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Brief Paper

H_2-suboptimal Stable Stabilization*†

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Key Words—H_2 optimization; strong stabilization; stable control; reduced-order control.

Abstract—In this paper we present two approaches for designing H_2-suboptimal stable controllers. Both full-order and reduced-order controllers are considered.

1. *Introduction*

ALTHOUGH LQG THEORY provides stabilizing controllers, these controllers may not be stable, even if the open-loop plant is stable. The problem of synthesizing stable stabilizing controllers has been of interest for many years (Youla *et al.*, 1974) and a variety of techniques have been proposed (Smith and Sondergeld, 1986; Boyd, 1987; Ganesh and Pearson, 1986, 1989; Jacobus, 1990; Jacobus *et al.*, 1990; Halevi *et al.*, 1991).

In this paper we present new results that are in the spirit of Jacobus (1990), Jacobus *et al.* (1990) and Halevi *et al.* (1991). Specifically, in these references the authors modify full- and reduced-order LQG theory (Hyland and Bernstein, 1984) to obtain suboptimal controllers that are stable. The new results given herein are based upon two different modifications of LQG theory that offer advantages over these earlier approaches. The first approach (Section 2) is based upon an a posteriori modification of LQG theory in the vein of Halevi *et al.* (1991). Unlike the technique of Halevi *et al.* (1991), our modification of LQG theory involves a third equation coupled to the regulator Riccati equation. The advantage of our approach over Halevi *et al.* (1991) is a unified treatment of the reduced-order case (Section 3).

Our second approach (Section 4) involves an a priori modification to LQG theory (that is, prior to optimization) in the vein of Jacobus (1990) and Jacobus *et al.* (1990). Our approach is an improvement over the approach of Jacobus (1990) and Jacobus *et al.* (1990) in that the modification to the design equations is less conservative, that is, sacrifices less H_2 performance in return for yielding a stable compensator.

2. *Full-order compensation*

Consider the nth-order plant

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t), \tag{1}$$

$$y(t) = Cx(t) + D_2 w(t), \tag{2}$$

with performance variables

$$z(t) = E_1 x(t) + E_2 u(t), \tag{3}$$

where $w(t)$ is standard white noise. Using the n_cth-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{4}$$

$$u(t) = C_c x_c(t), \tag{5}$$

we obtain the closed-loop system

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t), \tag{6}$$

$$z(t) = \tilde{E}\tilde{x}(t), \tag{7}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix},$$

$$\tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix}, \quad \tilde{E} \triangleq [E_1 \quad E_2 C_c].$$

The H_2 performance index is defined by

$$J(A_c, B_c, C_c) = \lim_{t\to\infty} \mathscr{E}[x^{\mathrm{T}}(t) R_1 x(t) + u^{\mathrm{T}}(t) R_2 u(t)], \tag{8}$$

where '$\mathscr{E}$' denotes expectation and $R_1 \triangleq E_1^{\mathrm{T}} E_1$, $R_{12} \triangleq E_1^{\mathrm{T}} E_2 = 0$, $R_2 \triangleq E_2^{\mathrm{T}} E_2 > 0$. For convenience, we define $V_1 \triangleq D_1 D_1^{\mathrm{T}}$, $V_{12} \triangleq D_1 D_2^{\mathrm{T}} = 0$, $V_2 \triangleq D_2 D_2^{\mathrm{T}} > 0$.

The H_2-optimal control problem can be stated as follows: minimize the H_2 performance $J(A_c, B_c, C_c)$ given in (8) or, equivalently

$$J(A_c, B_c, C_c) = \operatorname{tr} \tilde{Q}\tilde{R} \tag{9}$$

subject to

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{V}, \tag{10}$$

where

$$\tilde{R} \triangleq \tilde{E}^{\mathrm{T}}\tilde{E} = \begin{bmatrix} R_1 & 0 \\ 0 & C_c^{\mathrm{T}} R_2 C_c \end{bmatrix}, \quad \tilde{V} \triangleq \tilde{D}\tilde{D}^{\mathrm{T}} = \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^{\mathrm{T}} \end{bmatrix}.$$

