Counting Zeros Using Observability and Block-Toeplitz Matrices

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***Abstract*—Transmission zeros can be counted by using the Smith-McMillan form, pole/zero modules, or the dimension of the largest output nulling invariant subspace. This paper provides an alternative approach by showing that the number of transmission zeros of a MIMO transfer function is given in terms of the defect of a block-Toeplitz matrix and the defect of an augmented matrix consisting of an observability matrix and the block-Toeplitz matrix. It is also shown that the number of infinite zeros is related to the defect of a block-Toeplitz matrix. These results are illustrated with a numerical example.**

***Index Terms*—Transmission zeros, infinite zeros, block-Toeplitz matrix, Smith-McMillan form.**

I. Introduction

One of the cornerstone results of linear systems theory is the fact that the rank of a block-Hankel matrix of Markov parameters (that is, impulse-response coefficents) of a linear system is equal to the McMillan degree of the corresponding transfer function [1]. The rank of a block-Hankel matrix thus counts the number of poles of a minimal realization. Although the number of poles can also be counted by forming the Smith-McMillan form [2, Theorem 6.7.5, p. 514] of the transfer function, that approach is infeasible for many applications. These remarks apply to both continuous-time and discrete-time systems. For system identification within the context of discrete-time systems, decomposition of a block-Hankel matrix of Markov parameters using the SVD-based eigensystem realization algorithm [3] provides an estimate of the McMillan degree as well as a minimal realization.

It has been shown in [4], [5] that the number of poles of a transfer function is equal to the sum of the number of transmission zeros, the number of infinite zeros [6], and the number of generic zeros (also known as Kronecker indices). The number of transmission zeros of a transfer function can be counted by forming the Smith-McMillan form, and the number of infinite zeros of a transfer function can be counted by forming the Smith-McMillan form at infinity [7]. Another approach is to compute the number of transmission and infinite zeros by using pole and zero modules [5].

Within the context of discrete-time systems, the contribution of the present paper is to present alternative characterizations of the number of transmission zeros and the number of infinite zeros. These characterizations involve observability and block-Toeplitz matrices, and the numbers of transmission zeros and infinite zeros are given in terms of defects rather than ranks. For counting zeros, these results serve as duals to the counting of poles using a block-Hankel matrix.

It is well known that the convolution involving the input to a discrete-time linear system can be expressed in terms of a block-Toeplitz matrix. This formulation is used extensively in adaptive control [8], [9] as well as in system inversion [10], [11] and input estimation [12]–[14]. The response of the system is thus a linear combination of the free response, as determined by an observability matrix, and the forced response, as determined by a block-Toeplitz matrix. Furthermore, a transmission zero has the property that, for a specific initial condition and a specific input sequence, the output of the system is identically zero. This observation suggests that the number of transmission zeros of a linear system is related to the defect of an augmented matrix involving both observability and block-Toeplitz matrices. These observations provide the underlying motivation for the results obtained in this paper.

A closely related notion is given by the geometric characterization of the number of zeros. In particular, it is shown in [15]–[19] that the number of transmission zeros is equal to the dimension of the largest output nulling invariant subspace. This characterization captures a combination of the free and forced responses, and thus can be viewed as a geometric interpretation of the algebraic condition alluded to above.

The outline of the paper is as follows. Section II summarizes the notation used in the paper. Expressions for the number of transmission zeros and the number of infinite zeros are derived in Sections III and IV, respectively. Section V presents a numerical example. Conclusions are given in Section VI.

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II. Preliminaries

Let $\mathbb{R}[\mathbf{z}]^{p\times m}$ denote the set of $p\times m$ matrices each of whose entries is a polynomial with real coefficients, let $\mathbb{R}(\mathbf{z})^{p\times m}$ denote the set of $p\times m$ matrices each of whose entries is a rational function with real coefficients, and let $\mathbb{R}(\mathbf{z})^{p\times m}_{\rm prop}$ denote the proper transfer functions in $\mathbb{R}(\mathbf{z})^{p\times m}$. Let $\overset{\min}{\sim}$ denote a minimal realization of a transfer function, let $\dim V$ denotes the dimension of a vector space V, and, for a real matrix A, let $\mathcal{R}(A)$ denote the range of A and ${\rm def}\, A$ denote the defect of A.

