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Minimal-delay FIR delayed left inverses for systems with zero nonzero zeros

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ABSTRACT

This paper considers finite-time input reconstruction for discrete-time linear time-invariant systems in the case where the initial condition is unknown. There are three main results. First, a specific construction of finite-impulse-response (FIR) delayed left inverse with the minimal delay for systems with zero nonzero zeros is presented. Next, it is shown that, in the presence of an arbitrary unknown initial condition, finite-time input reconstruction is possible using a delayed left inverse H if and only if H is FIR. Finally, it is shown that a transfer function with full column normal rank has an FIR delayed left inverse with the minimal delay if and only if the system has zero nonzero zeros.

1. Introduction

The need to invert dynamical systems arises in many applications. One of these is input estimation, where the goal is to determine the input of a system based on measurements of its output. Within the context of linear systems, plant inversion is considered in the classic papers [1–4]. Input estimation for linear systems is considered in [5–8]. Additional references are cited in [9].

The present paper focuses on finite-time input reconstruction within the context of deterministic discrete-time linear time-invariant systems. For this problem there are three key issues. The first issue concerns the delay under which the input can be estimated. This question was resolved in [1], which showed that the minimal delay is the smallest index for which the difference of the ranks of two successive block-Toeplitz matrices is equal to the number of inputs. The second issue concerns the presence of zeros. Since zeros block inputs, it is reasonable to expect that the presence of zeros precludes the ability to unambiguously estimate the system input. The third issue concerns the effect of unknown, nonzero initial conditions. In particular, the free response of the system contributes to its output, thus making it difficult to determine the input simply by inverting the system.

Finite-time input reconstruction for discrete-time LTI systems was considered in [7,9,10], where the input was reconstructed based on a state space approach for systems with no zeros. The present paper has three key contributions relative to prior work. First, Theorem 4.2 uses the Smith–McMillan form at infinity to construct a finite-impulse-response (FIR) delayed left inverse with the minimal delay for systems with zero (that is, no) nonzero zeros. The significance of this construction is due to Theorem 5.1, which shows that, in the presence of arbitrary unknown initial conditions, finite-time input reconstruction is possible using a delayed left inverse H if and only if H is FIR. Theorem 5.1 also specifies a time step beyond which input reconstruction is achievable. Finally, Theorem 6.1 extends Theorem 4.2 by showing that a transfer function with full column normal rank has an FIR delayed left inverse with the minimal delay if and only if it has zero nonzero zeros. The explicit construction of a delayed left inverse was not given in [7,9,10]. In addition, unlike the present paper, which allows the system to have zeros at zero, the results of [7,9,10] assume that the system has no zeros.

Some preliminary results regarding finite-time input reconstruction were presented in [11], in which no proofs were provided. The current paper extends the results given in [11] in several ways. Necessary and sufficient conditions for finite-time input reconstruction in the presence of unknown, nonzero initial conditions are derived in this paper, whereas only sufficient conditions are provided in [11]. Proposition 6.1 was a conjecture in [11]. Finally, the construction of an FIR delayed left inverse with the minimal delay is not given in [11].

2. Preliminaries

Definition 2.1. Let $G \in \mathbb{R}(\mathbf{z})^{p\times m}_{\text{prop}}$, and, for each $i \geq 0$, let H_i be the ith Markov parameter of G. Then, for all $i \geq 0$, the ith *Toeplitz*

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matrix associated with G is defined by

$$\mathcal{T}_i \triangleq \begin{bmatrix} H_0 & 0 & 0 & \cdots & 0 \\ H_1 & H_0 & 0 & \cdots & 0 \\ H_2 & H_1 & H_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ H_i & H_{i-1} & \cdots & H_1 & H_0 \end{bmatrix} \in \mathbb{R}^{(i+1)p\times(i+1)m}.$$

In the case where i is a negative integer, $\mathcal{T}_i$ is an empty matrix.

Definition 2.2. Let $G \in \mathbb{R}(\mathbf{z})^{m\times p}_{\rm prop}$, and let d be a nonnegative integer. Then, G is *delayed left invertible with delay d* if there exists $H \in \mathbb{R}(\mathbf{z})^{m\times p}_{\rm prop}$ such that $H(\mathbf{z})G(\mathbf{z}) = \mathbf{z}^{-d}I_m$. In this case, H is a *delayed left inverse of G with delay d*. Furthermore, G is *delayed left invertible* if there exists $d \geq 0$ such that G is delayed left invertible with delay d, and H is a *delayed left inverse of G* if there exists $d \geq 0$ such that H is a delayed left inverse of G with delay d. Finally, H is a *left inverse of G* if H is a delayed left inverse of G with delay $d = 0$.

If H is a delayed left inverse of G with delay d, then the output of HG is equal to the d-step-delayed input of HG. However, HG does not account for the free response of the state space model formed by cascading state space models of G and H. The missing free response can be accounted for by specifying initial conditions of realizations of G and H. Let

$$G \overset{\min}{\sim} \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right], \quad H \overset{\min}{\sim} \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right], \tag{1}$$

and, for all $k \geq 0$, consider the state space equations

$$x_G(k+1) = A_G x_G(k) + B_G u(k), \tag{2}$$

$$y(k) = C_G x_G(k) + D_G u(k), \tag{3}$$

and

$$x_H(k+1) = A_H x_H(k) + B_H y(k), \tag{4}$$

$$z(k) = C_H x_H(k) + D_H y(k). \tag{5}$$

Then, the state space realization of the cascade HG is given by

$$x(k+1) = Ax(k) + Bu(k), \tag{6}$$

$$z(k) = Cx(k) + Du(k), \tag{7}$$

where

$$x \triangleq \begin{bmatrix} x_G \\ x_H \end{bmatrix}, \quad A \triangleq \begin{bmatrix} A_G & 0 \\ B_H C_G & A_H \end{bmatrix}, \quad B \triangleq \begin{bmatrix} B_G \\ B_H D_G \end{bmatrix}, \tag{8}$$

$$C \triangleq \begin{bmatrix} D_H C_G & C_H \end{bmatrix}, \quad D \triangleq D_H D_G. \tag{9}$$

Note that the realization (6), (7) of HG is not necessarily minimal.

Definition 2.3. Let $A \in \mathbb{R}^{n\times n}$. Then, the *index* of A, denoted by $\operatorname{ind} A$, is the smallest nonnegative integer ν such that $\operatorname{rank} A^\nu = \operatorname{rank} A^{\nu+1}$.

Note that, if A is nilpotent, then $\operatorname{ind} A$ is the smallest positive integer ν such that $A^\nu = 0$.

Definition 2.4. Let $G \in \mathbb{R}(\mathbf{z})^{p\times m}_{\rm prop}$, where $G \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $A \in \mathbb{R}^{n\times n}$. Then, the *index* of G, denoted by $\operatorname{ind} G$, is $\operatorname{ind} A$.

