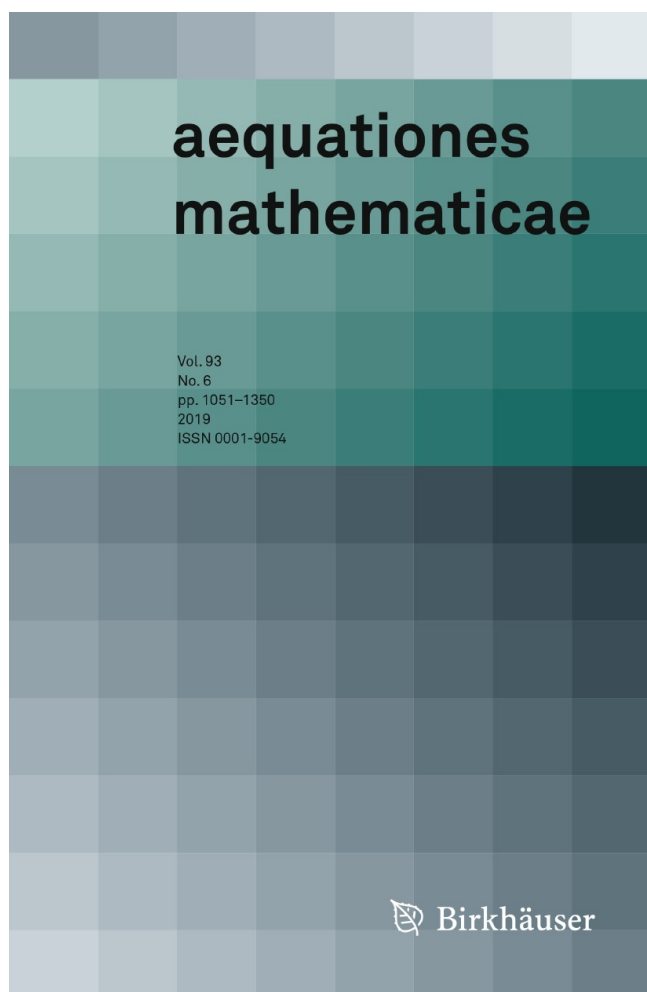
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Counting colorful necklaces and bracelets in three colors

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Abstract. A necklace or bracelet is *colorful* if no pair of adjacent beads are the same color. In addition, two necklaces are *equivalent* if one results from the other by permuting its colors, and two bracelets are *equivalent* if one results from the other by either permuting its colors or reversing the order of the beads; a bracelet is thus a necklace that can be turned over. This note counts the number $K(n)$ of non-equivalent colorful necklaces and the number $K'(n)$ of colorful bracelets formed with n -beads in at most three colors. Expressions obtained for $K'(n)$ simplify expressions given by OEIS sequence A114438, while the expressions given for $K(n)$ appear to be new and are not included in OEIS.

Mathematics Subject Classification. 05A05.

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1. Introduction

A *necklace* with n beads and c colors is an n -tuple, each of whose components can assume one of c values, where not all c colors need appear. Two necklaces are *equivalent* if one results from the other by either rotating it cyclically or permuting its colors. The classical necklace problem asks to determine the number of non-equivalent necklaces formed with n beads of c colors. The answer to this problem is given by

$$N(n, c) = \frac{1}{n} \sum_{d|n} \varphi(d) c^{n/d}, \quad (1.1)$$

where φ is the Euler totient function.

A *bracelet* with n beads and c colors is a necklace of n beads and c colors that can be turned over, and thus the order of its beads is reversed. The number of non-equivalent bracelets with n beads of c colors is given by

$$N'(n, c) = \frac{N(n, c) + R(n, c)}{2}, \quad (1.2)$$

where

$$R(n, c) = \begin{cases} c^{(n+1)/2} & \text{if } n \text{ is odd,} \\ \frac{1+c}{2}c^{n/2} & \text{if } n \text{ is even.} \end{cases} \tag{1.3}$$

As expected, $N'(n, c) \leq N(n, c)$. These results and further details are given in [9, 10] and the references therein.

In this work we consider a variation on this problem that arose from considering sequences of n coordinate-axis rotations defined by Euler angles, where $n = 7$ for aircraft [1]. For this problem, it is of interest to count the number of distinct coordinate-axis rotation sequences of length n that are closed in the sense of transforming the starting frame by a sequence of coordinate-axis rotations that lead back to the starting frame [2]. A pair of successive coordinate-axis rotations around the same axis can be combined into a single rotation, and the labeling of the axes of the starting frame is arbitrary. Counting the number of closed sequences consisting of n coordinate-axis rotations is thus equivalent to counting necklaces in 3 colors, where each color corresponds to an axis label. Furthermore, reversing a sequence of coordinate-axis rotations is equivalent to replacing each Euler angle in the sequence of coordinate-axis rotations by its negative. Hence, for the purpose of determining all feasible Euler angles for each closed sequence of coordinate-axis rotations, it suffices to count bracelets.

Motivated by the fact that successive coordinate-axis rotations around the same axis can be merged, the present paper considers colorful necklaces and bracelets formed with n -beads of three colors, where a necklace or bracelet is *colorful* if no pair of adjacent beads have the same color. Two colorful necklaces are *equivalent* if one results from the other by permuting its colors, and two colorful bracelets are *equivalent* if one results from the other by either permuting its colors or reversing the order of the beads; a colorful bracelet is thus a colorful necklace that can be turned over. In fact, the number of colorful bracelets with n beads in three colors appears in the On-line Encyclopedia of Integer Sequences (OEIS) as sequence [A114438](#), which is the “Number of Barlow packings that repeat after n (or a divisor of n) layers.” The provided references indicate that the problem of studying this sequence originates in crystallography [3, 5, 8].

The contribution of the present paper is twofold. First, we provide explicit formulas for the number of color bracelets that simplify those given by $P'(n)$ in [5, p. 272]. Furthermore, we provide expressions for the number of colorful necklaces with n beads in three colors; this sequence is currently unknown to OEIS.

2. n -periodic sequences

Instead of working with necklaces and bracelets of n beads we work with n -periodic sequences. To fix notation, let \mathbb{N}_q denote the set of natural numbers from 1 to q . In particular, \mathbb{N}_3 represents the set of the three colors under consideration.

Definition 2.1. For a positive integer n , a *colorful n -periodic sequence* is a function $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{N}_3$ defined on the set of integers modulo n , such that $f(i) \neq f(i + 1)$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. The set of colorful n -periodic sequences is denoted by \mathcal{A}_n .

The set \mathcal{A}_n represents the colorful n -bead necklaces or bracelets of three colors.

Next, we consider the permutations r and s on $\mathbb{Z}/n\mathbb{Z}$ defined by $s(i) = i + 1$ and $r(i) = -i$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. The group generated by s is

$$\langle s \rangle = \{id, s, s^2, \dots, s^{n-1}\},$$

which is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. The group generated by r is $\langle r \rangle = \{id, r\} \simeq \mathbb{Z}/2\mathbb{Z}$. Furthermore, since $rsr = s^{-1}$, the group generated by r and s is

$$\langle s, r \rangle = \{id, s, s^2, \dots, s^{n-1}, r, rs, rs^2, \dots, rs^{n-1}\},$$

which is isomorphic to the dihedral group D_n of order $2n$.