Let $G\in\mathbb{R}(\mathbf{z})^{p\times m}_{\rm prop}$, where $p\geq m$, $G\overset{\min}{\sim}\left[\begin{array}{c|c} A & B\\ \hline C & D\end{array}\right]$, and $A\in\mathbb{R}^{n\times n}$. Consider

$$x_{k+1}=\ Ax_k+Bu_k, \tag{1}$$
$$y_k=\ Cx_k+Du_k, \tag{2}$$

where, for all $k\geq 0$, $x_k\in\mathbb{R}^n$, $u_k\in\mathbb{R}^m$, and $y_k\in\mathbb{R}^p$. For all $l\geq 0$, define the $l^{\rm th}$ Markov parameter

$$H_l\triangleq\left\{\begin{array}{cc} D, & l=0,\\ CA^{l-1}B, & l\geq 1.\end{array}\right.$$

For all $l\geq 0$, define

$$\mathcal{Y}_l\triangleq\begin{bmatrix} y_0\\ y_1\\ \vdots\\ y_l\end{bmatrix}\in\mathbb{R}^{(l+1)p},\quad \mathcal{U}_l\triangleq\begin{bmatrix} u_0\\ u_1\\ \vdots\\ u_l\end{bmatrix}\in\mathbb{R}^{(l+1)m},$$

$$\Gamma_l\triangleq\begin{bmatrix} C\\ CA\\ \vdots\\ CA^l\end{bmatrix}\mathbb{R}^{(l+1)p\times n},$$

$$\mathcal{T}_l\triangleq\begin{bmatrix} H_0 & 0 & 0 & \cdots & 0\\ H_1 & H_0 & 0 & \cdots & 0\\ H_2 & H_1 & H_0 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ H_l & H_{l-1} & \cdots & H_1 & H_0\end{bmatrix}\in\mathbb{R}^{(l+1)p\times(l+1)m}.$$

Γ_l is the lth *observability matrix*, and $\mathcal{T}_l$ is the lth *block-Toeplitz matrix* associated with G. In the case where l is a negative integer, $\mathcal{T}_l$ is an empty matrix. It follows from (1), (2) that, for all $l\geq 0$,

$$\mathcal{Y}_l=\Gamma_l x_0+\mathcal{T}_l\mathcal{U}_l=\Psi_l\begin{bmatrix} x_0\\ \mathcal{U}_l\end{bmatrix}, \tag{3}$$

where

$$\Psi_l\triangleq\begin{bmatrix}\Gamma_l & \mathcal{T}_l\end{bmatrix}\in\mathbb{R}^{(l+1)p\times[n+(l+1)m]}.$$

For all $l\geq 0$, define

$$Q_l\triangleq\begin{bmatrix} H_0\\ H_1\\ \vdots\\ H_l\end{bmatrix}\in\mathbb{R}^{(l+1)p\times m},$$

$$P_l\triangleq\begin{bmatrix} 0\\ \mathcal{T}_{l-1}\end{bmatrix}\in\mathbb{R}^{(l+1)p\times lm},$$

so that $\mathcal{T}_l=\begin{bmatrix} Q_l & P_l\end{bmatrix}$. Define the following:

i) $\zeta\triangleq$ the number of transmission zeros of G.
ii) $\eta\triangleq\min\{l\geq 0:{\rm rank}\,\mathcal{T}_l=m+{\rm rank}\,\mathcal{T}_{l-1}\}$.

The parameter η plays a central role in system invertibility and input estimation and is discussed in detail in [12], [20].