3. Effect of zeros on input reconstruction

If the continuous-time system G has a transmission zero, then it follows from [12, p. 398] that there exist an initial condition and nonzero input such that the response of a minimal state space realization of G is identically zero. The following result is the discrete-time analogue.

Proposition 3.1. *Let $G \in \mathbb{R}(\mathbf{z})^{p\times m}_{\rm prop}$, where $G \in \mathbb{R}(\mathbf{z})^{p\times m}_{\rm prop}$, where $G \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $A \in \mathbb{R}^{n\times n}$, and, for all $k \geq 0$, consider*

$$x(k+1) = Ax(k) + Bu(k), \tag{10}$$

$$y(k) = Cx(k) + Du(k). \tag{11}$$

Assume that $\mathbf{z}_0 \in \mathbb{C}$ is a transmission zero of G, and let $\begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \in \mathcal{N}(\mathcal{Z}(\mathbf{z}_0))$ have nonzero real part, where $\mathcal{Z}(\mathbf{z}) \triangleq \begin{bmatrix} \mathbf{z}I - A & -B \\ C & D \end{bmatrix}$. Define the initial state $x(0) \triangleq \operatorname{Re}(\bar{x})$, and, for all $k \geq 0$, define the input sequence $u(k) \triangleq \operatorname{Re}(\mathbf{z}_0^k \bar{u})$, where $0^0 \triangleq 1$. Then, for all $k \geq 0$, $y(k) = 0$. Furthermore, $\bar{u} \neq 0$.

Proof. By assumption,

$$\begin{bmatrix} \mathbf{z}_0 I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = 0,$$

and thus

$$(\mathbf{z}_0 I - A)\bar{x} = B\bar{u}, \tag{12}$$

$$C\bar{x} + D\bar{u} = 0. \tag{13}$$

Using (12) and the fact that $\mathbf{z}_0^0 = 1$, it follows from (10) that $x(1) = A\operatorname{Re}(\bar{x}) + B\operatorname{Re}(\bar{u}) = A\operatorname{Re}(\bar{x}) + \operatorname{Re}(\mathbf{z}_0\bar{x}) - A\operatorname{Re}(\bar{x}) = \operatorname{Re}(\mathbf{z}_0\bar{x})$. Proceeding similarly, it follows that, for all $k \geq 0$, $x(k) = \operatorname{Re}(\mathbf{z}_0^k\bar{x})$. Thus (11) and (13) together imply that, for all $k \geq 0$, $y(k) = C\operatorname{Re}(\mathbf{z}_0^k\bar{x}) + D\operatorname{Re}(\mathbf{z}_0^k\bar{u}) = \operatorname{Re}(\mathbf{z}_0^k(C\bar{x} + D\bar{u})) = 0$.

Next, suppose that $\bar{u} = 0$. Hence (13) implies that $C\bar{x} = 0$. Then it follows from (10) and (11) that

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = \mathcal{O}_n \operatorname{Re}(\bar{x}),$$

where $\mathcal{O}_n$ is the observability matrix obtained from (A, C). Since $\mathcal{O}_n$ has full column rank and, for all $k \geq 0$, $y_k = 0$, it follows that $\operatorname{Re}(\bar{x}) = 0$, which is a contradiction. Thus $\bar{u} \neq 0$. □

Note that, in the case where $\mathbf{z}_0 \neq 0$, the input u that produces the zero output has the property that, for all $k \geq 0$, $u(k) \neq 0$. Since the zero input also produces the zero output, finite-time input reconstruction is impossible. However, in the case where $\mathbf{z}_0 = 0$, the input u that produces the zero output is $\{\operatorname{Re}(\bar{u}), 0, 0, \ldots.\}$. The fact that $u(k)$ is nonzero only at the initial time step suggests that finite-time input reconstruction may be possible in this case as long as G has zero nonzero zeros. In fact, a delayed left inverse for systems with this property is constructed in the next section.

4. Construction of an FIR delayed left inverse with the minimal delay

In this section, we use the Smith–McMillan form at infinity [13] to construct an FIR delayed left inverse with the minimal delay for systems with zero nonzero zeros. The main result is Theorem 4.2, which presents the expression for the constructed FIR inverse.

Definition 4.1. Let $U \in \mathbb{R}[\mathbf{z}]^{n\times n}$. Then U is *unimodular* if $\det U$ is a nonzero constant.

Definition 4.2. Let $W \in \mathbb{R}(\mathbf{z})^{m\times m}_{\rm prop}$. Then W is *biproper* if $W_\infty \triangleq \lim_{\mathbf{z}\to\infty} W(\mathbf{z})$ is nonsingular.

Lemma 4.1. *Let $U \in \mathbb{R}[\mathbf{z}]^{n\times n}$, assume that U is unimodular, and, for all $\mathbf{z} \neq 0$, define $V(\mathbf{z}) \triangleq U(1/\mathbf{z})$. Then V is biproper and FIR.*

Proof. Since U is a polynomial matrix, each entry of U is of the form $\alpha_k\mathbf{z}^k + \cdots + \alpha_1\mathbf{z} + \alpha_0$, where k is a nonnegative integer and $\alpha_0, \ldots, \alpha_k$ are real numbers. Then the corresponding entry of V has the form $\alpha_k\mathbf{z}^{-k} + \cdots + \alpha_1\mathbf{z}^{-1} + \alpha_0$, which is proper and FIR. Hence V is proper and FIR. Next, define the nonzero constant $\beta \triangleq \det U(\mathbf{z})$, and note that $\lim_{\mathbf{z}\to\infty} \det V(\mathbf{z}) = \lim_{\mathbf{z}\to\infty} \det U(1/\mathbf{z}) = \lim_{\mathbf{z}\to 0} \det U(\mathbf{z}) = \beta \neq 0$. Hence V is biproper. □

Lemma 4.2. *Let $G \in \mathbb{R}(\mathbf{z})^{p\times m}_{\mathrm{prop}}$ and, for all $\mathbf{z} \neq 0$, define $\hat{G}(\mathbf{z}) \triangleq G(1/\mathbf{z})$. Then the following statements hold:*

(i) $\hat{G}$ has no poles at zero.
(ii) If G has zero nonzero zeros, then $\hat{G}$ has zero nonzero zeros.