We also consider the symmetric group \mathfrak{S}_3 of \mathbb{N}_3 , where \mathfrak{S}_3 consists of the identity id , the three substitutions τ_{12} , τ_{13} , and τ_{23} (with τ_{ij} representing the 2-cycle (i, j)), and the two 3-cycles $c = (1, 2, 3)$ and $c^2 = c^{-1} = (1, 3, 2)$. We recall that two permutations σ and σ' are *conjugates* if there exists a permutation τ such that $\sigma' = \tau^{-1} \circ \sigma \circ \tau$; this is equivalent to the fact that σ and σ' have the same cyclic decomposition. Hence, for \mathfrak{S}_3 all transpositions are conjugates and all 3-cycles are conjugates, which can be checked directly.

The group acting on the elements of \mathcal{A}_n , considered as colorful n -bead necklaces, is $G = \mathfrak{S}_3 \times \langle s \rangle$, with group action defined by $\gamma f = \sigma \circ f \circ t^{-1}$ for $\gamma = (\sigma, t) \in G$ and $f \in \mathcal{A}_n$. Similarly, the group acting on the elements of \mathcal{A}_n , considered as colorful n -bead bracelets, is $G' = \mathfrak{S}_3 \times \langle s, r \rangle$, with group action defined again by $\gamma f = \sigma \circ f \circ t^{-1}$ for $\gamma = (\sigma, t) \in G'$ and $f \in \mathcal{A}_n$.

The action of G on \mathcal{A}_n defines an equivalence relation \sim_G on colorful n -bead necklaces given by

$$f \sim_G g \iff \text{there exists } \gamma \in G \text{ such that } \gamma f = g.$$

The set of equivalence classes of this relation is denoted by \mathcal{A}_n/G , and the desired number of non-equivalent colorful n -bead necklaces is exactly $K(n) = |\mathcal{A}_n/G|$. Similarly, $K'(n) = |\mathcal{A}_n/G'|$, where \mathcal{A}_n/G' is the set of equivalence classes of colorful n -bead bracelets defined by the action of G' on \mathcal{A}_n .

The basic tool in this investigation is the classical Burnside's Lemma [7, Theorem 3.22]. (While this lemma bears the name of Burnside, it seems that

it was well-known to Frobenius (1887) and before him to Cauchy (1845). An account on the history of this lemma can be found in [9], see also [11]).

Theorem 2.2. (Burnside's Lemma) *Let G be a finite group acting on a finite set X . Then*

$$|X/G| = \frac{1}{|G|} \sum_{\gamma \in G} |\mathfrak{F}(\gamma)|,$$

where $\mathfrak{F}(\gamma)$ is the set of elements $x \in X$ fixed by γ (i.e. $\gamma x = x$.)

Noting that G is a subgroup of G' , our task is to determine the numbers

$$f^{(n)}(\sigma, \varepsilon, j) = \left| \mathfrak{F}^{(n)}(\sigma, \varepsilon, j) \right| \tag{2.1}$$

with $\sigma \in \mathfrak{S}_3$, $\varepsilon \in \{0, 1\}$, and $j \in \{0, \dots, n - 1\}$, where

$$\mathfrak{F}^{(n)}(\sigma, \varepsilon, j) = \{f \in \mathcal{A}_n : f = \sigma \circ f \circ r^\varepsilon \circ s^j\}. \tag{2.2}$$

This paper is organized as follows: In Sect. 3, we gather some useful properties and lemmas. In Sect. 4 the case of necklaces is considered. Finally, in Sect. 5 we consider the case of bracelets.

3. Useful properties and lemmas

Lemma 3.1. *If n and m are positive integers, then $\mathcal{A}_n \cap \mathcal{A}_m = \mathcal{A}_{\gcd(n,m)}$.*

Proof. This result follows from the fact that $n\mathbb{Z} + m\mathbb{Z} = \gcd(n, m)\mathbb{Z}$. □

Our first step is to determine the number of n -periodic colorful sequences, that is $\alpha_n \stackrel{\text{def}}{=} |\mathcal{A}_n|$. This is the object of the next proposition.

Proposition 3.2. *For all $n \geq 1$,*

$$\alpha_n = |\mathcal{A}_n| = 2^n + 2(-1)^n. \tag{3.1}$$

Proof. Note that $\alpha_1 = 0$ and $\alpha_2 = 6$. Suppose that $n \geq 3$, and define

$$\begin{aligned} \mathcal{A}'_n &= \{f \in \mathcal{A}_n : f(n-1) \neq f(1)\}, \\ \mathcal{A}''_n &= \{f \in \mathcal{A}_n : f(n-1) = f(1)\}. \end{aligned}$$

The mapping $\mathcal{A}'_n \rightarrow \mathcal{A}_{n-1} : f \mapsto \tilde{f}$ with \tilde{f} defined by $\tilde{f}|_{\mathbb{N}_{n-1}} = f|_{\mathbb{N}_{n-1}}$ is bijective because $f(n)$ is uniquely defined by the knowledge of $f(n-1)$ and $f(1)$, indeed $\{f(1), f(n-1), f(n)\} = \mathbb{N}_3$. Hence $|\mathcal{A}'_n| = |\mathcal{A}_{n-1}| = \alpha_{n-1}$.

Also, the mapping $\mathcal{A}''_n \rightarrow \mathcal{A}_{n-2} : f \mapsto \hat{f}$ with \hat{f} defined by $\hat{f}|_{\mathbb{N}_{n-2}} = f|_{\mathbb{N}_{n-2}}$ is surjective and the pre-image of each $g \in \mathcal{A}_{n-2}$ consists of exactly two elements, namely f_1 and f_2 defined by $f_1|_{\mathbb{N}_{n-2}} = f_2|_{\mathbb{N}_{n-2}} = g|_{\mathbb{N}_{n-2}}$, $f_1(n) = \min(\mathbb{N}_3 \setminus \{f(1)\})$ and $f_2(n) = \max(\mathbb{N}_3 \setminus \{f(1)\})$. Hence $|\mathcal{A}''_n| = 2|\mathcal{A}_{n-2}| = 2\alpha_{n-2}$. But $\{\mathcal{A}'_n, \mathcal{A}''_n\}$ is a partition of \mathcal{A}_n , so

$$\alpha_n = |\mathcal{A}_n| = |\mathcal{A}'_n| + |\mathcal{A}''_n| = \alpha_{n-1} + 2\alpha_{n-2},$$

and the desired conclusion follows by induction. □

In particular, since the neutral element of G (or G') fixes the whole set \mathcal{A}_n , the next corollary is immediate.

Corollary 3.3.

$$f^{(n)}(id, 0, 0) = \alpha_n. \tag{3.2}$$

Corollary 3.4. *For distinct $i, j \in \mathbb{N}_3$, let $\mathcal{A}_n^{i \cdot j}$ denote the subset of \mathcal{A}_n consisting of functions f satisfying $f(1) = i$ and $f(n) = j$. Then*

$$|\mathcal{A}_n^{i \cdot j}| = \frac{\alpha_n}{6}. \tag{3.3}$$

Proof. Given i and j , there is a unique permutation $\sigma \in \mathfrak{S}_3$ such that $\sigma(i) = 1$ and $\sigma(j) = 2$, and with this σ the mapping $f \mapsto \sigma \circ f$ defines a bijection between $\mathcal{A}_n^{i \cdot j}$ and $\mathcal{A}_n^{1 \cdot 2}$. Thus

$$|\mathcal{A}_n^{i \cdot j}| = |\mathcal{A}_n^{1 \cdot 2}|.$$

The conclusion follows since $\{\mathcal{A}_n^{1 \cdot 2}, \mathcal{A}_n^{1 \cdot 3}, \mathcal{A}_n^{2 \cdot 1}, \mathcal{A}_n^{2 \cdot 3}, \mathcal{A}_n^{3 \cdot 1}, \mathcal{A}_n^{3 \cdot 2}\}$ constitutes a partition of \mathcal{A}_n . □

The next lemma helps to reduce the number of cases to be considered. The proof is immediate and left to the reader.