The *rank* of G is the maximum value of ${\rm rank}\,G(\mathbf{z})$ taken over the set of complex numbers z such that, for all $i=1,\ldots,p$ and $j=1,\ldots,m$, $\mathbf{z}$ is not a pole of the (i,j) entry of G.

We assume for the rest of the paper that G has full column normal rank, that is, ${\rm rank}\,G=m$. This assumption implies that G is square or tall, that is, $p\geq m$. However, since G and $G^{\rm T}$ have the same poles and zeros, the results in this paper can be used in the case where G has full row rank, that is, ${\rm rank}\,G=p$. In this case, G is square or wide, that is, $m\geq p$.

III. Counting Transmission Zeros

In this section, we relate the number of transmission zeros of G to the defect of an augmented matrix involving an observability matrix and a block-Toeplitz matrix. The concept of output nulling invariant subspaces [16] acts as a bridge in establishing this relationship. The main result is Theorem III.4, which provides an expression for the number of transmission zeros.

Definition III.1. Let $V\subseteq\mathbb{R}^n$ and let

$$\begin{bmatrix} A\\ C\end{bmatrix}V\subseteq\begin{bmatrix} I\\ 0\end{bmatrix}V+\mathcal{R}\left(\begin{bmatrix} B\\ D\end{bmatrix}\right). \tag{4}$$

Then V is an *output-nulling invariant subspace* of (A,B,C,D). The sum of all output-nulling invariant subspaces of (A,B,C,D) is the *maximal output-nulling invariant subspace* of (A,B,C,D).

The following result is given by Theorem 11 in [17].

Proposition III.2. Let V^* be the maximal output-nulling invariant subspace of a minimal realization of G. Then, $\dim\ V^*=\zeta$.

Lemma III.3. Let V^* be the maximal output-nulling invariant subspace of (1), (2), and let $x_0\in V^*$. Then there exists an input sequence $(u_k)_{k\geq 0}$ such that, for all $k\geq 0$, $y_k=0$.

Proof. Since $x_0 \in V^*$, it follows from (4) that there exists $u_0 \in \mathbb{R}^m$ such that

$$x_1 = Ax_0 + Bu_0,$$
$$0 = Cx_0 + Du_0,$$

where $x_1 \in V^*$. Since $x_1 \in V^*$, it follows from (4) that there exists $u_1 \in \mathbb{R}^m$ such that

$$x_2 = Ax_1 + Bu_1,$$
$$0 = Cx_1 + Du_1,$$

where $x_2 \in V^*$. By induction, it follows that there exists an input sequence $(u_k)_{k\geq 0}$ such that, for all $k \geq 0$, $y_k = 0$. □

The following result characterizes the number of transmission zeros in terms of the defect of a block-Toeplitz matrix and the defect of a matrix consisting of an observability matrix and a block-Toeplitz matrix.

Theorem III.4. For all $l \geq n-1$,

$$\operatorname{def}\Psi_l - \operatorname{def}\mathcal{T}_l = \dim(\mathcal{R}(\Gamma_l) \cap \mathcal{R}(\mathcal{T}_l)) = \zeta.$$

Proof. It follows from Fact 3.14.15 in [2] that, for all $l \geq 0$,

$$\operatorname{def}\Psi_l = \operatorname{def}\Gamma_l + \operatorname{def}\mathcal{T}_l + \dim(\mathcal{R}(\Gamma_l) \cap \mathcal{R}(\mathcal{T}_l)). \quad (5)$$

Note that, for all $l \geq n-1$, $\operatorname{def}\Gamma_l = 0$. Hence (5) implies that, for all $l \geq n-1$,

$$\operatorname{def}\Psi_l - \operatorname{def}\mathcal{T}_l = \dim(\mathcal{R}(\Gamma_l) \cap \mathcal{R}(\mathcal{T}_l)).$$