Proof. To prove *(i)*, suppose that $\hat{G}$ has at least one pole at zero. Then at least one entry of $\hat{G}$ is of the form $\frac{N(\mathbf{z})}{\mathbf{z}^k D(\mathbf{z})}$, where k is a positive integer, N and D are polynomials such that $N(0) \neq 0$, and $D(0) \neq 0$. Then the corresponding entry of G is $\frac{\mathbf{z}^k N(1/\mathbf{z})}{D(1/\mathbf{z})}$. Since $N(0) \neq 0$ and $D(0) \neq 0$, it follows that $\frac{N(1/\mathbf{z})}{D(1/\mathbf{z})}$ is exactly proper and hence $\frac{\mathbf{z}^k N(1/\mathbf{z})}{D(1/\mathbf{z})}$ is improper, which is a contradiction. Hence $\hat{G}$ has no poles at zero.

To prove *(ii)*, define $\rho \triangleq \operatorname{rank} G = \operatorname{rank} \hat{G}$. Suppose that $\mathbf{z}_0$ is a nonzero zero of $\hat{G}$. Then $\operatorname{rank} G(1/\mathbf{z}_0) = \operatorname{rank} \hat{G}(\mathbf{z}_0) < \rho$. Thus $1/\mathbf{z}_0$ is a nonzero zero of G, which is a contradiction. Hence $\hat{G}$ has zero nonzero zeros. □

The following result presents the *Smith–McMillan form at infinity* S_∞ of G. The proof presented here, which is different from the proof in [13], is constructive; this construction is also used to prove Theorem 4.2.

Theorem 4.1. *Let $G \in \mathbb{R}(\mathbf{z})^{p\times m}_{\mathrm{prop}}$, define $\rho \triangleq \operatorname{rank} G$, and define $\rho_0 \triangleq \rho - \operatorname{rank} G(\infty)$. Then there exist biproper transfer functions $W \in \mathbb{R}(\mathbf{z})^{p\times p}_{\mathrm{prop}}$ and $V \in \mathbb{R}(\mathbf{z})^{m\times m}_{\mathrm{prop}}$ and integers $\iota_1 \geq \iota_2 \geq \cdots \geq \iota_{\rho_0} > 0$ such that $G = WS_\infty V$, where*

$$S_\infty(\mathbf{z}) \triangleq \begin{bmatrix} \mathbf{z}^{-\iota_1} & & & & & & 0_{\rho\times(m-\rho)} \\ & \ddots & & & & & \\ & & \mathbf{z}^{-\iota_{\rho_0}} & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ 0_{(p-\rho)\times\rho} & & & & & & 0_{(p-\rho)\times(m-\rho)} \end{bmatrix}. \tag{14}$$

Proof. For all $\mathbf{z} \neq 0$, define $\hat{G}(\mathbf{z}) \triangleq G(1/\mathbf{z})$. Note that $\operatorname{rank} \hat{G} = \operatorname{rank} G = \rho$. Let $\hat{G} = \hat{S}_1\hat{S}\hat{S}_2$, where $\hat{S} \in \mathbb{R}(\mathbf{z})^{p\times m}$ is the Smith–McMillan form of $\hat{G}$, and $\hat{S}_1 \in \mathbb{R}(\mathbf{z})^{p\times p}$ and $\hat{S}_2 \in \mathbb{R}(\mathbf{z})^{m\times m}$ are unimodular matrices. Define $S_1(\mathbf{z}) \triangleq \hat{S}_1(1/\mathbf{z})$, $S(\mathbf{z}) \triangleq \hat{S}(1/\mathbf{z})$, and $S_2(\mathbf{z}) \triangleq \hat{S}_2(1/\mathbf{z})$. It follows from Lemma 4.2 that $\hat{G}$ has no poles at zero and thus $\hat{S}$ has no poles at zero. Hence $\hat{S}$ is of the form given in Box I, where $\iota_1 \geq \cdots \geq \iota_\kappa > 0$ and $\kappa \triangleq \rho - \operatorname{rank} \hat{G}(0)$. N_i and D_i, for $i = 1, \ldots, \rho$, are polynomials such that $N_i(0) \neq 0$, and $D_i(0) \neq 0$. Then S is as given in Box II. Since, for all $i = 1, \ldots, \rho$, $N_i(0) \neq 0$, and $D_i(0) \neq 0$, it follows that $N_i(1/\mathbf{z})$ and $D_i(1/\mathbf{z})$ are exactly proper and hence S is proper. Therefore, S can be factored as $S = S_0 D_v$, where

$$S_0(\mathbf{z}) \triangleq \begin{bmatrix} \mathbf{z}^{-\iota_1} & & & & & & 0_{\rho\times(m-\rho)} \\ & \ddots & & & & & \\ & & \mathbf{z}^{-\iota_\kappa} & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ 0_{(p-\rho)\times\rho} & & & & & & 0_{(p-\rho)\times(m-\rho)} \end{bmatrix}.$$

and

$$D_v(\mathbf{z}) \triangleq \begin{bmatrix} \frac{N_1(1/\mathbf{z})}{D_1(1/\mathbf{z})} & & & & & \\ & \ddots & & & & \\ & & \frac{N_\rho(1/\mathbf{z})}{D_\rho(1/\mathbf{z})} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}. \tag{15}$$

Note that $D_v \in \mathbb{R}(\mathbf{z})^{m\times m}_{\mathrm{prop}}$ is a biproper diagonal matrix. Since $\hat{G}(0) = G(\infty)$, it follows that $\kappa = \rho_0$ and thus $S_0 = S_\infty$. Now, for all $\mathbf{z} \neq 0$, $G(\mathbf{z}) = \hat{G}(1/\mathbf{z}) = \hat{S}_1(1/\mathbf{z})\hat{S}(1/\mathbf{z})\hat{S}_2(1/\mathbf{z}) = S_1(\mathbf{z})S(\mathbf{z})S_2(\mathbf{z})$. Since $\hat{S}_1$ and $\hat{S}_2$ are unimodular, it follows from Lemma 4.1 that S_1 and S_2 are biproper. Defining $W \triangleq S_1$ and $V \triangleq D_v S_2$, it follows that $G = S_1 S S_2 = S_1 S_0 D_v S_2 = S_1 S_\infty D_v S_2 = WS_\infty V$. □

Definition 4.3. Let $G \in \mathbb{R}(\mathbf{z})^{p\times m}_{\mathrm{prop}}$ and $i \geq 0$. Then $\beta_i(G) \triangleq \operatorname{rank} \mathcal{T}_i - \operatorname{rank} \mathcal{T}_{i-1}$.

The following result is given by Theorem 1 in [14].

Lemma 4.3. *Let $G_1 \in \mathbb{R}(\mathbf{z})^{p\times m}_{\mathrm{prop}}$ and $G_2 \in \mathbb{R}(\mathbf{z})^{p\times m}_{\mathrm{prop}}$ be such that $G_2 = WG_1V$, where $W \in \mathbb{R}(\mathbf{z})^{p\times p}_{\mathrm{prop}}$ and $V \in \mathbb{R}(\mathbf{z})^{m\times m}_{\mathrm{prop}}$ are biproper. Then, for all $i \geq 0$, $\beta_i(G_1) = \beta_i(G_2)$.*

The following result is given by Theorem 4 in [1].