Lemma 3.5. (Reduction) *Suppose that a group G acts on a set X , and consider two elements g and g' from G . If there is $h \in G$ such that $g' = h^{-1}gh$, then the mapping $x \mapsto hx$ defines a bijection from $\mathfrak{F}(g')$ onto $\mathfrak{F}(g)$. In particular, if X and G are finite and if g and g' are conjugate elements from G then $|\mathfrak{F}(g)| = |\mathfrak{F}(g')|$.*

Remark 3.6. Another simple remark from group theory is that if $G = A \times B$ is the direct product of two groups A and B , and if a and a' are conjugate elements from A , then (a, e_B) and (a', e_B) , (with e_B denoting the neutral element of B), are also conjugate elements in G .

With Lemma 3.5 and Remark 3.6 at hand, the next corollary is immediate:

Corollary 3.7.

(a) *For all $\varepsilon \in \{0, 1\}$ and all $\ell \in \{0, 1, \dots, n - 1\}$ we have*

$$f^{(n)}(\tau_{12}, \varepsilon, \ell) = f^{(n)}(\tau_{13}, \varepsilon, \ell) = f^{(n)}(\tau_{23}, \varepsilon, \ell). \tag{3.4}$$

(b) *For all $\varepsilon \in \{0, 1\}$ and all $\ell \in \{0, 1, \dots, n - 1\}$ we have*

$$f^{(n)}(c, \varepsilon, \ell) = f^{(n)}(c^2, \varepsilon, \ell). \tag{3.5}$$

(c) *For all $\sigma \in \mathfrak{S}_3$ and all $\ell \in \{0, 1, \dots, \lfloor (n - 1)/2 \rfloor\}$ we have*

$$f^{(n)}(\sigma, 1, 2\ell) = f^{(n)}(\sigma, 1, 0). \tag{3.6}$$

(d) For all $\sigma \in \mathfrak{S}_3$ and all $\ell \in \{0, 1, \dots, \lfloor n/2 - 1 \rfloor\}$ we have

$$f^{(n)}(\sigma, 1, 2\ell + 1) = f^{(n)}(\sigma, 1, 1). \tag{3.7}$$

Proof. Both (a) and (b) follow from the fact that all permutations of the same cycle structure are conjugate. On the other hand, since $rs^{2\ell} = s^{-\ell}rs^\ell$ and $rs^{2\ell+1} = s^{-\ell}(rs)s^\ell$ for all ℓ , both (c) and (d) follow from Corollary 3.7. \square

The final result in this preliminary section is a simple formula concerning sums involving Euler's totient function φ (see [4, Chapter V, Section 5.5]), recall that $\varphi(n)$ is the number of integers in \mathbb{N}_n coprime to n .

Lemma 3.8. *For every positive integer n we have*

$$\sum_{d|n} (-1)^d \varphi\left(\frac{n}{d}\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If n is odd then all its divisors are odd and using [4, Theorem 63], we get

$$\sum_{d|n} (-1)^d \varphi\left(\frac{n}{d}\right) = -\sum_{d|n} \varphi\left(\frac{n}{d}\right) = -n.$$

Now, if $n = 2m$ for some positive integer m , then

$$\begin{aligned} \sum_{d|n} (-1)^d \varphi\left(\frac{n}{d}\right) &= \sum_{\substack{d|n \\ d \text{ is even}}} \varphi\left(\frac{n}{d}\right) - \sum_{\substack{d|n \\ d \text{ is odd}}} \varphi\left(\frac{n}{d}\right) \\ &= 2 \sum_{\substack{d|n \\ d \text{ is even}}} \varphi\left(\frac{n}{d}\right) - \sum_{d|n} \varphi\left(\frac{n}{d}\right) \\ &= 2 \sum_{d'|m} \varphi\left(\frac{m}{d'}\right) - \sum_{d|n} \varphi\left(\frac{n}{d}\right) \\ &= 2m - n = 0, \end{aligned}$$

where we used again [4, Theorem 63]. \square

4. Counting colorful necklaces

In this section we consider $G = \mathfrak{S}_3 \times \langle s \rangle$. According to Corollary 3.7 we need to determine $f^{(n)}(\sigma, 0, \ell)$, for $\sigma \in \{id, \tau_{12}, c\}$ and $\ell \in \mathbb{Z}/n\mathbb{Z}$. The next proposition gives the answer.

Proposition 4.1.

(a) If $\gcd(\ell, n) = d$, then $\mathfrak{F}^{(n)}(id, 0, \ell) = \mathfrak{F}^{(n)}(id, 0, d) = \mathcal{A}_d$. In particular,

$$f^{(n)}(id, 0, \ell) = \alpha_{\gcd(\ell, n)}. \tag{4.1}$$

Thus, $\gcd(n, \ell) = 1$ implies $\mathfrak{F}^{(n)}(id, 0, \ell) = \emptyset$.

(b) *i.* Suppose that $3 \nmid n$, then for all $\ell \in \mathbb{Z}/n\mathbb{Z}$ we have

$$f^{(n)}(c, 0, \ell) = 0. \tag{4.2}$$

ii. Suppose that $n = 3m$ for some positive integer m , then for all $\ell \in \mathbb{Z}/n\mathbb{Z}$ we have

$$f^{(n)}(c, 0, \ell) = \begin{cases} 0 & \text{if } 3 \mid (\ell/d), \\ 2^d - (-1)^d & \text{if } 3 \nmid (\ell/d), \end{cases} \tag{4.3}$$

where $d = \gcd(m, \ell)$.

(c) *i.* Suppose that $2 \nmid n$, then for all $\ell \in \mathbb{Z}/n\mathbb{Z}$ we have

$$f^{(n)}(\tau_{12}, 0, \ell) = 0. \tag{4.4}$$

ii. Suppose that $n = 2m$ for some positive integer m , then for all $\ell \in \mathbb{Z}/n\mathbb{Z}$ we have

$$f^{(n)}(\tau_{12}, 0, \ell) = \begin{cases} 0 & \text{if } 2 \mid (\ell/d), \\ 2^d & \text{if } 2 \nmid (\ell/d), \end{cases} \tag{4.5}$$

where $d = \gcd(m, \ell)$.

Proof. (a) A sequence $f \in \mathfrak{F}^{(n)}((id, 0, \ell))$ satisfies $f \circ s^\ell = f$, so it belongs to \mathcal{A}_ℓ . Thus, by Lemma 3.1, we have

$$\mathfrak{F}^{(n)}((id, 0, \ell)) \subset \mathcal{A}_n \cap \mathcal{A}_\ell = \mathcal{A}_d.$$

The converse inclusion: $\mathcal{A}_d \subset \mathfrak{F}^{(n)}((id, 0, \ell))$ is trivial, because both ℓ and n are multiples of d .