In the following development, we assume that both (A, B) and (A, D_1) are stabilizable and both (C, A) and (E_1, A) are detectable. Then, it is well known that the optimal full-order controller (4), (5) is given by

$$A_c = A + BC_c - B_c C, \tag{11}$$

$$B_c = QC^{\mathrm{T}} V_2^{-1}, \tag{12}$$

$$C_c = -R_2^{-1} B^{\mathrm{T}} P, \tag{13}$$

where Q, P are nonnegative-definite matrices satisfying

$$0 = A^{\mathrm{T}} P + PA + R_1 - P\Sigma P, \tag{14}$$

$$0 = AQ + QA^{\mathrm{T}} + V_1 - Q\bar{\Sigma}Q, \tag{15}$$

where $\Sigma \triangleq BR_2^{-1}B^{\mathrm{T}}$, $\bar{\Sigma} \triangleq C^{\mathrm{T}} V_2^{-1} C$. Note that (11) can be written as

$$A_c = A - Q\bar{\Sigma} - \Sigma P. \tag{16}$$

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Since $A - \Sigma P$ and $A - Q\bar{\Sigma}$ are asymptotically stable, there exist nonnegative-definite matrices $\hat{Q}$ and $\hat{P}$ such that

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^{\mathrm{T}} + Q\bar{\Sigma}Q, \tag{17}$$

$$0 = (A - Q\bar{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P. \tag{18}$$

The optimal cost (8) is thus given by either of the expressions

$$\begin{aligned} J(A_c, B_c, C_c) &= \mathrm{tr}\,[(Q + \hat{Q})R_1 + \hat{Q}P\Sigma P] \\ &= \mathrm{tr}\,[(P + \hat{P})V_1 + \hat{P}Q\bar{\Sigma}Q] \\ &= \mathrm{tr}\,[QR_1 + PQ\bar{\Sigma}Q] \\ &= \mathrm{tr}\,[PV_1 + QP\Sigma P]. \end{aligned} \tag{19}$$

Furthermore, the state cost is given by

$$J_s(A_c, B_c, C_c) \triangleq \lim_{t\to\infty} \mathscr{E}[x^{\mathrm{T}}(t)R_1x(t)] = \mathrm{tr}\,(Q + \hat{Q})R_1,$$

while the control cost is given by

$$J_c(A_c, B_c, C_c) \triangleq \lim_{t\to\infty} \mathscr{E}[u^{\mathrm{T}}(t)R_2u(t)] = \mathrm{tr}\,\hat{Q}P\Sigma P.$$

In general, the LQG result does not guarantee that A_c is asymptotically stable. The goal of the following result is to obtain a suboptimal controller (4), (5) such that $\tilde{A}$ is asymptotically stable and A_c is either Lyapunov stable or asymptotically stable.

Theorem 2.1. Suppose there exist $\alpha, \beta > 0$ and nonnegative-definite matrices Q, P and $\hat{P}$ satisfying

$$0 = AQ + QA^{\mathrm{T}} + V_1 - Q\bar{\Sigma}Q, \tag{20}$$

$$\begin{aligned} 0 = A^{\mathrm{T}}P + PA + R_1 - P\Sigma P + (\alpha P - \alpha^{-1}\hat{P}) \\ \times \Sigma(\alpha P - \alpha^{-1}\hat{P}) + \beta A^{\mathrm{T}}A + \beta^{-1}P^2, \end{aligned} \tag{21}$$

$$0 = (A - Q\bar{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P, \tag{22}$$

and let (A_c, B_c, C_c) be given by (11)–(13). Then $\tilde{A}$ is asymptotically stable, A_c is Lyapunov stable, and the closed-loop cost (8) is given by (19) where $\hat{Q}$ satisfies (17). If, in addition, $R_1 > 0$, then A_c is asymptotically stable.