Proposition 4.1. *Let $G \in \mathbb{R}(\mathbf{z})^{p\times m}_{\mathrm{prop}}$ and $d \geq 0$. Then G is delayed left invertible with delay d if and only if $\operatorname{rank} \mathcal{T}_d - \operatorname{rank} \mathcal{T}_{d-1} = m$.*

Definition 4.4. Let $G \in \mathbb{R}(\mathbf{z})^{p\times m}_{\mathrm{prop}}$, and assume that G has full column normal rank. Then η_G denotes the smallest nonnegative integer d for which there exists a delayed left inverse of G with delay d.

The following result is based on the discussion of the pole/zero structure at infinity given in [14].

Proposition 4.2. *Let $G \in \mathbb{R}(\mathbf{z})^{p\times m}_{\mathrm{prop}}$, assume that G has full column normal rank, and define ι_1 as in Theorem 4.1. Then $\eta_G = \iota_1$.*

Proof. Let $H_{\infty,i}$ be the ith Markov parameter of S_∞ and $\mathcal{T}_{\infty,i}$ be the ith Toeplitz matrix associated with S_∞, where S_∞ is the Smith–McMillan form at infinity of G. Define the multiset $F \triangleq \{\iota_1, \ldots, \iota_{\rho_0}, 0, \ldots, 0\}$ with ρ elements, where $\iota_1, \ldots, \iota_{\rho_0}$ and ρ are defined in Theorem 4.1. For all $i \geq 0$, let F_i be the multiset consisting of all elements of F that are less than or equal to i, and let $|F_i|$ denote the cardinality of F_i.

Note that, for all $i \geq 0$, each row of $\mathcal{T}_{\infty,i}$ is either zero or has exactly one nonzero entry that is equal to one, and the nonzero rows of $\mathcal{T}_{\infty,i}$ are linearly independent. It thus follows that $\beta_i(S_\infty) = \operatorname{rank} \mathcal{T}_{\infty,i} - \operatorname{rank} \mathcal{T}_{\infty,i-1} = \operatorname{rank} \begin{bmatrix} H_{\infty,0} & \cdots & H_{\infty,i} \end{bmatrix} = |F_i|$. Hence, Theorem 4.1 and Lemma 4.3 imply that, for all $i \geq 0$, $\beta_i(G) = \beta_i(S_\infty) = |F_i|$.

$$\hat{S}(\mathbf{z}) = \begin{bmatrix} \frac{\mathbf{z}^{\iota_1} N_1(\mathbf{z})}{D_1(\mathbf{z})} & & & & & 0_{\rho\times(m-\rho)} \\ & \ddots & & & & \\ & & \frac{\mathbf{z}^{\iota_\kappa} N_\kappa(\mathbf{z})}{D_\kappa(\mathbf{z})} & & & \\ & & & \frac{N_{\kappa+1}(\mathbf{z})}{D_{\kappa+1}(\mathbf{z})} & & \\ & & & & \ddots & \\ & & & & \frac{N_\rho(\mathbf{z})}{D_\rho(\mathbf{z})} & \\ 0_{(p-\rho)\times\rho} & & & & & 0_{(p-\rho)\times(m-\rho)} \end{bmatrix}$$

Box I.

$$S(\mathbf{z}) = \begin{bmatrix} \frac{\mathbf{z}^{-\iota_1} N_1(1/\mathbf{z})}{D_1(1/\mathbf{z})} & & & & & 0_{\rho\times(m-\rho)} \\ & \ddots & & & & \\ & & \frac{\mathbf{z}^{-\iota_\kappa} N_\kappa(1/\mathbf{z})}{D_\kappa(1/\mathbf{z})} & & & \\ & & & \frac{N_{\kappa+1}(1/\mathbf{z})}{D_{\kappa+1}(1/\mathbf{z})} & & \\ & & & & \ddots & \\ & & & & \frac{N_\rho(1/\mathbf{z})}{D_\rho(1/\mathbf{z})} & \\ 0_{(p-\rho)\times\rho} & & & & & 0_{(p-\rho)\times(m-\rho)} \end{bmatrix}$$

Box II.

Note that $\max_{i\geq 0}\beta_i(S_\infty) = \max_{i\geq 0}|F_i| = |F| = \rho$. Since ι_1 is the largest element in F, it follows that the smallest i such that $|F_i| = \rho$ is ι_1. Thus $\rho = |F_{\iota_1}| = \beta_{\iota_1}(G) = \operatorname{rank}\mathcal{T}_{\iota_1} - \operatorname{rank}\mathcal{T}_{\iota_1-1}$, where $\mathcal{T}_i$ is the ith Toeplitz matrix associated with G. Since G has full column rank, it follows that $\rho = m$, and thus ι_1 is the smallest i such that $\operatorname{rank}\mathcal{T}_i - \operatorname{rank}\mathcal{T}_{i-1} = m$. Hence Proposition 4.1 implies that $\eta_G = \iota_1$. □

The following result constructs an FIR delayed left inverse of G with the minimal delay.

Theorem 4.2. *Let $G \in \mathbb{R}(\boldsymbol{z})^{p\times m}_{\rm prop}$, assume that G has full column rank, and assume that G has zero nonzero zeros. Then there exist biproper transfer functions $W \in \mathbb{R}(\boldsymbol{z})^{p\times p}_{\rm prop}$ and $V \in \mathbb{R}(\boldsymbol{z})^{m\times m}_{\rm prop}$ such that*

$$H_\infty(\mathbf{z}) \triangleq \boldsymbol{z}^{-\eta_G} V^{-1}(\mathbf{z}) S_\infty^{\rm T}(1/\mathbf{z}) W^{-1}(\mathbf{z}) \tag{16}$$

is an FIR delayed left inverse of G with delay η_G, where

$$S_\infty(\boldsymbol{z}) \triangleq \begin{bmatrix} \boldsymbol{z}^{-\iota_1} & & & & \\ & \ddots & & & \\ & & \boldsymbol{z}^{-\iota_{\rho_0}} & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ & & 0_{(p-m)\times m} & & \end{bmatrix} \tag{17}$$

is the Smith–McMillan form at infinity of G, $\iota_1 \geq \iota_2 \geq \cdots \geq \iota_{\rho_0} > 0$ are integers, and $\rho_0 \triangleq m - \operatorname{rank} G(\infty)$.

