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Brief Paper

A Riccati Equation Approach to the Singular LQG Problem*

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Key Words—Linear optimal regulator; singular control; system order reduction.

Abstract—The problem of optimal fixed-order dynamic compensation for the singular LQG problem is considered. Necessary conditions characterizing the optimal compensator for the case involving both singular measurement noise and singular control weighting are given. The solution consists of a set of two algebraic Riccati equations and two Lyapunov equations coupled by three projection matrices. One projection is the standard order reduction projection while the other two projections reflect the two types of singularity that exist in the system. The three projections are shown to satisfy disjointness conditions. In addition to order reduction, an advantage of the fixed-structure approach is that differentiation, which is often undesirable from a practical point of view and which may exist in the unconstrained optimal control, can be avoided. It is shown that the fixed-order compensator agrees with the unconstrained solution when the latter possesses the same number of differentiations as are included in the prespecified controller structure and when the order is selected appropriately.

1. *Introduction*

THE SINGULAR LQG CONTROL problem has been of considerable interest for almost two decades (e.g. Friedland, 1971; Clements and Anderson, 1978; Francis, 1979; Haas, 1982; O'Reilly, 1983; Willems *et al.*, 1986; Halevi and Palmor, 1986; Soroka and Shaked, 1988; Bernstein and Zeidan, 1990). For a more complete list of references see Bernstein and Zeidan (1990). Such problems arise when some of the measurements are noise free or when some of the control signals are unweighted. This will be the case, for example, if the sensor noise is colored or if actuator dynamics are present. Augmentation of the plant dynamics by means of noise shaping filters or actuator dynamics thus leads directly to a singular problem formulation.

Most of the literature on the singular LQG problem is based upon either limiting procedures in which suitable weighting matrices and noise intensities approach zero, or differentiation of noise-free signals. These results demonstrate that the compensator that arises in the limiting solution may include differentiators. Although the dimension of the optimal compensator depends on the order of singularity of the problem, the total number of differentiators and integrators is generally equal to the dimension of the system minus the number of noise-free measurements (unweighted control signals in the dual problem). In practical applications, however, it is often of interest to limit the number of differentiators included in the compensator or possibly eliminate differentiation entirely. Thus, the approach of the present paper differs significantly from the prior literature in that we no longer seek the optimal singular LQG controller, but rather the optimal controller constrained to be a member of a prespecified class of fixed-structure controllers.

The fixed structure approach to control design originated by Johnson and Athans (1970) and Levine *et al.* (1971) and has since undergone extensive development (see Hyland and Bernstein (1984) and the references therein). To apply the fixed-structure approach to the singular LQG problem, we fix the order of the contoller as in Hyland and Bernstein (1984) and optimize over the gains associated with the noisy and non-noisy measurements. If, in applications, we wish to include differentiations within the feedback compensator, non-noisy measurements can be differentiated and then the resulting signals can be treated as "original" measurements. Thus our approach can be used to obtain improper compensators with a bound on the number of differentiators.

Preliminary results for the singular LQG problem using the fixed structure approach were obtained in Bernstein (1987). The results given there are incomplete, however, in that the gains associated with certain feedback paths were not given explicitly. For the corresponding singular estimation problem (Haddad and Bernstein, 1987) this defect was remedied in Halevi (1989) where all feedback gains were explicitly characterized. In addition, the solution obtained there was shown to agree completely with results obtained using standard limiting methods when the (unconstrained) optimal singular estimator does not possess differentiators (Friedland, 1971; Halevi, 1988). For certain cases the results of Halevi (1989) thus provide an alternative approach to the singular estimation problem considered in Bryson and Johansen (1965), Friedland (1971), Schumacher (1985), Soroka and Shaked (1987) and Halevi (1988) and the numerous references therein. Preliminary and partial results of the present paper were reported in Halevi *et al.* (1989) where only the singular measurement noise case was considered.