Proof. Defining

$$\begin{aligned} \bar{R}_1 \triangleq R_1 + (\alpha P - \alpha^{-1}\hat{P})\Sigma(\alpha P - \alpha^{-1}\hat{P}) \\ + \beta A^{\mathrm{T}}A + \beta^{-1}P^2 \geq 0, \end{aligned}$$

it is seen that (20) and (21) are in the form of the standard LQG Riccati equations, (14) and (15), with R_1 replaced by $\bar{R}_1$. Thus $\tilde{A}$ is asymptotically stable. Now combining (21) and (22) yields

$$\begin{aligned} A_c^{\mathrm{T}}\hat{P} + \hat{P}A_c = -[R_1 + (\beta^{1/2}A + \beta^{-1/2}P)^{\mathrm{T}} \\ \times (\beta^{1/2}A + \beta^{-1/2}P)] \leq 0, \end{aligned}$$

which shows that A_c is Lyapunov stable. If $R_1 > 0$, then $A_c^{\mathrm{T}}\hat{P} + \hat{P}A_c < 0$ which further implies that A_c is asymptotically stable. □

Note that unlike the standard LQG result and its modification by Halevi *et al.* (1991) to stable controllers, Theorem 2.1 involves three matrix equations. Equation (20) is the standard estimator Riccati equation, while equations (21) and (22) are coupled in P and $\hat{P}$. Note that Theorem 2.1 does not assume that A is asymptotically stable. Hence, there may not exist a stable compensator that stabilizes the plant (Youla *et al.*, 1974). Furthermore, even if a stable stabilizer exists, its order may be greater than that of the plant. Nevertheless, Theorem 2.1 provides a constructive sufficient condition for stable, full-order compensation.

3. *Reduced-order dynamic compensation*

In this section, we focus on the reduced-order case $n_c < n$. First we recall from Hyland and Bernstein (1984) the necessary conditions for H_2-optimal reduced-order compensation.

Theorem 3.1. Let $n_c \leq n$, suppose (A_c, B_c, C_c) minimizes $J(A_c, B_c, C_c)$ and assume that (A_c, B_c) is stabilizable. Then there exist $n \times n$ nonnegative-definite matrices Q, P, $\hat{Q}$, $\hat{P}$ such that A_c, B_c, C_c are given by

$$A_c = \Gamma(A - Q\bar{\Sigma} - \Sigma P)G^{\mathrm{T}}, \tag{23}$$

$$B_c = \Gamma QC^{\mathrm{T}}V_2^{-1}, \tag{24}$$

$$C_c = -R_2^{-1}B^{\mathrm{T}}PG^{\mathrm{T}}, \tag{25}$$

where Q, P, $\hat{Q}$, $\hat{P}$, Γ and G satisfy

$$0 = AQ + QA^{\mathrm{T}} + V_1 - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}}, \tag{26}$$

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^{\mathrm{T}} + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}}, \tag{27}$$

$$0 = A^{\mathrm{T}}P + PA + R_1 - P\Sigma P + \tau_\perp^{\mathrm{T}}P\Sigma P\tau_\perp, \tag{28}$$

$$0 = (A - Q\bar{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P - \tau_\perp^{\mathrm{T}}P\Sigma P\tau_\perp, \tag{29}$$

$$\mathrm{rank}\,\hat{Q} = \mathrm{rank}\,\hat{P} = \mathrm{rank}\,\hat{Q}\hat{P} = n_c, \tag{30}$$

$$\hat{Q}\hat{P} = G^{\mathrm{T}}M\Gamma, \quad \Gamma G^{\mathrm{T}} = I_{n_c}, \quad M \in \mathscr{R}^{n_c \times n_c}, \tag{31}$$

$$\tau \triangleq G^{\mathrm{T}}\Gamma, \quad \tau_\perp \triangleq I_n - \tau, \tag{32}$$

$$\hat{Q} = \tau\hat{Q}, \quad \hat{P} = \hat{P}\tau. \tag{33}$$

Furthermore, the closed-loop cost (8) is given by either of the expressions

$$\begin{aligned} J(A_c, B_c, C_c) &= \mathrm{tr}\,[(Q + \hat{Q})R_1 + \Gamma\hat{Q}P\Sigma PG^{\mathrm{T}}] \\ &= \mathrm{tr}\,[(P + \hat{P})V_1 + G\hat{P}Q\bar{\Sigma}Q\Gamma^{\mathrm{T}}] \\ &= \mathrm{tr}\,[QR_1 + P(Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}})] \\ &= \mathrm{tr}\,[PV_1 + Q(P\Sigma P - \tau_\perp^{\mathrm{T}}P\Sigma P\tau_\perp)]. \end{aligned} \tag{34}$$

As in the full-order case, Theorem 3.1 does not guarantee that the controller is asymptotically stable. To construct an asymptotically stable A_c, we introduce the following extension of Theorem 3.1.