Proof. Define, for all $\mathbf{z} \neq 0$, $\hat{G}(\mathbf{z}) \triangleq G(1/\mathbf{z})$. Note that $\operatorname{rank}\hat{G} = \operatorname{rank} G = m$. Let $\hat{G} = \hat{S}_1\hat{S}\hat{S}_2$, where $\hat{S}$ is the Smith–McMillan form of $\hat{G}$, and $\hat{S}_1$ and $\hat{S}_2$ are unimodular matrices. Define $S_1(\mathbf{z}) \triangleq \hat{S}_1(1/\mathbf{z})$, $S(\mathbf{z}) \triangleq \hat{S}(1/\mathbf{z})$, and $S_2(\mathbf{z}) \triangleq \hat{S}_2(1/\mathbf{z})$. Following the same steps given in the proof of Theorem 4.1 yields $G = WS_\infty V$, where $W \triangleq S_1$, $V \triangleq D_v S_2$, S_∞ is given by (17), and D_v is given by (15) with ρ replaced by m. Since $\hat{S}_1$ and $\hat{S}_2$ are unimodular, it follows that $\hat{S}_1^{-1}$ and $\hat{S}_2^{-1}$ are unimodular and thus Lemma 4.1 implies that $W^{-1} = S_1^{-1}$ and S_2^{-1} are FIR. Since G has zero nonzero zeros, it follows from Lemma 4.2 that $\hat{G}$ has zero nonzero zeros. Hence, $\hat{S}$ has zero nonzero zeros. Hence, for all $i = 1,\ldots,m$, $N_i = 1$ in (15). Hence D_v^{-1} is FIR. Thus $V^{-1} = S_2^{-1}D_v^{-1}$ is FIR, and hence H_∞ is FIR. Next, it follows from Proposition 4.2 that $\eta_G = \iota_1$. Hence, $\mathbf{z}^{-\eta_G}S_\infty^{\rm T}(1/\mathbf{z})$ is proper. Note that W^{-1} and V^{-1} are biproper and thus H_∞ is proper. Since $H_\infty(\mathbf{z})G(\mathbf{z}) = \mathbf{z}^{-\eta_G}V^{-1}(\mathbf{z})S_\infty^{\rm T}(1/\mathbf{z})W^{-1}(\mathbf{z})W(\mathbf{z})S_\infty(\mathbf{z})V(\mathbf{z}) = \mathbf{z}^{-\eta_G}I_m$, it follows that H_∞ is an FIR delayed left inverse of G with delay η_G. □

5. Input reconstruction using FIR delayed left inverse

The main result in this section shows that, in the presence of an arbitrary unknown initial condition, finite-time input reconstruction is possible using a delayed left inverse H if and only if H is FIR. The following lemma will be needed.

Lemma 5.1. *Let $G \in \mathbb{R}(\boldsymbol{z})^{p\times m}_{\rm prop}$ and $H \in \mathbb{R}(\boldsymbol{z})^{m\times p}_{\rm prop}$, with minimal state space realizations (1)–(5). Assume that H is FIR and that H is a delayed left inverse of G with delay d. Define $K(\boldsymbol{z}) \triangleq H(\boldsymbol{z})C_G(\boldsymbol{z}I - A_G)^{-1}$. Then K is FIR.*

Proof. For the state space realization of HG given by (6)–(9), note that $\operatorname{spec}(A) = \operatorname{spec}(A_G) \cup \operatorname{spec}(A_H)$. Since H is FIR, it follows that $\operatorname{spec}(A_H) = \{0\}$. Therefore, each nonzero eigenvalue of A is an eigenvalue of A_G. Since HG is FIR, it follows that each nonzero eigenvalue of A (including multiplicity) is either an uncontrollable eigenvalue of (A, B) or an unobservable eigenvalue of (A, C). However, since (A_G, B_G) is controllable, each nonzero eigenvalue of A is contained in $\operatorname{spec}(A_G)$, and A is lower triangular, it follows from the PBH test that each nonzero eigenvalue of A is a controllable eigenvalue of (A, B) and thus an unobservable eigenvalue of (A, C). Defining

$$B_0 \triangleq \begin{bmatrix} I_{n_G} \\ 0 \end{bmatrix}, \quad D_0 \triangleq 0,$$

where $n_G \triangleq \operatorname{Mcdeg} G$, note that (A, B_0, C, D_0) is a state space realization of K. Since each nonzero eigenvalue of A is an unobservable eigenvalue of (A, C), it follows that none of the nonzero eigenvalues of A are poles of K. Hence, every pole of K is zero, and thus K is FIR. $\square$

Theorem 5.1. *Let $G \in \mathbb{R}(\boldsymbol{z})^{p\times m}_{\text{prop}}$ and $H \in \mathbb{R}(\boldsymbol{z})^{m\times p}_{\text{prop}}$ with minimal state space realizations (1)–(5), assume that H is a delayed left inverse of G with delay d, and define $K(\mathbf{z}) \triangleq H(\mathbf{z})C_G(\mathbf{z}I - A_G)^{-1}$. Then the following statements hold:*

(i) If there exists a nonnegative integer ν such that, for all $k \geq \nu$ and all initial conditions $x_G(0)$ and $x_H(0)$, $z(k) = u(k-d)$, then H is FIR.

(ii) If H is FIR, then for all $k \geq \nu = \max\{\operatorname{ind} H, \operatorname{ind} K, d\}$ and all initial conditions $x_G(0)$ and $x_H(0)$, $z(k) = u(k-d)$. If, in addition, $x_H(0) = 0$, then $\nu = \max\{\operatorname{ind} K, d\}$.

Proof. Note that, for all $k \geq 0$, $z(k) = z_{\text{free}}(k) + z_{\text{forced}}(k)$, where z_{free} and z_{forced} denote the free response and forced response, respectively, of (6)–(9). Since $H(\mathbf{z})G(\mathbf{z}) = \mathbf{z}^{-d}I_m$, it follows that, for all $k \geq d$, $z_{\text{forced}}(k) = u(k-d)$. Next, note that, for all $k \geq 0$,

$$\begin{aligned} z_{\text{free}}(k) &= CA^k x(0) \\ &= \begin{bmatrix} D_H C_G & C_H \end{bmatrix} \begin{bmatrix} A_G^k & 0 \\ \sum_{i=0}^{k-1} A_H^i B_H C_G A_G^{k-i-1} & A_H^k \end{bmatrix} \begin{bmatrix} x_G(0) \\ x_H(0) \end{bmatrix} \\ &= z_G(k) + z_H(k), \end{aligned} \tag{18}$$

where

$$z_G(k) \triangleq \left(D_H C_G A_G^k + C_H \sum_{i=0}^{k-1} A_H^i B_H C_G A_G^{k-i-1} \right) x_G(0),$$

$$z_H(k) \triangleq C_H A_H^k x_H(0).$$

To prove *(i)*, note that there exists a nonnegative integer ν such that, for all $k \geq \nu$ and all $x_G(0)$, $x_H(0)$, $z_{\text{free}}(k) = 0$. Hence it follows from (18) that, for all $k \geq \nu$,

$$\begin{bmatrix} D_H C_G & C_H \end{bmatrix} \begin{bmatrix} A_G^k & 0 \\ \sum_{i=0}^{k-1} A_H^i B_H C_G A_G^{k-i-1} & A_H^k \end{bmatrix} = 0,$$

and thus, for all $k \geq \nu$, $C_H A_H^k = 0$. Hence H is FIR.