Next, let V^* be the maximal output-nulling invariant subspace of (1), (2). Then Proposition III.2 implies that $\dim\ V^* = \zeta$. Let $x_{1,0}, x_{2,0}, \ldots, x_{\zeta,0}$ be a basis for V^*. It follows from Lemma III.3 that, for all $l \geq n-1$ and $i = 1, \ldots, \zeta$, there exists $\mathcal{U}_{l,i} \in \mathbb{R}^{(l+1)m}$ such that, when substituted for $\mathcal{U}_l$ in (3), yields $\mathcal{Y}_l = 0$. Thus, for all $l \geq n-1$ and $i = 1, \ldots, \zeta$, it follows that

$$\Gamma_l x_{i,0} + \mathcal{T}_l \mathcal{U}_{l,i} = 0.$$

For all $l \geq n-1$ and $i = 1, \ldots, \zeta$, define $z_{l,i} \stackrel{\triangle}{=} \Gamma_l x_{i,0} = -\mathcal{T}_l \mathcal{U}_{l,i}$. For all $l \geq n-1$, let $\alpha_{l,1}, \ldots, \alpha_{l,\zeta}$ be real numbers such that $\sum_{i=1}^{\zeta} \alpha_{l,i} z_{l,i} = 0$. Then, for all $l \geq n-1$,

$$0 = \sum_{i=1}^{\zeta} \alpha_{l,i} z_{l,i} = \sum_{i=1}^{\zeta} \alpha_{l,i} \Gamma_l x_{i,0} = \Gamma_l \sum_{i=1}^{\zeta} \alpha_{l,i} x_{i,0}.$$

Since, for all $l \geq n-1$, Γ_l has full column rank, it follows that $\sum_{i=1}^{\zeta} \alpha_{l,i} x_{i,0} = 0$ and thus $\alpha_{l,i} = 0$. Hence, for all $l \geq n-1$, $z_{l,1}, \ldots, z_{l,\zeta}$ are linearly independent. Now, for all $l \geq n-1$, define $z_l \stackrel{\triangle}{=} \Gamma_l x_0$, where $x_0 \stackrel{\triangle}{=} \sum_{i=1}^{\zeta} \beta_i x_{i,0}$. It follows that, for all $l \geq n-1$,

$$z_l = \Gamma_l \sum_{i=1}^{\zeta} \beta_i x_{i,0} = \sum_{i=1}^{\zeta} \beta_i \Gamma_l x_{i,0} = \sum_{i=1}^{\zeta} \beta_i z_{l,i}.$$

Thus, for all $l \geq n-1$, $\operatorname{span}\{z_{l,1}, \ldots, z_{l,\zeta}\} = \mathcal{R}(\Gamma_l) \cap \mathcal{R}(\mathcal{T}_l)$, and hence $\dim(\mathcal{R}(\Gamma_l) \cap \mathcal{R}(\mathcal{T}_l)) = \zeta$. □

IV. Counting Infinite Zeros

Infinite zeros extend the notion of relative degree to MIMO systems; in fact, for a SISO system, the number of infinite zeros is the relative degree of the transfer function. The main result in this section, Theorem IV.8, establishes a relationship between the number of infinite zeros and the defect of a block-Toeplitz matrix. All of the definitions and results given below support the main result.

Definition IV.1. Let $G \in \mathbb{R}(\mathbf{z})^{m\times p}_{\text{prop}}$, and let d be a nonnegative integer. Then, G is *delayed left invertible with delay* d if there exists $H \in \mathbb{R}(\mathbf{z})^{m\times p}_{\text{prop}}$ such that $H(\mathbf{z})G(\mathbf{z}) = \mathbf{z}^{-d} I_m$. In this case, H is a *delayed left inverse of* G *with delay* d.

Proposition IV.2. η is finite.