Theorem 3.2. Suppose there exist $\alpha, \beta > 0$ and $n \times n$ nonnegative-definite matrices Q, P, $\hat{Q}$ and $\hat{P}$ satisfying (30)–(33) and

$$0 = AQ + QA^{\mathrm{T}} - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}} + V_1, \tag{35}$$

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^{\mathrm{T}} + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}}, \tag{36}$$

$$\begin{aligned} 0 = A^{\mathrm{T}}P + PA - P\Sigma P + \tau_\perp^{\mathrm{T}}P\Sigma P\tau_\perp + R_1 + (\alpha P - \alpha^{-1}\hat{P}) \\ \times \Sigma(\alpha P - \alpha^{-1}\hat{P}) + \beta A^{\mathrm{T}}A + \beta^{-1}P^2, \end{aligned} \tag{37}$$

$$0 = (A - Q\bar{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P - \tau_\perp^{\mathrm{T}}P\Sigma P\tau_\perp, \tag{38}$$

and let (A_c, B_c, C_c) be given by (23)–(25). Then $\tilde{A}$ and A_c are Lyapunov stable and the closed-loop cost (8) is given by (34). If, in addition, (A_c, B_c) is stabilizable, then $\tilde{A}$ is asymptotically stable, while if $R_1 > 0$ and (C_c, A_c) is observable, then A_c is asymptotically stable.

Proof. Defining $\bar{R}_1$ as in the proof of Theorem 2.1, it can be seen that (35)–(38) are in the form of the reduced-order LQG synthesis equations with R_1 replaced by $\bar{R}_1$ in (37). It thus follows that

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \bar{\tilde{R}},$$

where

$$\tilde{Q} \triangleq \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^{\mathrm{T}} \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^{\mathrm{T}} \end{bmatrix} \geq 0, \quad \bar{\tilde{R}} = \begin{bmatrix} \bar{R}_1 & 0 \\ 0 & C_c^{\mathrm{T}}R_2C_c \end{bmatrix} \geq 0,$$

and thus $\tilde{A}$ is Lyapunov stable. Now if $(\tilde{A}, \bar{\tilde{R}})$ is stabilizable, then it follows from Lemma 2.1 that $\tilde{A}$ is asymptotically stable. Adding (37) to (38) yields

$$\begin{aligned} 0 = (A - Q\bar{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + A^{\mathrm{T}}P + PA \\ + R_1 + (\alpha P - \alpha^{-1}\hat{P})\Sigma(\alpha P - \alpha^{-1}\hat{P}) \\ + \beta A^{\mathrm{T}}A + \beta^{-1}P^2, \end{aligned}$$

which can be rewritten as

$$0 = (A - Q\bar{\Sigma} - \Sigma P)^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma} - \Sigma P) + R_1 + \alpha^2 P\Sigma P + \alpha^{-2}\hat{P}\Sigma\hat{P} + (\beta^{1/2}A + \beta^{-1/2}P)^{\mathrm{T}}(\beta^{1/2}A + \beta^{-1/2}P).$$

Using the fact that $\hat{P}\tau = \hat{P}$ and letting $P_2 \triangleq G\hat{P}G^{\mathrm{T}} \geq 0$ (Hyland and Bernstein, 1984), it follows that

$$A_c^{\mathrm{T}}P_2 + P_2A_c = -G[R_1 + \alpha^2 P\Sigma P + \alpha^{-2}\hat{P}\Sigma\hat{P} + (\beta^{1/2}A + \beta^{-1/2}P)^{\mathrm{T}}(\beta^{1/2}A + \beta^{-1/2}P)]G^{\mathrm{T}} \leq 0,$$

which implies that A_c is Lyapunov stable. Furthermore, if $R_1 > 0$ and (C_c, A_c) is observable, then $GR_1G^{\mathrm{T}} > 0$ and $P_2 > 0$. Thus A_c is asymptotically stable. □

Remark 3.1. Note that by setting $n_c = n$ and thus $\tau = I$, we recover Theroem 2.1.