To prove *(ii)*, note that since H is FIR and thus A_H is nilpotent, it follows that, for all $k \geq \operatorname{ind} H$, $z_H(k) = 0$. Noting that z_G is the output of (6), (7) in the case where $u \equiv 0$ and $x_H(0) = 0$, it follows that the Z transform of z_G is given by

$$\begin{aligned} \hat{z}_G(\mathbf{z}) &= C_H \hat{x}_H(\mathbf{z}) + D_H \hat{y}(\mathbf{z}) \\ &= \big(C_H(\mathbf{z}I - A_H)^{-1}B_H + D_H\big)\hat{y}(\mathbf{z}) \\ &= \big(C_H(\mathbf{z}I - A_H)^{-1}B_H + D_H\big)C_G\hat{x}_G(\mathbf{z}) \\ &= z\big(C_H(\mathbf{z}I - A_H)^{-1}B_H + D_H\big)C_G(\mathbf{z}I - A_G)^{-1}x_G(0) \\ &= \mathbf{z}K(\mathbf{z})x_G(0) \\ &= \mathbf{z}\hat{w}_G(\mathbf{z}), \end{aligned}$$

where $\hat{w}_G(\mathbf{z}) \triangleq K(\mathbf{z})x_G(0)$. Note that the inverse Z transform w_G of $\hat{w}_G$ is a linear combination of the single-channel impulse responses of K. Lemma 5.1 implies that K is FIR and thus, for all $k \geq \operatorname{ind} K + 1$, $w_G(k) = 0$. Since $z_G(k) = w_G(k+1)$, it follows that, for all $k \geq \operatorname{ind} K$, $z_G(k) = 0$. Hence, for all $k \geq \nu = \max\{\operatorname{ind} H, \operatorname{ind} K, d\}$, $z(k) = u(k-d)$.

Finally, consider the case where $x_H(0) = 0$. In this case, it follows that, for all $k \geq 0$, $z_H(k) = 0$, and thus, for all $k \geq 0$, $z(k) = z_{\text{free}}(k) + z_{\text{forced}}(k) = z_G(k) + z_H(k) + z_{\text{forced}}(k) = z_G(k) + z_{\text{forced}}(k)$. Therefore, for all $k \geq \max\{\operatorname{ind} K, d\}$, $z(k) = u(k-d)$. $\square$

Theorem 5.1 shows that, for all $k \geq \max\{\operatorname{ind} H, \operatorname{ind} K, d\}$, the output z is equal to the input u delayed by d steps. However, if $\max\{\operatorname{ind} H, \operatorname{ind} K\} > d$, then, for all $k = 0, \ldots, \max\{\operatorname{ind} H, \operatorname{ind} K\} - d - 1$, the input $u(k)$ is not reconstructed. Note that Theorem 5.1 does not assume any stability condition, and thus the result holds even in the case where both G and H are unstable.

6. Existence of FIR delayed left inverse

The following result restates part of Theorem 4.2 and provides its converse. In particular, Theorem 6.1 shows that a transfer function with full column normal rank has an FIR delayed left inverse with the minimal delay if and only if it has zero nonzero zeros. It follows from this fact and Theorem 5.1 that finite-time input reconstruction is possible if and only if the system has zero nonzero zeros.

Theorem 6.1. *Let $G \in \mathbb{R}(\boldsymbol{z})^{p\times m}_{\text{prop}}$, and assume that G has full column normal rank. Then, for all $d \geq \eta_G$, there exists an FIR $H \in \mathbb{R}(\boldsymbol{z})^{m\times p}_{\text{prop}}$ such that H is a delayed left inverse of G with delay d if and only if G has zero nonzero zeros.*

Proof. Sufficiency follows from Theorem 4.2. To prove necessity, suppose that $\mathbf{z}_0$ is a nonzero zero of G. Since H is FIR, it follows that $\mathbf{z}_0$ is not a pole of H. Note that $\operatorname{rank} H(\mathbf{z}_0)G(\mathbf{z}_0) = \operatorname{rank} \mathbf{z}_0^{-d}I_m = m$. Since $\mathbf{z}_0$ is a nonzero zero of G, it follows that $\operatorname{rank} G(\mathbf{z}_0) < m$. Hence $\operatorname{rank} H(\mathbf{z}_0)G(\mathbf{z}_0) \leq \min\{\operatorname{rank} H(\mathbf{z}_0), \operatorname{rank} G(\mathbf{z}_0)\} < m$, which is a contradiction. Hence G has zero nonzero zeros. $\square$

Consider the case where G has at least one zero zero and zero nonzero zeros. With $\mathbf{z}_0 = 0$, it follows from Proposition 3.1 that, if $y \equiv 0$, then either u is an impulse or $u \equiv 0$. Hence, the initial input $u(0)$ cannot be reconstructed. However, the inability to reconstruct the initial input cannot be inferred from Theorem 5.1. As discussed at the end of this section, the following result strengthens Theorem 5.1 by implying that $u(0)$ cannot be reconstructed.

Proposition 6.1. *Let $G \in \mathbb{R}(\boldsymbol{z})^{p\times m}_{\text{prop}}$ and $H \in \mathbb{R}(\boldsymbol{z})^{m\times p}_{\text{prop}}$ with minimal state space realizations (1)–(5). Assume that H is an FIR left inverse*

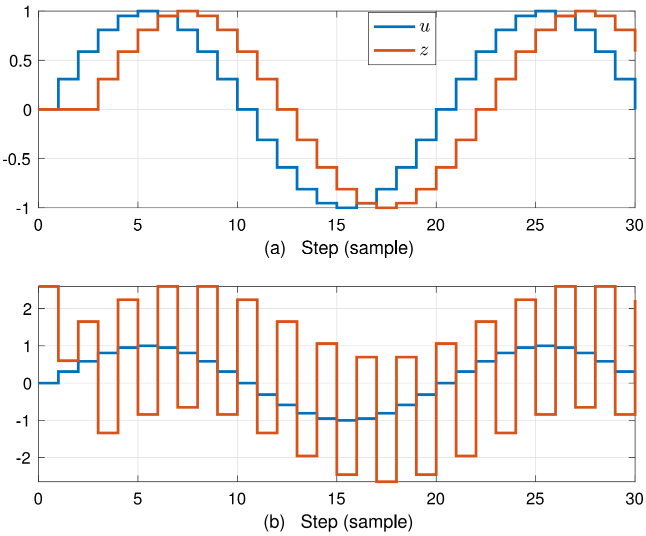

Fig. 1. (a) shows the input and output of (6), (7) with zero initial conditions. (b) shows the input and output of (6), (7) with nonzero initial conditions.