Proof. Note that, since G has full column rank, $[G(\mathbf{z})^{\mathrm{T}} G(\mathbf{z})]^{-1} G(\mathbf{z})^{\mathrm{T}}$ is a left inverse of G and thus there exists $d \geq 0$ such that $H(\mathbf{z}) = \mathbf{z}^{-d}[G(\mathbf{z})^{\mathrm{T}} G(\mathbf{z})]^{-1} G(\mathbf{z})^{\mathrm{T}}$ is a delayed left inverse of G with delay d. Hence G is delayed left invertible with delay d. Then Theorem 4 in [10] implies that $\operatorname{rank}\mathcal{T}_d - \operatorname{rank}\mathcal{T}_{d-1} = m$ and hence η is finite. □

Lemma IV.3. Let $l_0 \geq 0$. The following statements are equivalent:

i) $\operatorname{rank}\mathcal{T}_{l_0} - \operatorname{rank}\mathcal{T}_{l_0-1} = m$.
ii) $\operatorname{rank} Q_{l_0} = m$ and $\dim(\mathcal{R}(Q_{l_0}) \cap \mathcal{R}(P_{l_0})) = 0$.
iii) For all $l \geq l_0$, $\operatorname{rank}\mathcal{T}_l - \operatorname{rank}\mathcal{T}_{l-1} = m$.

Proof. To prove *i)* $\Longrightarrow$ *ii)*, note that it follows from Fact 3.14.15 in [2, p. 322] that $m = \operatorname{rank}\mathcal{T}_{l_0} - \operatorname{rank}\mathcal{T}_{l_0-1} = \operatorname{rank} Q_{l_0} - \dim(\mathcal{R}(Q_{l_0}) \cap \mathcal{R}(P_{l_0}))$. Thus, $m + \dim(\mathcal{R}(Q_{l_0}) \cap \mathcal{R}(P_{l_0})) = \operatorname{rank} Q_{l_0} \leq m$. Hence, $\operatorname{rank} Q_{l_0} = m$, and $\dim(\mathcal{R}(Q_{l_0}) \cap \mathcal{R}(P_{l_0})) = 0$.

To prove *ii)* $\Longrightarrow$ *iii)*, note that, for all $l \geq 0$,

$$Q_{l+1} = \begin{bmatrix} Q_l \\ H_{l+1} \end{bmatrix}, \quad P_{l+1} = \left[\begin{array}{ccc|c} & P_l & & 0 \\ \hline H_l & \cdots & H_1 & H_0 \end{array}\right].$$

Furthermore, for all $l \geq l_0$, $\operatorname{rank} Q_{l+1} = \operatorname{rank} Q_{l_0} = m$. Since $\operatorname{rank} Q_{l_0} = m$ and $\dim(\mathcal{R}(Q_{l_0}) \cap \mathcal{R}(P_{l_0})) = 0$, it follows from Lemma A in [12] that $\dim(\mathcal{R}(Q_{l_0+1}) \cap \mathcal{R}(P_{l_0+1})) = 0$. By induction, it thus follows that, for all $l \geq l_0$, $\dim(\mathcal{R}(Q_l) \cap \mathcal{R}(P_l)) = 0$. Thus, for all $l \geq l_0$, Fact 3.14.15 in [2, p. 322] implies that $\operatorname{rank} \mathcal{T}_l - \operatorname{rank} \mathcal{T}_{l-1} = \operatorname{rank} Q_l - \dim(\mathcal{R}(Q_l) \cap \mathcal{R}(P_l)) = m$.

The proof of *iii*) $\implies$ *i*) is immediate. □

Definition IV.4. Let $W \in \mathbb{R}(\mathbf{z})^{m\times m}_{\rm prop}$. Then W is *biproper* if $W_\infty \stackrel{\triangle}{=} \lim_{\mathbf{z}\to\infty} W(\mathbf{z})$ is nonsingular.

The following result, given by Theorem 2 in [7], presents the *Smith-McMillan form at infinity* S_∞ of G.