4. *An alternative approach based upon cost modification*

The cost modification approach for obtaining a stable compensator was introduced by Jacobus (1990) and Jacobus *et al.* (1990). This approach addresses the minimization problem

$$\mathscr{J}(A_c, B_c, C_c) = \operatorname{tr} \mathscr{Q}\tilde{R} \qquad (39)$$

subject to

$$0 = \tilde{A}\mathscr{Q} + \mathscr{Q}\tilde{A}^{\mathrm{T}} + \tilde{V} + \Omega(\mathscr{Q}), \qquad (40)$$

where $\Omega(\cdot)$ is a matrix function that satisfies $\Omega(\mathscr{Q}) \geq 0$ for all $\mathscr{Q} \geq 0$ while guaranteeing that A_c is Lyapunov or asymptotically stable.

Note that if $\Omega(\mathscr{Q}) = 0$, then (40) is the standard covariance Lyapunov equation and we recover the standard H_2 problem. If $\tilde{Q}$ denotes the solution of (40) with $\Omega(\mathscr{Q}) = 0$, then it follows that

$$\mathscr{Q} - \tilde{Q} = \int_0^\infty e^{\tilde{A}t}\Omega(\mathscr{Q})e^{\tilde{A}^{\mathrm{T}}t}\,dt \geq 0, \qquad (41)$$

where $\mathscr{Q}$ satisfies (40). This shows that $\mathscr{Q}$ is a bound for $\tilde{Q}$. Consequently, the modified covariance matrix $\mathscr{Q}$ leads to a suboptimal controller. Several approaches were developed by Jacobus (1990) and Jacobus *et al.* (1990) for constructing $\Omega(\mathscr{Q})$ to obtain stable compensators. However, those approaches were found to be quite conservative by unnecessarily sacrificing H_2 performance to obtain a stable compensator.

Here we introduce a less conservative choice for $\Omega(\mathscr{Q})$. Specifically, we choose

$$\Omega(\mathscr{Q}) = \begin{bmatrix} 0 & 0 \\ 0 & Q_{12}^{\mathrm{T}}\bar{\Sigma}Q_{12} \end{bmatrix}, \qquad (42)$$

where

$$\mathscr{Q} \triangleq \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_2 \end{bmatrix}.$$

By using the Lagrange multiplier method to minimize (39) subject to (40), it follows that B_c, C_c are given by (12), (13) for the full-order case and (24), (25) for the reduced-order case. Furthermore, it can be shown that in the full-order case A_c satisfies

$$A_c\hat{Q} + \hat{Q}A_c^{\mathrm{T}} = -[(Q + \hat{Q})\bar{\Sigma}(Q + \hat{Q})] \leq 0, \qquad (43)$$

while in the reduced-order case,

$$A_cQ_2 + Q_2A_c^{\mathrm{T}} = -\Gamma[(Q + \hat{Q})\bar{\Sigma}(Q + \hat{Q})]\Gamma^{\mathrm{T}} \leq 0. \qquad (44)$$

Thus, in both cases, (40) guarantees that A_c is Lyapunov stable. The above steps yield the following result:

Theorem 4.1. Suppose there exist nonnegative-definite matrices Q, P, $\hat{Q}$, $\hat{P}$ satisfying

$$0 = AQ + QA^{\mathrm{T}} - Q\bar{\Sigma}Q + \hat{Q}\bar{\Sigma}\hat{Q} + V_1, \qquad (45)$$