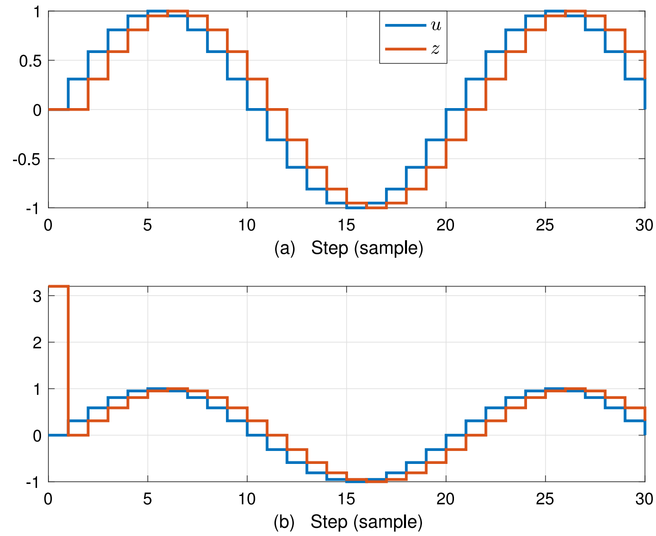

Fig. 2. (a) shows the input and output of (6), (7) with zero initial conditions. (b) shows the input and output of (6), (7) with nonzero initial conditions.

of G, define $K(\mathbf{z}) \triangleq H(\mathbf{z})C_G(\mathbf{z}I - A_G)^{-1}$*, and assume that G has at least one zero zero. Then* $K \neq 0$.

Proof. Since $HG = I_m$, it follows that $D_H D_G = I_m$ and hence $\operatorname{rank} D_H = \operatorname{rank} D_G = m$. Thus there exists a nonsingular matrix $S \in \mathbb{R}^{p\times p}$ such that $\hat{D}_H \triangleq D_H S = \begin{bmatrix} I_m & 0 \end{bmatrix}$. Define $n_G \triangleq$ Mcdeg G, and define $\hat{C}_G \triangleq S^{-1}C_G = \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \end{bmatrix}$, where $\hat{C}_1 \in \mathbb{R}^{m\times n_G}$ and $\hat{C}_G \in \mathbb{R}^{(p-m)\times n_G}$. Similarly, define $\hat{D}_G \triangleq S^{-1}D_G = \begin{bmatrix} \hat{D}_1 \\ \hat{D}_2 \end{bmatrix}$, where $\hat{D}_1 \in \mathbb{R}^{m\times m}$ and $\hat{D}_2 \in \mathbb{R}^{(p-m)\times m}$. Let $\hat{G} \in \mathbb{R}(\mathbf{z})^{p\times m}$, where $\hat{G} \sim \left[\begin{array}{c|c} A_G & B_G \\ \hline \hat{C}_G & \hat{D}_G \end{array}\right]$. Let $\mathcal{O}$ and $\hat{\mathcal{O}}$ denote the observability matrices corresponding to (A_G, C_G) and $(A_G, \hat{C}_G)$, respectively. Note that

$$\operatorname{rank} \hat{\mathcal{O}} = \operatorname{rank} \begin{bmatrix} S^{-1}C_G \\ S^{-1}C_G A_G \\ \vdots \\ S^{-1}C_G A_G^{n_G-1} \end{bmatrix} = \operatorname{rank}\,(I_{n_G} \otimes S^{-1})\mathcal{O} = \operatorname{rank} \mathcal{O} = n_G.$$

Thus $\hat{G} \overset{\min}{\sim} \left[\begin{array}{c|c} A_G & B_G \\ \hline \hat{C}_G & \hat{D}_G \end{array}\right]$. Note that

$$\hat{D}_1 = \begin{bmatrix} I_m & 0 \end{bmatrix} \begin{bmatrix} \hat{D}_1 \\ \hat{D}_2 \end{bmatrix} = \hat{D}_H \hat{D}_G = D_H S S^{-1} D_G = I_m. \tag{19}$$

Now, suppose that $K = 0$. Since $K(\mathbf{z}) = H(\mathbf{z})C_G(\mathbf{z}I - A_G)^{-1} = 0$, it follows that $(C_H(\mathbf{z}I - A_H)^{-1}B_H + D_H)C_G = H(\mathbf{z})C_G = 0$. Letting $\mathbf{z} \to \infty$ implies that $D_H C_G = 0$. Then

$$\hat{C}_1 = \begin{bmatrix} I_m & 0 \end{bmatrix} \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \end{bmatrix} = \hat{D}_H \hat{C}_G = D_H S S^{-1} C_G = 0. \tag{20}$$

Let $\mathcal{Z}$ denote the Rosenbrock system matrix of the minimal realization (1) of G. Since G has at least one zero zero, it follows that

$$n_G + m > \operatorname{rank} \mathcal{Z}(0) = \operatorname{rank} \begin{bmatrix} -A_G & B_G \\ C_G & -D_G \end{bmatrix}$$
$$= \operatorname{rank} \begin{bmatrix} I & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} -A_G & B_G \\ C_G & -D_G \end{bmatrix}$$
$$= \operatorname{rank} \begin{bmatrix} -A_G & B_G \\ \hat{C}_G & -\hat{D}_G \end{bmatrix}. \tag{21}$$

It follows from (19)–(21) that

$$n_G + m > \operatorname{rank} \begin{bmatrix} -A_G & B_G \\ \hat{C}_G & -\hat{D}_G \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -A_G & B_G \\ 0 & -I_m \\ \hat{C}_2 & -\hat{D}_2 \end{bmatrix}$$
$$= \operatorname{rank} \begin{bmatrix} -A_G & 0 \\ 0 & I_m \\ \hat{C}_2 & 0 \end{bmatrix}. \tag{22}$$

Since $(A_G, \hat{C}_G)$ is observable, it follows from the PBH test that $\operatorname{rank} \begin{bmatrix} -A_G \\ \hat{C}_G \end{bmatrix} = n_G$. Hence $\operatorname{rank} \begin{bmatrix} -A_G & 0 \\ 0 & I_m \\ \hat{C}_2 & 0 \end{bmatrix} = n_G + m$, which contradicts (22). Therefore, $K \neq 0$. □

It can be noted from Theorem 5.1 that, if $d \geq 1$, then it is not possible to reconstruct $u(0)$. In the case where $d = 0$ and G has at least one zero zero, Proposition 6.1 implies that $\operatorname{ind} K \geq 1$ and thus it follows from Theorem 5.1 that $u(0)$ cannot be reconstructed.