Theorem IV.5. Define $\rho_0 \stackrel{\triangle}{=} m - \operatorname{rank} G(\infty)$. Then there exist biproper transfer functions $W \in \mathbb{R}(\mathbf{z})^{p\times p}_{\rm prop}$ and $V \in \mathbb{R}(\mathbf{z})^{m\times m}_{\rm prop}$ and integers $\iota_1 \geq \iota_2 \geq \cdots \geq \iota_{\rho_0} > 0$ such that $G = W S_\infty V$, where

$$S_\infty(\mathbf{z}) \stackrel{\triangle}{=} \begin{bmatrix} \mathbf{z}^{-\iota_1} & & & & & \\ & \ddots & & & & \\ & & \mathbf{z}^{-\iota_{\rho_0}} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ & & 0_{(p-m)\times m} & & & \end{bmatrix}. \quad (6)$$

Definition IV.6. Let $\iota_1, \ldots, \iota_{\rho_0}$ be as defined in Theorem IV.5. Then ρ_0 is the *number of infinite zero directions*, for all $i = 1, \ldots, \rho_0$, ι_i is the *number of infinite zeros in the ith direction*, and $\iota \stackrel{\triangle}{=} \sum_{j=1}^{\rho_0} \iota_j$ is the *number of infinite zeros* of G.

The following result is given by Theorem 1 in [21].

Lemma IV.7. Let $G_1 \in \mathbb{R}(\mathbf{z})^{p\times m}_{\rm prop}$ and $G_2 \in \mathbb{R}(\mathbf{z})^{p\times m}_{\rm prop}$ be such that $G_2 = W G_1 V$, where $W \in \mathbb{R}(\mathbf{z})^{p\times p}_{\rm prop}$ and $V \in \mathbb{R}(\mathbf{z})^{m\times m}_{\rm prop}$ are biproper. Then, for all $i \geq 0$, $\operatorname{rank} \mathcal{T}_{1,i} - \operatorname{rank} \mathcal{T}_{1,i-1} = \operatorname{rank} \mathcal{T}_{2,i} - \operatorname{rank} \mathcal{T}_{2,i-1}$, where $\mathcal{T}_{1,i}$ and $\mathcal{T}_{2,i}$ are the $i^{\rm th}$ block-Toeplitz matrices associated with G_1 and G_2, respectively.

The following result characterizes the number of infinite zeros in terms of the defect of a block-Toeplitz matrix.

Theorem IV.8. For all $l \geq \eta - 1$, $\operatorname{def} \mathcal{T}_l = \iota$.

Proof. Note that it follows from Proposition IV.2 that η is finite. Next, Fact 3.14.15 in [2, p. 322] implies that, for all $l \geq 0$,

$$\operatorname{def} \mathcal{T}_l = \operatorname{def} Q_l + \operatorname{def} P_l + \dim(\mathcal{R}(Q_l) \cap \mathcal{R}(P_l)). \quad (7)$$

For all $l \geq \eta$, Lemma IV.3 implies that $\operatorname{rank} Q_l = m$, and $\dim(\mathcal{R}(Q_l) \cap \mathcal{R}(P_l)) = 0$. Therefore, it follows from (7) that, for all $l \geq \eta$, $\operatorname{def} \mathcal{T}_l = \operatorname{def} P_l = \operatorname{def} \begin{bmatrix} 0 \\ \mathcal{T}_{l-1} \end{bmatrix} = \operatorname{def} \mathcal{T}_{l-1}$. Hence, for all $l \geq \eta$, $\operatorname{def} \mathcal{T}_l = \operatorname{def} \mathcal{T}_{\eta-1}$.