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^{\mathrm{T}} + Q\bar{\Sigma}Q - \hat{Q}\bar{\Sigma}\hat{Q}, \qquad (46)$$

$$0 = A^{\mathrm{T}}P + PA - P\Sigma P + R_1 + \hat{P}\hat{Q}\bar{\Sigma} + \bar{\Sigma}\hat{Q}\hat{P}, \qquad (47)$$

$$0 = (A - (Q + \hat{Q})\bar{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A - (Q + \hat{Q})\bar{\Sigma}) + P\Sigma P, \qquad (48)$$

and let (A_c, B_c, C_c) be given by

$$A_c = A - (Q + \hat{Q})\bar{\Sigma} - \Sigma P, \qquad (49)$$

(12) and (13). Then A_c is Lyapunov stable and the modified cost (39) is given by (19). Furthermore, if (A_c, B_c) is stabilizable, then $\tilde{A}$ is asymptotically stable.

Remark 4.1. Note that $A - (Q + \hat{Q})\bar{\Sigma}$ in (48) is not necessarily asymptotically stable.

Using $\Omega(\mathscr{Q})$ given in (42), we obtain the following sufficient condition for reduced-order stable stabilization.

Theorem 4.2. Suppose there exist nonnegative-definite matrices Q, P, $\hat{Q}$, $\hat{P}$ satisfying (30)–(33) and

$$0 = AQ + QA^{\mathrm{T}} - Q\bar{\Sigma}Q + V_1 + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}} + \hat{Q}\bar{\Sigma}\hat{Q}, \qquad (50)$$

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^{\mathrm{T}} + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}} - \hat{Q}\bar{\Sigma}\hat{Q}, \qquad (51)$$

$$0 = A^{\mathrm{T}}P + PA - P\Sigma P + R_1 + \tau_\perp^{\mathrm{T}}P\Sigma P\tau_\perp + \hat{P}\hat{Q}\bar{\Sigma} + \bar{\Sigma}\hat{Q}\hat{P}, \qquad (52)$$

$$0 = (A - (Q + \hat{Q})\bar{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A - (Q + \hat{Q})\bar{\Sigma}) + P\Sigma P - \tau_\perp^{\mathrm{T}}P\Sigma P\tau_\perp, \qquad (53)$$

and let (A_c, B_c, C_c) be given by

$$A_c = \Gamma(A - (Q + \hat{Q})\bar{\Sigma} - \Sigma P)G^{\mathrm{T}}, \qquad (54)$$

(24) and (25). Then, A_c is Lyapunov stable and the modified cost (39), (40) is given by (34). Furthermore, if (A_c, B_c) is stabilizable, then $\tilde{A}$ is asymptotically stable.

Note that Theorems 4.1 and 4.2 guarantee that A_c is Lyapunov stable but not necessarily asymptotically stable.

5. *Numerical algorithm and illustrative examples*

We implement Newton's method to solve (21), (22). The method involves a first-order parameter variation in the unknown parameters P, $\hat{P}$. Hence let $\delta P = P_1 - P_0$ and $\delta\hat{P} = \hat{P}_1 - \hat{P}_0$, where P_0, $\hat{P}_0$ and P_1, $\hat{P}_1$ represent the current and the updated points, respectively. Letting $0 = \mathscr{F}(P, \hat{P})$ and $0 = \mathscr{G}(P, \hat{P})$ represent (21), (22) and applying a first-order parameter variation in P, $\hat{P}$ at the current point P_0, $\hat{P}_0$ yields

$$0 = \operatorname{vec}\mathscr{F}(P_0, \hat{P}_0) + \mathscr{F}_P(P_0, \hat{P}_0)\operatorname{vec}\delta P + \mathscr{F}_{\hat{P}}(P_0, \hat{P}_0)\operatorname{vec}\delta\hat{P}, \qquad (55)$$

$$0 = \operatorname{vec}\mathscr{G}(P_0, \hat{P}_0) + \mathscr{G}_P(P_0, \hat{P}_0)\operatorname{vec}\delta P + \mathscr{G}_{\hat{P}}(P_0, \hat{P}_0)\operatorname{vec}\delta\hat{P}, \qquad (56)$$