The contribution of the present paper is thus to complete the development of Bernstein (1987) by incorporating the methods used in Halevi (1989). Accordingly, we derive a coupled system of modified Riccati and Lyapunov equations that explicitly characterize the feedback gains of the fixed-structure singular LQG controller. For generality we consider partial or total singularity in both the control weighting and measurement noise intensity matrices, and we allow the dynamic compensator to be of arbitrary dimension less than or equal to the number of plant states minus the sum of the number of noise-free measurements and the

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number of unweighted control signals. In the special case in which only the measurement noise is singular, the order of the dynamic compensator is equal to the number of plant states minus the number of noise-free measurements (i.e. the quasi full-order case), and a certain matrix is nonsingular, then we show that the optimal solution decomposes (separates) into a reduced-order observer followed by state feedback. Furthermore, as in Halevi (1989) we demonstrate connections with earlier results by showing that the fixed structure solution agrees with the standard limiting solution when the latter possesses the same number of differentiators as are included in the prespecified controller structure. The improvement of the derivation here over that given in Halevi (1989) is that we show explicitly how the equations in Friedland (1971), Halevi and Palmor (1986) and Halevi (1988) are obtained from our results and not just the equivalence between them.

In view of the flexibility of the fixed-structure approach in constraining the order of the controller and limiting the number of differentiators, it should not be surprising that the optimality conditions for the fixed-structure controller are more complex than those obtained in the prior literature for the unconstrained solution. As shown by Hyland and Bernstein (1984), the fixed-order constraint on the controller leads to a generalization of LQG theory requiring the solution of a coupled system consisting of two algebraic Riccati equations (in variables Q, P) and two algebraic Lyapunov equations (in variables $\hat{Q}$, $\hat{P}$). The coupling is due to the presence of a "dynamic" order-reduction projection τ. To address the singular LQG problem, the present paper goes beyond those results by determining gains associated with non-noisy measurements and unweighted controls. These gains are characterized by means of "static" projections v_1 and v_2 whose structure is familiar from least squares analysis. The use of projections in singular optimal control and estimation is not a new idea, e.g. Friedland (1971), Lewis (1981, 1982). The distinction of the results of this paper is that unlike the quasi full-order case, v_1 and v_2 are not given in terms of the problem data but have to be solved together with all other parameters for the compensator. The equations we derive involve all three projections τ, v_1 and v_2. The structure of these equations is quite intricate, which leads us to believe that the construction of optimal fixed-structure controllers without the aid of these equations would be a formidable task. In spite of the complexity of these equations, however, we obtain useful insights into the structure of the compensator that arises as a direct result of optimality. For example, we show that τ, v_1, v_2 satisfy disjointness conditions of the form

$$0 = \tau v_1 = v_2 \tau = v_2 v_1.$$

The material is organized as follows: in Section 2 the problem is stated, followed by some preliminary derivations. The main results of the paper are given in Section 3. The quasi full-order case is considered in Section 4. The results of the paper are summarized in Section 5. The proof of Theorem 3.1 is given in the Appendix.

2. *Problem statement and preliminaries*

The basic block diagram considered in this paper is given in Fig. 1 where $w \in R^q$ is the signal of exogenous inputs (process and measurement noises in the LQG setting) $u \in R^m$ is the control input, $y \in R^r$ is the measured output and $z \in R^p$ is the generalized error signal. The state vector is $x \in R^n$. We partition $u = [u_1^T u_2^T]^T$ where $u_1 \in R^{m_1}$ and $u_2 \in R^{m_2}$ are the weighted and unweighted control inputs, respectively, and $y_1 \in R^{r_1}$ and $y_2 \in R^{r_2}$ are the noisy and non-noisy measurements, respectively. The generalized plant $P(s)$ realization is then given as

$$P(s) \approx \left[\begin{array}{c|c:c} A & D_1 & B \\ \hline E_1 & 0 & E_2 \\ \hdashline C & D_2 & 0 \end{array}\right] = \left[\begin{array}{c|c:cc} A & D_1 & B_1 & B_2 \\ \hline E_1 & 0 & E_{21} & 0 \\ \hdashline C_1 & D_{21} & 0 & 0 \\ C_2 & 0 & 0 & 0 \end{array}\right]. \qquad (2.1)$$