(b,c) *i.* Let σ be any permutation from \mathfrak{S}_3 , and suppose that $\mathfrak{F}^{(n)}(\sigma, 0, \ell) \neq \emptyset$ so there is $f \in \mathfrak{F}^{(n)}(\sigma, 0, \ell)$. From

$$f \circ s^\ell = \sigma^{-1} \circ f, \tag{4.6}$$

we conclude by an easy induction that for all integers p we have

$$f \circ s^{p\ell} = \sigma^{-p} \circ f. \tag{4.7}$$

- If $\sigma = c$ and $3 \nmid n$, we have $c^3 = id$, so (4.7) implies that $f \circ s^{3p\ell} = f$ for all integers p . But, because $3 \nmid n$ there is $r \in \{1, 2\}$ such that $n - r = 3p$ for some p . Consequently, $f \circ s^{(n-r)\ell} = f$, or equivalently $f = f \circ s^{r\ell} = c^{-r} \circ f$ because f is n -periodic. This is a contradiction because neither c nor c^2 has fixed points. Thus $\mathfrak{F}^{(n)}(c, 0, \ell) = \emptyset$. This proves (b) *i.*
- If $\sigma = \tau_{12}$ and n is odd, we have $\tau_{12}^2 = id$, so (4.7) implies that $f \circ s^{2p\ell} = f$ for all integers p . But, because $n = 2p + 1$ for some p we conclude that $f \circ s^{(n-1)\ell} = f$, or equivalently $f = f \circ s^\ell = \tau_{12} \circ f$. This is a contradiction because f takes two different values, and τ_{12} has only one fixed point. Thus $\mathfrak{F}^{(n)}(\tau_{12}, 0, \ell) = \emptyset$. This proves (c) *i.*

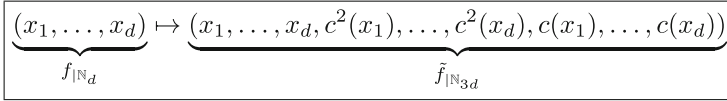


FIGURE 1. The bijection $\Phi : \mathcal{A}_{d+1}^{1..3} \cup \mathcal{A}_{d+1}^{2..1} \cup \mathcal{A}_{d+1}^{3..2} \rightarrow \mathfrak{F}^{(3d)}(c, 0, d)$

(b) *ii.* Assume that $\mathfrak{F}^{(n)}(c, 0, \ell) \neq \emptyset$ and consider $f \in \mathfrak{F}^{(n)}(c, 0, \ell)$. From $c \circ f \circ s^\ell = f$ we conclude that $f \circ s^{3\ell} = f$. Thus $f \in \mathcal{A}_{3j} \cap \mathcal{A}_{3m} = \mathcal{A}_{3d}$, with $d = \gcd(m, \ell)$. Further, if $\ell/d = 3q + r$ with $r \in \{0, 1, 2\}$, then $\ell = 3dq + dr$ and consequently

$$f \circ s^\ell = f \circ s^{rd} = c^2 \circ f. \tag{4.8}$$

- If $r = 0$, then (4.8) implies $f = c^2 \circ f$, which is impossible since f is not constant. Thus $\mathfrak{F}^{(n)}(c, 0, \ell) = \emptyset$ in this case.
- If $r = 1$, then (4.8) shows that $f \in \mathfrak{F}^{(3d)}(c, 0, d)$. Conversely, it is easy to check that any $f \in \mathfrak{F}^{(3d)}(c, 0, d)$ belongs to $\mathfrak{F}^{(n)}(c, 0, \ell)$. Thus, we have shown that

$$\mathfrak{F}^{(n)}(c, 0, \ell) = \mathfrak{F}^{(3d)}(c, 0, d). \tag{4.9}$$

Now, when f belongs to $\mathfrak{F}^{(3d)}(c, 0, d)$ it is completely determined by its restriction to \mathbb{N}_d , and the mapping (see Fig. 1):

$$\Phi : \mathcal{A}_{d+1}^{1..3} \cup \mathcal{A}_{d+1}^{2..1} \cup \mathcal{A}_{d+1}^{3..2} \rightarrow \mathfrak{F}^{(3d)}(c, 0, d), f \mapsto \tilde{f},$$

where \tilde{f} is the unique sequence from $\mathfrak{F}^{(3d)}(c, 0, d)$ which coincides with f on \mathbb{N}_d , is a bijection.

We conclude, according to Corollary 3.4 that

$$f^{(3d)}(c, 0, d) = |\mathcal{A}_{d+1}^{1..3}| + |\mathcal{A}_{d+1}^{2..1}| + |\mathcal{A}_{d+1}^{3..2}| = \frac{\alpha_{d+1}}{2}.$$

- If $r = 2$, then a similar argument to the previous one (with c replaced by c^2), yields the desired conclusion. This completes the proof of (b) *ii.*

(c) *ii.* For simplicity we write τ for τ_{12} . Suppose that $\mathfrak{F}^{(n)}(\tau, 0, \ell) \neq \emptyset$ and consider $f \in \mathfrak{F}^{(n)}(\tau, 0, \ell)$. We have

$$f \circ s^\ell = \tau \circ f. \tag{4.10}$$

Hence $f \circ s^{2\ell} = f$ and consequently $f \in \mathcal{A}_{2\ell}$, which implies that $f \in \mathcal{A}_{2m} \cap \mathcal{A}_{2\ell} = \mathcal{A}_{2d}$ where $d = \gcd(m, \ell)$. Now write $\ell/d = 2q + r$ with $r \in \{0, 1\}$, then $\ell = 2dq + dr$ and consequently

$$f \circ s^\ell = f \circ s^{rd} = \tau \circ f. \tag{4.11}$$

- If $r = 0$, then (4.11) implies $f = \tau \circ f$, which is impossible since f is not constant. Thus $\mathfrak{F}^{(n)}(\tau, 0, \ell) = \emptyset$ in this case.

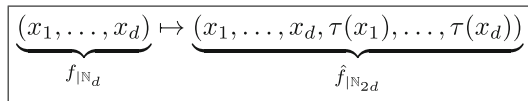


FIGURE 2. The bijection $\Psi : \mathcal{A}_{d+1}^{1\cdot 2} \cup \mathcal{A}_{d+1}^{2\cdot 1} \cup \mathcal{A}_d^{3\cdot 1} \cup \mathcal{A}_d^{3\cdot 2} \rightarrow \mathfrak{F}^{(2d)}(\tau, 0, d)$

- If $r = 1$, then (4.11) implies $f \circ s^d = \tau \circ f$, that is $f \in \mathfrak{F}^{(2d)}(\tau, 0, d)$. Conversely, it is easy to check that any $f \in \mathfrak{F}^{(2d)}(\tau, 0, d)$ belongs to $\mathfrak{F}^{(n)}(\tau, 0, \ell)$. Thus, we have shown that in this case

$$\mathfrak{F}^{(n)}(\tau, 0, \ell) = \mathfrak{F}^{(2d)}(\tau, 0, d). \tag{4.12}$$

Clearly, if $d = 1$, then $\mathfrak{F}^{(2d)}(\tau, 0, d)$ consists exactly of two elements: namely f_1 , defined by $f_1(1) = 1, f_1(2) = 2$, and $f_2 = \tau \circ f_1$. So,

$$f^{(2)}(\tau, 0, 1) = 2.$$

Now suppose that $d > 1$. Any $f \in \mathfrak{F}^{(2d)}(\tau, 0, d)$ is completely determined by its restriction to \mathbb{N}_d (note that $f(d)$ should be different from $\tau(f(1))$ and $f(d - 1)$), so considering the different possibilities for $f(1)$ we see that the mapping Ψ , (see Fig. 2):

$$\Psi : \mathcal{A}_{d+1}^{1\cdot 2} \cup \mathcal{A}_{d+1}^{2\cdot 1} \cup \mathcal{A}_d^{3\cdot 1} \cup \mathcal{A}_d^{3\cdot 2} \rightarrow \mathfrak{F}^{(2d)}(\tau, 0, d), f \mapsto \hat{f},$$

where \hat{f} is the unique sequence from $\mathfrak{F}^{(2d)}(\tau, 0, d)$ that coincides with f on \mathbb{N}_d , is a bijection. Thus, according to Corollary 3.4, we have

$$f^{(2d)}(\tau, 0, d) = \frac{\alpha_{d+1} + \alpha_d}{3} = 2^d.$$

This concludes the proof of (c) *ii*, in view of (4.12). □

The final step is to put all the pieces together to get the expression of K_n in terms of n using Burnside's Lemma.