Next, let S_∞ be the Smith-McMillan form at infinity of G, and let $\iota_1, \ldots, \iota_{\rho_0}$ be as defined in Theorem IV.5. Let $H_{\infty,j}$ be the jth Markov parameter of S_∞. It follows from Proposition 4.2 in [22] that $\eta = \iota_1$, and hence

$$\iota = \sum_{j=1}^{\rho_0} \iota_j = \sum_{j=1}^{\iota_1} j \operatorname{rank} H_{\infty,j} = \sum_{j=1}^{\eta} j \operatorname{rank} H_{\infty,j}. \quad (8)$$

Since G has full column normal rank, it follows that

$$m = \sum_{j=0}^{\eta} \operatorname{rank} H_{\infty,j}. \quad (9)$$

Let $\mathcal{T}_{\infty,i}$ be the ith block-Toeplitz matrix associated with S_∞. Note that, for all $i \geq 0$, each row of $\mathcal{T}_{\infty,i}$ is either zero or has exactly one nonzero entry that is equal to one, and the nonzero rows of $\mathcal{T}_{\infty,i}$ are linearly independent. It thus follows that, for all $i \geq 0$,

$$\operatorname{rank} \mathcal{T}_{\infty,i} = \sum_{j=0}^{i} (i - j + 1) \operatorname{rank} H_{\infty,j}. \quad (10)$$

$$S_\infty(\mathbf{z}) = \begin{bmatrix} \frac{1}{\mathbf{z}} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad V(\mathbf{z}) = \begin{bmatrix} \frac{1-\mathbf{z}}{\mathbf{z}} & \frac{(\mathbf{z}-1)(5\mathbf{z}^2 - 6\mathbf{z} - 9)}{4\mathbf{z}^3} \\ -\frac{6\mathbf{z}}{\mathbf{z}^2 + 4\mathbf{z} + 3} & \frac{6\mathbf{z} - 9}{2\mathbf{z}} \end{bmatrix}, \quad (15)$$

$$W(\mathbf{z}) = \begin{bmatrix} -\frac{6\mathbf{z}+9}{2\mathbf{z}} & \frac{4\mathbf{z}^4 + 21\mathbf{z}^3 + 21\mathbf{z}^2 - 27\mathbf{z} - 27}{12\mathbf{z}^4} & 1 \\ -\frac{2\mathbf{z}+3}{2\mathbf{z}} & \frac{7\mathbf{z}^3 + 7\mathbf{z}^2 - 9\mathbf{z} - 9}{12\mathbf{z}^4} & \frac{1}{3} \\ -\frac{10\mathbf{z}^2 + 15\mathbf{z} + 9}{4\mathbf{z}^2} & \frac{8\mathbf{z}^5 + 37\mathbf{z}^4 + 38\mathbf{z}^3 - 18\mathbf{z}^2 - 54\mathbf{z} - 27}{24\mathbf{z}^5} & \frac{\mathbf{z}+1}{2\mathbf{z}} \end{bmatrix}, \quad (16)$$

Hence (8), (9), and (10) imply that

$$
\begin{aligned}
\operatorname{def} \mathcal{T}_{\infty,\eta-1} &= \eta m - \sum_{j=0}^{\eta-1} (\eta - j) \operatorname{rank} H_{\infty,j} \\
&= \eta(m - \operatorname{rank} H_{\infty,0}) - \eta \sum_{j=1}^{\eta-1} \operatorname{rank} H_{\infty,j} \\
&\quad + \sum_{j=1}^{\eta-1} j \operatorname{rank} H_{\infty,j} \\
&= \eta(m - \operatorname{rank} H_{\infty,0}) \\
&\quad - \eta(m - \operatorname{rank} H_{\infty,0} - \operatorname{rank} H_{\infty,\eta}) \\
&\quad + \sum_{j=1}^{\eta-1} j \operatorname{rank} H_{\infty,j} \\
&= \sum_{j=1}^{\eta} j \operatorname{rank} H_{\infty,j} = \iota.
\end{aligned}
$$