It is assumed that (A, B) and (A, D_1) are stabilizable and that (C, A) and (E_1, A) are detectable. It is further assumed

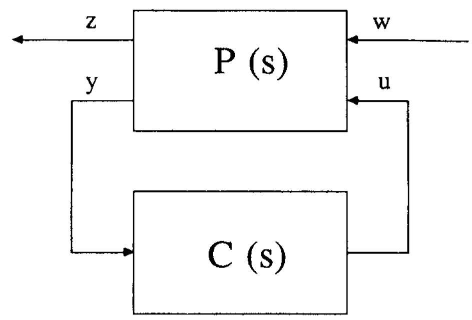

FIG. 1. The closed-loop system.

that E_{21} has full column-rank, which entails no loss of generality since otherwise additional unweighted control inputs can be identified by input transformation. The same argument applies to D_{21} which is assumed to have full row-rank. The H_2 cost is defined as

$$J = \|H(s)\|_2^2 \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}(H^*(j\omega)H(j\omega))\, d\omega, \qquad (2.2)$$

where $H(s)$ is the closed loop transfer function from w to z, 'tr' is the trace operator and $(\ )^*$ denotes conjugate transpose. We treat the problem from the stochastic point of view and consider w as a unit intensity white noise in R^q. The intensities of the process noise, measurement noise, and the cross-intensity are given by

$$V_1 = D_1 D_1^T, \quad V_2 = D_{21} D_{21}^T, \quad V_{12} = D_1 D_{21}^T.$$

Similarly the state, control and cross-term weighting matrices are given by

$$R_1 = E_1^T E_1, \quad R_2 = E_{21}^T E_{21}, \quad R_{12} = E_1^T E_{21}.$$

It is well known (Kwakernaak and Sivan, 1972) that

$$\begin{aligned} J &= \lim_{t\to\infty} E\{z^T(t) z(t)\} \\ &= \lim_{t\to\infty} E\{x^T(t) R_1 x(t) + 2x^T(t) R_{12} u_1(t) + u_1^T(t) R_2 u_1(t)\}1. \end{aligned} \qquad (2.3)$$

The optimization problem is as follows: Find an n_cth order compensator

$$C(s) \approx \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{c|cc} A_c & B_{c1} & B_{c2} \\ \hline C_{c1} & D_{c11} & D_{c12} \\ C_{c2} & D_{c21} & D_{c22} \end{array}\right], \qquad (2.4)$$

that minimizes J.

The compensator (2.4) is constrained to be proper, which is distinct from unconstrained optimal singular control which may require derivatives of the noise-free measurement y_2. A fixed level of differentiation can be accommodated in our formulation, however, by carrying it out first and then redefining y_1 and y_2. The proper compensator acting on the modified output is equivalent to an improper compensator with the original output as its input. From an implementation point of view this is advantageous because the level of allowed differentiation (which is often zero) is also prespecified.

Since J is independent of the internal realization of the compensator we restrict our attention to minimal realizations. Hence without loss of generality we invoke the following.

Assumption 1. (A_c, B_c) is controllable (A_c, C_c) is observable.

Combining the states of the plant and the compensator we obtain the augmented system

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}w, \\ z &= \tilde{C}\tilde{x} + \tilde{D}w, \end{aligned} \qquad (2.5)$$

where

$$\bar{x} = [x^T \quad x_c^T]^T, \quad \bar{A} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} D_1 + BD_cD_2 \\ B_cD_2 \end{bmatrix},$$

$$\bar{C} = [E_1 + E_2D_cC \quad E_2C_c],$$

$$\bar{D} = E_{21}D_{c11}D_{21}.$$

To guarantee the finiteness of J, $\bar{D}$ must be zero. Since D_{21} and E_{21} have full column and row rank, respectively, it follows that $D_{c11} = 0$, i.e. there is no direct transmission from the noisy measurement to the weighted input. As a result of the assumed detectability and stabilizability of (A, B, C) and (A, D_1, E_1) internal and external stability of the closed loop system (2.5) are equivalent. Therefore, to guarantee a finite cost J, the following assumption is made.