Theorem 4.2. *The number of non-equivalent colorful n -bead necklaces with three colors is given by*

$$K(n) = \frac{1}{6n} \sum_{d|n} (1 + \mathbb{I}_{2\mathbb{Z} \setminus 3\mathbb{Z}}(d)) \gcd(d, 6) \varphi(d) 2^{n/d} - \frac{1}{3^{1+\nu_3(n)}} \mathbb{I}_{\mathbb{Z} \setminus 2\mathbb{Z}}(n) \tag{4.13}$$

$$= \left\lfloor \frac{1}{6n} \sum_{d|n} (1 + \mathbb{I}_{2\mathbb{Z} \setminus 3\mathbb{Z}}(d)) \gcd(d, 6) \varphi(d) 2^{n/d} \right\rfloor \tag{4.14}$$

where \mathbb{I}_X is the indicator function of the set X , (i.e. $\mathbb{I}_X(k) = 1$ if $k \in X$ and $\mathbb{I}_X(k) = 0$ if $k \notin X$), and $3^{\nu_3(n)}$ is the largest power of 3 dividing n .

Proof. According to Corollary 3.7 and Burnside's Lemma 2.2 we have

$$K(n) = \frac{A_n + 3B_n + 2C_n}{6n} \tag{4.15}$$

with

$$A_n = \sum_{\ell=0}^{n-1} f^{(n)}(id, 0, \ell), \tag{4.16}$$

$$B_n = \sum_{\ell=0}^{n-1} f^{(n)}(\tau_{12}, 0, \ell), \tag{4.17}$$

$$C_n = \sum_{\ell=0}^{n-1} f^{(n)}(c, 0, \ell). \tag{4.18}$$

Using part (a) of Proposition 4.1 we have

$$\begin{aligned} A_n &= \sum_{\ell=0}^{n-1} \alpha_{\gcd(\ell,n)} = \sum_{d|n} \alpha_d |\{\ell : 0 \leq \ell < n, \gcd(\ell, n) = d\}| \\ &= \sum_{d|n} \alpha_d \left| \left\{ \ell' : 0 \leq \ell' < \frac{n}{d}, \gcd\left(\ell', \frac{n}{d}\right) = 1 \right\} \right| = \sum_{d|n} \varphi\left(\frac{n}{d}\right) \alpha_d. \end{aligned}$$

Thus, using the expression of α_n from Proposition 3.2, we get

$$A_n = \sum_{d|n} (2^d + 2(-1)^d) \varphi\left(\frac{n}{d}\right). \tag{4.19}$$

Similarly, according to part (c) of Proposition 4.1 we know that $B_n = 0$ if n is odd, while we have the following when $n = 2m$:

$$\begin{aligned} B_n &= \sum_{\ell=0}^{n-1} 2^{\gcd(\ell,m)} \mathbb{I}_{\mathbb{Z} \setminus 2\mathbb{Z}}\left(\frac{\ell}{\gcd(\ell,m)}\right) \\ &= \sum_{d|m} 2^d |\{0 \leq \ell < 2m : \gcd(\ell, m) = d, \text{ and } \ell/d \text{ is odd}\}| \\ &= \sum_{d|m} 2^d \left| \left\{ 0 \leq \ell' < 2\frac{m}{d} : \gcd(\ell', \frac{m}{d}) = 1, \text{ and } \ell' \text{ is odd} \right\} \right| \\ &= \sum_{d|m} 2^d \left| \left\{ 0 \leq \ell' < 2\frac{m}{d} : \gcd(\ell', 2\frac{m}{d}) = 1 \right\} \right| = \sum_{d|m} 2^d \varphi\left(\frac{n}{d}\right). \end{aligned}$$

Finally we get

$$B_n = \mathbb{I}_{2\mathbb{Z}}(n) \sum_{d|(n/2)} 2^d \varphi\left(\frac{n}{d}\right). \tag{4.20}$$

Now, we come to C_n . According to part **(b)** of Proposition 4.1 we know that $C_n = 0$ if n is not a multiple of 3 while if $n = 3m$, we have

$$\begin{aligned}
 C_n &= \sum_{\ell=0}^{n-1} \left(2^{\gcd(\ell,m)} - (-1)^{\gcd(\ell,m)} \right) \mathbb{I}_{\mathbb{Z} \setminus 3\mathbb{Z}} \left(\frac{\ell}{\gcd(\ell,m)} \right) \\
 &= \sum_{d|m} (2^d - (-1)^d) |\{0 \leq \ell < 3m : \gcd(\ell,m) = d, \text{ and } 3 \nmid \ell/d\}| \\
 &= \sum_{d|m} (2^d - (-1)^d) |\{0 \leq \ell' < 3\frac{m}{d} : \gcd(\ell', \frac{m}{d}) = 1, \text{ and } 3 \nmid \ell'\}| \\
 &= \sum_{d|m} (2^d - (-1)^d) |\{0 \leq \ell' < 3\frac{m}{d} : \gcd(\ell', 3\frac{m}{d}) = 1\}| \\
 &= \sum_{d|m} (2^d - (-1)^d) \varphi \left(\frac{n}{d} \right).
 \end{aligned}$$

Thus,

$$C_n = \mathbb{I}_{3\mathbb{Z}}(n) \sum_{d|(n/3)} (2^d - (-1)^d) \varphi \left(\frac{n}{d} \right). \tag{4.21}$$

Replacing (4.19), (4.20) and (4.21) in (4.15) we get

$$K(n) = b_n + \varepsilon_n, \tag{4.22}$$

with

$$b_n = \frac{1}{6n} \left(\sum_{d|n} 2^d \varphi \left(\frac{n}{d} \right) + 3\mathbb{I}_{2\mathbb{Z}}(n) \sum_{d|(n/2)} 2^d \varphi \left(\frac{n}{d} \right) + 2\mathbb{I}_{3\mathbb{Z}}(n) \sum_{d|(n/3)} 2^d \varphi \left(\frac{n}{d} \right) \right) \tag{4.23}$$

and

$$\varepsilon_n = \frac{1}{3n} \left(\sum_{d|n} (-1)^d \varphi \left(\frac{n}{d} \right) - \mathbb{I}_{3\mathbb{Z}}(n) \sum_{d|(n/3)} (-1)^d \varphi \left(\frac{n}{d} \right) \right). \tag{4.24}$$

In order to reduce a little bit the expression of $K(n)n$ we use Lemma 3.8. Indeed, suppose that $n = 3^\nu m$ where $\nu = \nu_3(n)$ is the exponent of 3 in the prime factorization of n , thus $3 \nmid m$. Clearly if $\nu = 0$, then using Lemma 3.8 we get

$$\varepsilon_n = \frac{1}{3n} \sum_{d|n} (-1)^d \varphi \left(\frac{n}{d} \right) = -\frac{1}{3} \mathbb{I}_{\mathbb{Z} \setminus 2\mathbb{Z}}(n). \tag{4.25}$$