7. Illustrative examples

Example 7.1. Let

$$G(\mathbf{z}) = \begin{bmatrix} \dfrac{1}{\mathbf{z}^2} \\ \dfrac{1}{\mathbf{z}+1} \end{bmatrix}, \quad H(\mathbf{z}) = \begin{bmatrix} \dfrac{\mathbf{z}}{\mathbf{z}+1} & \dfrac{1}{\mathbf{z}^2} \end{bmatrix}, \tag{23}$$

so that $H(\mathbf{z})G(\mathbf{z}) = \mathbf{z}^{-2}$ and thus H is a delayed left inverse of G with delay 2. Fig. 1 shows the input and output of (6), (7) with zero initial conditions and with nonzero initial conditions. Note that H, which is an IIR transfer function, fails to reconstruct the input in the case where the initial conditions are nonzero. Now, let

$$H(\mathbf{z}) = \begin{bmatrix} 0 & \dfrac{\mathbf{z}+1}{\mathbf{z}} \end{bmatrix}, \tag{24}$$

$$H_\infty(\mathbf{z}) = \begin{bmatrix} -\dfrac{2\mathbf{z}^3+7\mathbf{z}^2+7\mathbf{z}+2}{2\mathbf{z}^3} & \dfrac{4\mathbf{z}^4+11\mathbf{z}^3+3\mathbf{z}^2-8\mathbf{z}-4}{2\mathbf{z}^4} & \dfrac{2\mathbf{z}^3+7\mathbf{z}^2+7\mathbf{z}+2}{2\mathbf{z}^4} \\ -\dfrac{1}{2} & -\dfrac{\mathbf{z}+2}{2\mathbf{z}} & \dfrac{2\mathbf{z}+1}{2\mathbf{z}} \end{bmatrix}$$

Box III.

so that $H(\mathbf{z})G(\mathbf{z}) = \mathbf{z}^{-1}$, and thus H is a delayed left inverse of G with delay 1. Fig. 2 shows the input and output of (6), (7) with zero and nonzero initial conditions. Note that H, which is an FIR transfer function, correctly reconstructs the input in the case where the initial conditions are nonzero.

Example 7.2. Let $G \in \mathbb{R}(\mathbf{z})^{3\times 2}_{\mathrm{prop}}$, where

$$G(\mathbf{z}) = \begin{bmatrix} \dfrac{\mathbf{z}}{\mathbf{z}+1} & \dfrac{1}{\mathbf{z}} \\ \dfrac{\mathbf{z}}{\mathbf{z}+2} & 0 \\ \dfrac{\mathbf{z}}{\mathbf{z}+1} & 1 \end{bmatrix}. \quad (25)$$

Then

$$S_\infty(\mathbf{z}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, W(\mathbf{z}) = \begin{bmatrix} \dfrac{\mathbf{z}+2}{2\mathbf{z}} & \dfrac{2(\mathbf{z}+1)}{\mathbf{z}^2} & \dfrac{2(2\mathbf{z}+1)}{\mathbf{z}} \\ \dfrac{\mathbf{z}+1}{2\mathbf{z}} & \dfrac{\mathbf{z}+1}{\mathbf{z}^2} & \dfrac{2\mathbf{z}+1}{\mathbf{z}} \\ \dfrac{\mathbf{z}+2}{2\mathbf{z}} & \dfrac{\mathbf{z}^2+\mathbf{z}+2}{\mathbf{z}^2} & \dfrac{3\mathbf{z}+2}{\mathbf{z}} \end{bmatrix},$$

$$V(\mathbf{z}) = \begin{bmatrix} 1 & -\dfrac{\mathbf{z}^2+3\mathbf{z}+2}{\mathbf{z}^3} \\ 0 & 1 \end{bmatrix}, \quad (26)$$

where S_∞ is the Smith–McMillan form at infinity of G and $G = WS_\infty V$. It follows from Proposition 4.1 that $\eta_G = 0$. Evaluating the expression for H_∞ given in Theorem 4.2 yields the equation given in Box III, which satisfies $H_\infty(\mathbf{z})G(\mathbf{z}) = I_2$. Hence, H_∞ is an FIR left inverse of G. Constructing minimal realizations of H_∞ and K shows that $\operatorname{ind} H_\infty = \operatorname{ind} K = 4$, where K is defined in Theorem 5.1 with H replaced by H_∞. Theorem 5.1 thus implies that $\nu = \max\{\operatorname{ind} H_\infty, \operatorname{ind} K, d\} = 4$ and hence, for all $k \geq 4$, $z(k) = u(k)$, where u and z are defined in (6), (7). Fig. 3 shows the input and output of (6), (7) with nonzero initial conditions. Note that, in this example, G is unstable and has a zero at zero.

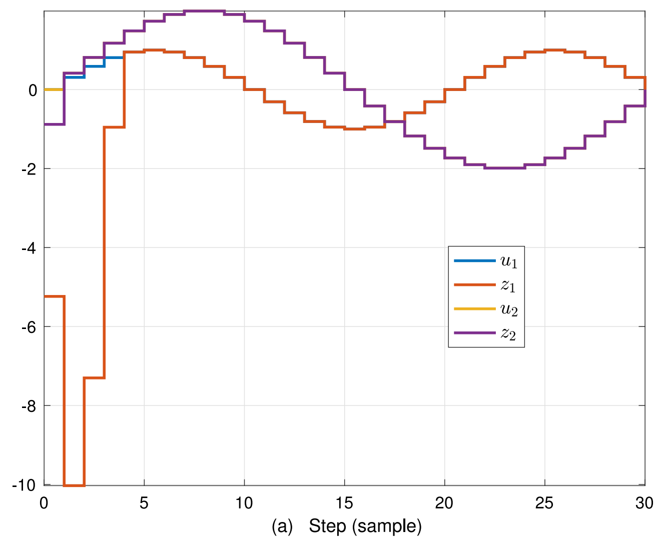

Fig. 3. Input and output of (6), (7), where $u = [u_1\ u_2]^\mathrm{T}$ and $z = [z_1\ z_2]^\mathrm{T}$. Note that, for all $k \geq 4$, $z_1(k) = u_1(k)$, and, for all $k \geq 1$, $z_2(k) = u_2(k)$. Hence, for all $k \geq 4$, $z(k) = u(k)$.

8. Conclusions

It was shown that, in the presence of an arbitrary unknown initial condition, finite-time input reconstruction is possible using a delayed left inverse H if and only if H is FIR. It was also shown that an FIR delayed left inverse with the minimal delay exists for systems with full column normal rank if and only if the system has zero nonzero zeros. A procedure for constructing an FIR delayed left inverse with the minimal delay was presented. Examples were provided for illustration. As part of future work, robustness of the obtained results to parameter uncertainties and noise will be analyzed.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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