where 'vec' denotes the column stacking operator (Brewer, 1976). If

$$\begin{bmatrix} \mathscr{F}_P & \mathscr{F}_{\hat{P}} \\ \mathscr{G}_P & \mathscr{G}_{\hat{P}} \end{bmatrix}_{P_0, \hat{P}_0}$$

is invertible, then

$$\begin{bmatrix} \operatorname{vec}\delta P \\ \operatorname{vec}\delta\hat{P} \end{bmatrix} = -\begin{bmatrix} \mathscr{F}_P & \mathscr{F}_{\hat{P}} \\ \mathscr{G}_P & \mathscr{G}_{\hat{P}} \end{bmatrix}_{P_0, \hat{P}_0}^{-1} \begin{bmatrix} \operatorname{vec}\mathscr{F} \\ \operatorname{vec}\mathscr{G} \end{bmatrix}_{P_0, \hat{P}_0}. \qquad (57)$$

We can then solve for the updated point P_1, $\hat{P}_1$. The LQG result provides the initial condition for this algorithm.

Example 1. Consider the two mass system shown in Fig. 1 with $m_1 = m_2 = k = 1$ such that

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0 \quad 0],$$

with disturbance weighting matrices given by

$$D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ .68 & 0 \end{bmatrix}, \quad D_2 = [0 \quad 1],$$

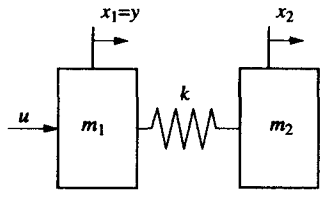

FIG. 1. Two mass system.

and performance weighting matrices given by

$$E_1=\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\quad E_2=\begin{bmatrix}0\\ 0.01\end{bmatrix}.$$

The eigenvalues of A are $\{0, 0, \pm j1.4142\}$. In the full-order case, LQG design yields the closed-loop poles $\{-99.985, -1.0277 \pm j2.8161, -2.4941 \pm j1.1604, -1.0001, -0.0035 \pm j1.0\}$, while the eigenvalues of A_c are $\{-99.6374, -8.399, 0.0003 \pm j1.0396\}$. The closed-loop LQG cost is 261.6534. By performing a simple parameter search on α and β, we choose $\alpha = 4.51$, $\beta = 1.175$ in Theorem 2.1. The modified design yields the closed-loop poles $\{-99.9796, -1.0277 \pm j2.8161, -2.4941 \pm j1.1604, -1.0014, -0.0035 \pm j1.0\}$, while the eigenvalues of A_c are $\{-99.632, -8.3993, -0.0002 \pm j1.0396\}$. The closed-loop cost for the modified design is 332.148. The cost increment for the design of an asymptotically stable compensator is thus 26.94%.

Example 2. Next we consider Example 2 of Halevi *et al.* (1991). Specifically,

$$A=\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & -3 & 0\\ 0 & 0 & 0 & -4\end{bmatrix},\quad B=\begin{bmatrix}1\\ 1\\ 1\\ 1\end{bmatrix},\quad C=[1\ \ 1\ \ 1\ \ 1],$$

$V_1 = D_1D_1^T = 10^3 I_4$, $V_2 = D_2D_2^T = 1$, $R_1 = 10^4 I_4$ and $R_2 = 1$. In the full-order case, the LQG cost is 7.2156×10^6, while the eigenvalues of A_c are $\{-264.4913, 1.3969, -2.2669, -3.5039\}$. The H_2-suboptimal result given by Halevi *et al.* (1991) has a total cost of 7.2215×10^6. Now choosing $\alpha = 1.0059$, $\beta = 1.45$ and applying the computational procedure described in this section to solve equations (20)–(22) in Theorem 2.1, we obtain the closed-loop cost 7.2190×10^6, which shows that the cost increment is slightly less than the cost increment obtained in Halevi *et al.* (1991). The resulting eigenvalues of A_c are $\{-235.389, -1.6989, -4.2047, -2.8711\}$.

6. *Conclusion*

Two approaches were developed for obtaining stable compensators. One approach involves modifying the standard LQG Riccati equations to guarantee stability of the compensator, while the other approach is based upon bounding the closed-loop covariance.

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