Now if $\nu > 0$, then

$$\begin{aligned} \varepsilon_n &= \frac{1}{3n} \left(\sum_{d|(3^\nu m)} (-1)^d \varphi\left(\frac{n}{d}\right) - \sum_{d|(3^{\nu-1}m)} (-1)^d \varphi\left(\frac{n}{d}\right) \right) \\ &= \frac{1}{3n} \sum_{d|(3^\nu m), d \nmid (3^{\nu-1}m)} (-1)^d \varphi\left(\frac{n}{d}\right) \\ &= \frac{1}{3n} \sum_{d=3^\nu q, q|m} (-1)^d \varphi\left(\frac{n}{d}\right) \\ &= \frac{1}{3n} \sum_{q|m} (-1)^{3^\nu q} \varphi\left(\frac{m}{q}\right) = \frac{1}{3n} \sum_{q|m} (-1)^q \varphi\left(\frac{m}{q}\right) \\ &= -\frac{m}{3n} \mathbb{I}_{\mathbb{Z} \setminus 2\mathbb{Z}}(m) = -\frac{1}{3^{1+\nu}} \mathbb{I}_{\mathbb{Z} \setminus 2\mathbb{Z}}(m). \end{aligned}$$

Finally, noting that $n = m \pmod 2$, we obtain the following formula for ε_n , which is also valid when $\nu = 0$ according to (4.25):

$$\varepsilon_n = -\frac{1}{3^{1+\nu_3(n)}} \mathbb{I}_{\mathbb{Z} \setminus 2\mathbb{Z}}(n). \tag{4.26}$$

Now note that b_n can be written as follows

$$b_n = \frac{1}{6n} \sum_{d|n} 2^d \lambda(n, d) \varphi\left(\frac{n}{d}\right) \tag{4.27}$$

with

$$\lambda(n, d) = 1 + 3J(n, d) + 2K(n, d)$$

where

$$J(n, d) = \begin{cases} 1 & \text{if } 2|n \text{ and } d|(n/2), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$K(n, d) = \begin{cases} 1 & \text{if } 3|n \text{ and } d|(n/3), \\ 0 & \text{otherwise,} \end{cases}$$

equivalently

$$J(n, d) = \mathbb{I}_{2\mathbb{Z}}\left(\frac{n}{d}\right), \quad \text{and} \quad K(n, d) = \mathbb{I}_{3\mathbb{Z}}\left(\frac{n}{d}\right).$$

Thus

$$\lambda(n, d) = 1 + 3\mathbb{I}_{2\mathbb{Z}}\left(\frac{n}{d}\right) + 2\mathbb{I}_{3\mathbb{Z}}\left(\frac{n}{d}\right). \tag{4.28}$$

So, we may write b_n in the following form

$$b_n = \frac{1}{6n} \sum_{d|n} 2^d \chi\left(\frac{n}{d}\right) \varphi\left(\frac{n}{d}\right) \tag{4.29}$$

with $\chi : \mathbb{Z} \rightarrow \mathbb{N}_6$ defined by

$$\chi(k) = (1 + 3\mathbb{I}_{2\mathbb{Z}}(k) + 2\mathbb{I}_{3\mathbb{Z}}(k)) = \begin{cases} 1 & \text{if } \gcd(k, 6) = 1, \\ 4 & \text{if } \gcd(k, 6) = 2, \\ 3 & \text{if } \gcd(k, 6) = 3, \\ 6 & \text{if } \gcd(k, 6) = 6. \end{cases} \quad (4.30)$$

This can also be written in the form $\chi(k) = (1 + \mathbb{I}_{2\mathbb{Z} \setminus 3\mathbb{Z}}(k)) \gcd(k, 6)$, and the announced expression (4.13) for $K(n)$ is obtained. Finally, the formula $K(n) = \lfloor b_n \rfloor$ follows from the fact that $-\frac{1}{3} \leq \varepsilon_n \leq 0$. □

We conclude our discussion of the case of necklaces by noting that there are some simple cases where the formula for $K(n)$ is particularly appealing. For example, if $n = p > 3$ is prime, then

$$K(p) = \frac{2^p - 2}{6p},$$

and if $\gcd(n, 6) = 1$, then

$$K(n) = \left\lfloor \frac{1}{6n} \sum_{d|n} \varphi(d) 2^{n/d} \right\rfloor = \frac{1}{6n} \sum_{d|n} \varphi(d) 2^{n/d} - \frac{1}{3}.$$

Remark 4.3. If 6 and n are coprime, then $K(n)$ is related to the number $N(n, 2)$ of n -bead necklaces of two colors (1.1) by the formula

$$K(n) = \lfloor N(n, 2)/6 \rfloor = (N(n, 2) - 2)/6.$$

Remark 4.4. An equivalent formula for $K(n)$ that does not use the indicator function of the set $2\mathbb{Z} \setminus 3\mathbb{Z}$ is the following

$$K(n) = \left\lfloor \frac{1}{6n} \sum_{d|n} \left(1 + \frac{4}{3} \cos^2 \left(\frac{d\pi}{2} \right) \sin^2 \left(\frac{d\pi}{3} \right) \right) \gcd(d, 6) \varphi(d) 2^{n/d} \right\rfloor.$$

Table 1 lists the first 40 terms of the sequence $(K(n))_{n \geq 1}$.

5. Counting colorful bracelets

As we explained before, bracelets are turnover necklaces. It is the action of the group $G' = \mathfrak{S}_3 \times \langle r, s \rangle$ on the set of n -periodic colorful sequences \mathcal{A}_n that is considered.

We are interested in the number of orbits \mathcal{A}_n/G' denoted by $K'(n)$. Again Burnside's Lemma comes to our rescue. We need to determine the numbers $f^{(n)}(\sigma, \varepsilon, \ell)$ with $\sigma \in \mathfrak{S}_3$, $\varepsilon \in \{0, 1\}$ and $\ell \in \mathbb{Z}/n\mathbb{Z}$, but we have already done this in the case $\varepsilon = 0$ in the previous section.

TABLE 1. List of $K(1), \dots, K(40)$, which counts colorful necklaces

n	$K(n)$	n	$K(n)$	n	$K(n)$	n	$K(n)$
1	0	11	31	21	16,651	31	11,545,611
2	1	12	64	22	31,838	32	22,371,000
3	1	13	105	23	60,787	33	43,383,571
4	2	14	202	24	116,640	34	84,217,616
5	1	15	367	25	223,697	35	163,617,805
6	4	16	696	26	430,396	36	318,150,720
7	3	17	1285	27	828,525	37	619,094,385
8	8	18	2452	28	1,598,228	38	1,205,614,054
9	11	19	4599	29	3,085,465	39	2,349,384,031
10	20	20	8776	30	5,966,000	40	4,581,315,968

Further, based on Corollary 3.7, we only need to determine $f^{(n)}(\sigma, 1, 0)$ and $f^{(n)}(\sigma, 1, 1)$ for σ in $\{id, \tau_{12}, c\}$. This is the object of the next proposition.

Proposition 5.1.

- (a) *i.* If n is odd, then $f^{(n)}(id, 1, 0) = 0$, otherwise $f^{(n)}(id, 1, 0) = 3 \times 2^{n/2}$.
- ii.* $f^{(n)}(id, 1, 1) = 0$.
- (b) *i.* $f^{(n)}(\tau_{12}, 1, 0) = \alpha_{\lfloor (n+1)/2 \rfloor} / 3$.
- ii.* $f^{(n)}(\tau_{12}, 1, 1) = \alpha_{\lfloor n/2 + 1 \rfloor} / 3$.
- (c) $f^{(n)}(c, 1, 0) = f^{(n)}(c, 1, 1) = 0$.