Next, since $\mathcal{T}_{-1}$ is an empty matrix, it follows from Lemma IV.7 and Theorem IV.5 that, for all $l \geq 0$, $\operatorname{rank} \mathcal{T}_l = \operatorname{rank} \mathcal{T}_{\infty,l}$. Hence, $\operatorname{def} \mathcal{T}_{\eta-1} = \operatorname{def} \mathcal{T}_{\infty,\eta-1} = \iota$. □

V. Numerical Example

Let

$$
G = \begin{bmatrix} \frac{1}{\mathbf{z}+1} & 1 \\ \frac{1}{\mathbf{z}+3} & \frac{1}{2\mathbf{z}} \\ \frac{1}{2\mathbf{z}} & 1 \end{bmatrix}. \tag{11}
$$

Numerical computation using Matlab yields

i) $n = 4$, $\eta = 1$.
ii) $\operatorname{def} \mathcal{T}_0 = 1$.
iii) $\operatorname{def} \Psi_0 = 3$, and $\operatorname{def} \Psi_l = 2$, for $l = 1, 2, 3$.

Theorem IV.8 thus implies that, for all $l \geq 0$, $\operatorname{def} \mathcal{T}_l = 1$, and thus $\iota = 1$. Similarly, Theorem III.4 implies that, for all $l \geq 3$, $\operatorname{def} \Psi_l = 2$, and thus $\zeta = 1$.

As a check, the numbers of infinite and transmission zeros are calculated from the Smith-McMillan form at infinity and the Smith-McMillan form, respectively, as follows. Note that $G = W S_\infty V$, where S_∞, W, and V are given by (15), (16), S_∞ is the Smith-McMillan form at infinity of G, and W and V are biproper transfer functions. It can be seen from S_∞ that $\iota = 1$. Next, note that $G = S_1 S S_2$, where

$$
S(\mathbf{z}) = \begin{bmatrix} \frac{1}{\mathbf{z}(\mathbf{z}+1)(\mathbf{z}+3)} & 0 \\ 0 & \frac{\mathbf{z}-1}{\mathbf{z}} \\ 0 & 0 \end{bmatrix}, \tag{12}
$$

$$
S_1(\mathbf{z}) = \begin{bmatrix} \mathbf{z}(\mathbf{z}+3) & -\frac{\mathbf{z}(2\mathbf{z}+9)}{12} & -\frac{\mathbf{z}+6}{6} \\ \mathbf{z}(\mathbf{z}+1) & -\frac{2\mathbf{z}^2+5\mathbf{z}+6}{12} & -\frac{\mathbf{z}+4}{6} \\ \frac{\mathbf{z}^2+4\mathbf{z}+3}{2} & \frac{-2\mathbf{z}^2-11\mathbf{z}+3}{24} & -\frac{\mathbf{z}+7}{12} \end{bmatrix}, \tag{13}
$$

$$
S_2(\mathbf{z}) = \begin{bmatrix} 1 & \frac{2\mathbf{z}^3+9\mathbf{z}^2+10\mathbf{z}+3}{12} \\ 0 & 1 \end{bmatrix}, \tag{14}
$$

S is the Smith-McMillan form of G, and S_1 and S_2 are unimodular matrices. It can be seen from S that $\zeta = 1$.

VI. Conclusions

It was shown that the number of transmission zeros of a MIMO transfer function is given in terms of the defect of a block-Toeplitz matrix and the defect of an augmented matrix consisting of an observability matrix and the block-Toeplitz matrix. It was also proved that the number of infinite zeros is related to the defect of a block-Toeplitz matrix. A numerical example was given to illustrate these results.

Future research will focus on numerically estimating the numbers of infinite zeros and transmission zeros in the presence of noisy data. In particular, by applying the singular value decomposition and nuclear norm minimization [23], [24] to the matrices Ψ and $\mathcal{T}$ obtained from subspace identification [25], it may be possible to estimate the number of zeros. The application of these results to improving the accuracy of the computation of zeros using standard methods [26] is another promising topic for future work.

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