Proof. (a) Suppose that $\mathfrak{F}^{(n)}(id, 1, 0) \neq \emptyset$ and consider $f \in \mathfrak{F}^{(n)}(id, 1, 0)$. Write $n = 2m + t$ with $t \in \{0, 1\}$. Because $f(k) = f(-k) = f(n - k)$ for every k , we conclude by considering $k = m$ that $f(m + t) = f(m)$. But $f(m) \neq f(m + 1)$, so we must have $t = 0$ and $n = 2m$. Now, from the fact that $f(2m - k) = f(k)$ for every k we conclude that

$$\underbrace{(f(0), \dots, f(m), f(m + 1), \dots, f(2m - 1))}_{\text{a period of } n = 2m} = (f(0), \dots, f(m), f(m - 1), \dots, f(1)).$$

So, the mapping

$$f \mapsto (f(0), f(1), \dots, f(m))$$

defines a bijection between $\mathfrak{F}^{(n)}(id, 1, 0)$ and the set

$$\{(x_0, \dots, x_m) \in \mathbb{N}_3 : x_{i+1} \neq x_i, i = 0, \dots, m - 1\}.$$

Now, x_0 may take any one of three possible values and each other x_i has two possible values. So, the cardinality of this set is 3×2^m . Thus (a) *i.* is proved.

Now suppose that $\mathfrak{F}^{(n)}(id, 1, 1) \neq \emptyset$ and consider f from $\mathfrak{F}^{(n)}(id, 1, 1)$. We have $f(-k-1) = f(k)$ for every k , in particular, for $k = 0$ we get $f(-1) = f(0)$, which is absurd, and **(a) ii.** follows.

(b) i. We write τ for τ_{12} . Suppose that $\mathfrak{F}^{(n)}(\tau, 1, 0) \neq \emptyset$ and consider $f \in \mathfrak{F}^{(n)}(\tau, 1, 0)$. We have

$$\forall k \in \mathbb{Z}, \quad f(-k) = \tau(f(k)).$$

Taking $k = 0$ we get $f(0) = \tau(f(0))$, and this implies that $f(0) = 3$.

- If $n = 2m$, then $f(m) = f(m - n) = f(-m) = \tau(f(m))$ and consequently $f(m) = 3$. The restriction of f to the period $\{-m + 1, \dots, m - 1, m\}$ has the form

$$(\tau(f(m - 1)), \dots, \tau(f(1)), 3, f(1), \dots, f(m - 1), 3).$$

So, f is completely determined by the knowledge of $(f(1), \dots, f(m - 1))$ and consequently there is a bijection between $\mathfrak{F}^{(2m)}(\tau, 1, 0)$ and $\mathcal{A}_m^{3 \cdot 1} \cup \mathcal{A}_m^{3 \cdot 2}$. Thus, by Corollary 3.4, we have

$$f^{(n)}(\tau, 1, 0) = \frac{\alpha_m}{3} = \frac{1}{3} \alpha_{\lfloor (n+1)/2 \rfloor}.$$

- If $n = 2m + 1$, then $f(m + 1) = f(m + 1 - n) = f(-m) = \tau(f(m))$ and consequently $f(m) \neq 3$. The restriction of f to the period $\{-m, \dots, m\}$ takes the form

$$(\tau(f(m)), \tau(f(m - 1)), \dots, \tau(f(1)), 3, f(1), \dots, f(m)).$$

So, f is completely determined by the knowledge of $(f(1), \dots, f(m))$ and consequently there is a bijection between $\mathfrak{F}^{(2m+1)}(\tau, 1, 0)$ and $\mathcal{A}_{m+1}^{3 \cdot 1} \cup \mathcal{A}_{m+1}^{3 \cdot 2}$. Thus

$$f^{(n)}(\tau, 1, 0) = \frac{\alpha_{m+1}}{3} = \frac{1}{3} \alpha_{\lfloor (n+1)/2 \rfloor}.$$

(b) ii. Now suppose that $\mathfrak{F}^{(n)}(\tau, 1, 1) \neq \emptyset$ and consider $f \in \mathfrak{F}^{(n)}(\tau, 1, 1)$. We have

$$\forall k \in \mathbb{Z}, \quad f(-k - 1) = \tau(f(k)).$$

Taking $k = 0$ we get $f(-1) = \tau(f(0))$, but $f(-1) \neq f(0)$ thus $f(0) \in \{1, 2\}$.

- If $n = 2m$, then $f(m - 1) = f(m - 1 - n) = f(-m - 1) = \tau(f(m))$ but $f(m - 1) \neq f(m)$ thus $f(m - 1) \in \{1, 2\}$. The restriction of f to the period $\{-m, \dots, m - 1\}$ takes the form

$$(\tau(f(m - 1)), \dots, \tau(f(0)), f(0), f(1), \dots, f(m - 1)).$$

So, f is completely determined by the knowledge of $(f(0), \dots, f(m - 1))$. We can partition the set $\mathfrak{F}^{(2m)}(\tau, 1, 1)$ according to the values taken by

$(f(0), f(m - 1))$, and we have obvious bijective mappings:

$$\begin{aligned} \mathfrak{F}^{(2m)}(\tau, 1, 1) \cap \{f : f(0) = 1, f(m - 1) = 1\} &\rightarrow \mathcal{A}_{m-1}^{1 \cdot \cdot 2} \cup \mathcal{A}_{m-1}^{1 \cdot \cdot 3} \\ \mathfrak{F}^{(2m)}(\tau, 1, 1) \cap \{f : f(0) = 2, f(m - 1) = 2\} &\rightarrow \mathcal{A}_{m-1}^{2 \cdot \cdot 1} \cup \mathcal{A}_{m-1}^{2 \cdot \cdot 3} \\ \mathfrak{F}^{(2m)}(\tau, 1, 1) \cap \{f : f(0) = 1, f(m - 1) = 2\} &\rightarrow \mathcal{A}_{m-1}^{1 \cdot \cdot 2} \\ \mathfrak{F}^{(2m)}(\tau, 1, 1) \cap \{f : f(0) = 2, f(m - 1) = 1\} &\rightarrow \mathcal{A}_{m-1}^{2 \cdot \cdot 1}, \end{aligned}$$

Thus

$$f^{(n)}(\tau, 1, 1) = \frac{1}{6}(4\alpha_{m-1} + 2\alpha_m) = \frac{2}{3}(2^m - (-1)^m) = \frac{1}{3}\alpha_{\lfloor n/2+1 \rfloor}.$$

- If $n = 2m + 1$, then $f(m) = f(m - n) = f(-m - 1) = \tau(f(m))$ and consequently $f(m) = 3$. The restriction of f to the set $\{-m, \dots, m\}$ takes the form

$$(\tau(f(m - 1)), \dots, \tau(f(0)), f(0), f(1), \dots, f(m - 1), 3).$$

So, f is completely determined by the knowledge of $(f(0), \dots, f(m - 1))$ and there is an obvious bijective mapping between $\mathfrak{F}^{(2m+1)}(\tau, 1, 1)$ and $\mathcal{A}_{m+1}^{1 \cdot \cdot 3} \cup \mathcal{A}_{m+1}^{2 \cdot \cdot 3}$. Thus

$$f^{(n)}(\tau, 1, 1) = \frac{\alpha_{m+1}}{3} = \frac{1}{3}\alpha_{\lfloor n/2+1 \rfloor}.$$

This concludes the proof of part **(b)**.

(c) First, suppose that $\mathfrak{F}^{(n)}(c, 1, 0) \neq \emptyset$ and consider $f \in \mathfrak{F}^{(n)}(c, 1, 0)$. We have $f(k) = c(f(-k))$ for all $k \in \mathbb{Z}$. In particular, $f(0) = c(f(0))$, which is absurd because c has no fixed points.

Next suppose that $\mathfrak{F}^{(n)}(c, 1, 1) \neq \emptyset$ and consider $f \in \mathfrak{F}^{(n)}(c, 1, 1)$. We have $f(k) = c(f(-k - 1))$ for all $k \in \mathbb{Z}$.

- If $n = 2m + 1$, then

$$f(m) = f(m - n) = f(-m - 1) = c^{-1}(f(m)),$$

which is absurd because c has no fixed points.

- If $n = 2m$, then

$$\begin{aligned} f(m) &= f(m - n) = f(-m) = c^{-1}(f(m - 1)) \\ &= c^{-1}(f(m - 1 - n)) = c^{-1}(f(-m - 1)) \\ &= c^{-2}(f(m)), \end{aligned}$$

which is also absurd because c^2 has no fixed points.

This finishes the proof of the proposition. □

Finally we arrive at the main theorem of this section.

Theorem 5.2. *The number of non-equivalent colorful n -bead bracelets with three colors is given by*

$$K'(n) = \frac{K(n) + R(n)}{2} \tag{5.1}$$

with

$$R(n) = \begin{cases} 2^{n/2-1} & \text{if } n \text{ is even,} \\ \frac{1}{3}(2^{(n-1)/2} - (-1)^{(n-1)/2}) & \text{if } n \text{ is odd,} \end{cases} \tag{5.2}$$

where $K(n)$ is given by Theorem 4.2.

Proof. We only need to put things together. We know that

$$K'(n) = \frac{1}{12n} \left(\sum_{(\sigma,j) \in \mathfrak{S}_3 \times \mathbb{Z}/n\mathbb{Z}} f^{(n)}(\sigma, 0, j) + \sum_{(\sigma,j) \in \mathfrak{S}_3 \times \mathbb{Z}/n\mathbb{Z}} f^{(n)}(\sigma, 1, j) \right).$$

Thus $K'(n) = (K(n) + R(n))/2$ with

$$\begin{aligned} R(n) &= \frac{1}{6n} \sum_{(\sigma,j) \in \mathfrak{S}_3 \times \mathbb{Z}/n\mathbb{Z}} f^{(n)}(\sigma, 1, j) \\ &= \frac{1}{6n} \left(\sum_{\substack{\sigma \in \mathfrak{S}_3 \\ 0 \leq 2j \leq n-1}} f^{(n)}(\sigma, 1, 2j) + \sum_{\substack{\sigma \in \mathfrak{S}_3 \\ 0 \leq 2j+1 \leq n-1}} f^{(n)}(\sigma, 1, 2j+1) \right) \\ &= \frac{1}{6n} \left(\sum_{\substack{\sigma \in \mathfrak{S}_3 \\ 0 \leq 2j \leq n-1}} f^{(n)}(\sigma, 1, 0) + \sum_{\substack{\sigma \in \mathfrak{S}_3 \\ 0 \leq 2j+1 \leq n-1}} f^{(n)}(\sigma, 1, 1) \right) \\ &= \frac{1}{6n} \left(\left\lfloor \frac{n+1}{2} \right\rfloor \sum_{\sigma \in \mathfrak{S}_3} f^{(n)}(\sigma, 1, 0) + \left\lfloor \frac{n}{2} \right\rfloor \sum_{\sigma \in \mathfrak{S}_3} f^{(n)}(\sigma, 1, 1) \right), \end{aligned}$$

where we used Corollary 3.7. Now using Proposition 5.1 we get

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_3} f^{(n)}(\sigma, 1, 0) &= f^{(n)}(id, 1, 0) + 3f^{(n)}(\tau_{12}, 1, 0) \\ &= \begin{cases} 3 \times 2^m + \alpha_m & \text{if } n = 2m, \\ \alpha_{m+1} & \text{if } n = 2m + 1, \end{cases} \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_3} f^{(n)}(\sigma, 1, 1) &= 3f^{(n)}(\tau_{1,2}, 1, 1) \\ &= \begin{cases} \alpha_{m+1} & \text{if } n = 2m, \\ \alpha_{m+1} & \text{if } n = 2m + 1. \end{cases} \end{aligned} \tag{5.4}$$

TABLE 2. List of $K'(1), \dots, K'(40)$, which counts colorful bracelets

n	$K'(n)$	n	$K'(n)$	n	$K'(n)$	n	$K'(n)$
1	0	11	21	21	8496	31	5,778,267
2	1	12	48	22	16,431	32	11,201,884
3	1	13	63	23	30,735	33	21,702,708
4	2	14	133	24	59,344	34	42,141,576
5	1	15	205	25	112,531	35	81,830,748
6	4	16	412	26	217,246	36	159,140,896
7	3	17	685	27	415,628	37	309,590,883
8	8	18	1354	28	803,210	38	602,938,099
9	8	19	2385	29	1,545,463	39	1,174,779,397
10	18	20	4644	30	2,991,192	40	2,290,920,128

Replacing in the expression of $R(n)$ we obtain

$$R(n) = \begin{cases} 2^{m-1} & \text{if } n = 2m, \\ \alpha_{m+1}/6 & \text{if } n = 2m + 1, \end{cases}$$

and the announced result follows. □

Table 2 lists the first 40 terms of the sequence $(K'(n))_n$.

Remark 5.3. Although $K(n) \leq K'(n)$ for all $n \geq 1$, a surprising fact about $(K(n))_{n \geq 1}$ and $(K'(n))_{n \geq 1}$ is that they coincide for the first 8 values!

Remark 5.4. The equality $2K'(n) = K(n) + R(n)$ and the easy-to-prove fact that $R(n) = n \pmod 2$ for $n \geq 3$, allow us to find the parity pattern of the $K(n)$'s. The fact that $K(n) = n \pmod 2$ for $n \geq 3$ seems difficult to prove directly.

6. Related combinatorial sequences

Colorful necklaces or bracelets with n beads and two colors are easy to determine. There are none when n is odd and just one equivalence class when n is even. Thus the sequences $(K^*(n))_{n \geq 1}$ and $(K'^*(n))_{n \geq 1}$ defined by

$$K^*(n) = K(n) - \frac{1 + (-1)^n}{2}, \quad \text{and} \quad K'^*(n) = K'(n) - \frac{1 + (-1)^n}{2} \quad (6.1)$$

represent the number of non-equivalent colorful necklaces in n beads with exactly 3 colors and the number of non-equivalent colorful bracelets in n beads with exactly 3 colors, respectively. Both sequences $(K^*(n))_{n \geq 1}$, and $(K'^*(n))_{n \geq 1}$ are currently not recognized by the OEIS.

Further, if we are interested in periodic colorful sequences of *exact* period n in at most 3 colors, then the number $\tilde{K}(n)$ of non-equivalent such sequences assuming that reversing is not allowed is given by

$$\tilde{K}(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) K(d),$$

where μ is the well known Moebius function. Indeed, this follows from the classical result [6, Theorem 1.5], because clearly $K(n) = \sum_{d|n} \tilde{K}(d)$.

OEIS recognizes $(\tilde{K}(n))_{n \geq 1}$ as the “Number of ZnS polytypes that repeat after n layers” [A011957](#).

Similarly, if we are interested in periodic colorful sequences of *exact* period n in at most 3 colors, then the number $\tilde{K}'(n)$ of non-equivalent such sequences assuming that reversing is allowed is given by

$$\tilde{K}'(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) K'(d).$$

OEIS recognizes $(\tilde{K}'(n))_{n \geq 1}$ as the “Number of Barlow packings that repeat after exactly n layers” [A011768](#).

7. Future research

This paper has counted non-equivalent colorful necklaces and non-equivalent colorful bracelets in n beads with 3 colors. An open problem is to count non-equivalent colorful necklaces and colorful bracelets in n beads with $c \geq 4$ colors.

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