Assumption 2. $\bar{A}$ is a stable matrix.

We define now the controllability and observability Gramians of the system (2.5) which satisfy

$$\bar{A}\bar{Q} + \bar{Q}\bar{A}^T + \bar{B}\bar{B}^T = 0, \tag{2.6}$$

and

$$\bar{A}^T\bar{P} + \bar{P}\bar{A} + \bar{C}^T\bar{C} = 0, \tag{2.7}$$

and partition them as

$$\bar{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \quad \bar{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}, \quad Q_1, P_1 \in R^{n \times n}.$$

For the solution in the next section in the general case where $C_2 \neq 0$, $B_2 \neq 0$, we need the following assumption.

Assumption 3. The following matrices are positive definite:

(i) Q_2, (ii) $C_2(Q_1 - Q_{12}Q_2^{-1}Q_{12}^T)C_2^T$,
(iii) P_2, (iv) $B_2^T(P_1 - P_{12}P_2^{-1}P_{12}^T)B_2$.

First notice that by their definitions these matrices are nonnegative definite and the assumption is required to guarantee their invertibility. If $(\bar{A}, \bar{B}, \bar{C})$ is minimal then $\bar{Q}, \bar{P} > 0$ and (i)–(iv) follow immediately. Also if $C_2 = 0$ then (i) can be proved and (ii) becomes meaningless. Dual results exist for $B_2 = 0$. The physical interpretation of the assumption is as follows: (i) and (iii) mean that the compensator is minimal in the closed loop setting, i.e. there exists no lower order compensator that yields the same $H(s)$ (that by itself does not mean that $(\bar{A}, \bar{B}, \bar{C})$ is minimal). (ii) and (iv) guarantee that in the closed-loop there is no redundancy in the non-noisy measurements y_2 and the unweighted inputs u_2.

It is well known (Kwakernaak and Sivan, 1972) that

$$\bar{Q} = \lim_{t \to \infty} E\{\bar{x}(t)\bar{x}^T(t)\}. \tag{2.8}$$

Hence the optimization problem can be restated as follows:

$$\text{minimize tr}\, \bar{C}\bar{Q}\bar{C}^T \quad \text{s.t.} \quad \bar{A}\bar{Q} + \bar{Q}\bar{A}^T + \bar{B}\bar{B}^T = 0.$$

The following lemma is required for the main results of the paper.

Lemma 2.1 (Hyland and Bernstein, 1984). Suppose $\hat{Q}, \hat{P}$ are $n \times n$ nonnegative definite matrices. Then $\hat{Q}\hat{P}$ is diagonalizable with nonnegative eigenvalues. If, in addition, rank $\hat{Q}\hat{P} = n_c$, then there exist $n_c \times n$, G, Γ and $n_c \times n_c$ invertible M such that $\hat{Q}\hat{P} = G^TM\Gamma$; $\Gamma G^T = I_{n_c}$. $\tau \triangleq G^T\Gamma$ is an oblique projection.

3. *Main results*

The following theorem gives the main result of this paper.

Theorem 3.1. Suppose (A_c, B_c, C_c, D_c) satisfy Assumptions (1)–(3) and minimize J. Then they are given by

$$\left[\begin{array}{c|cc} A_c & B_{c1} & B_{c2} \\ \hline C_{c1} & D_{c11} & D_{c12} \\ C_{c2} & D_{c21} & D_{c22} \end{array}\right] = \left[\begin{array}{c|cc} \Gamma(\hat{A} - \hat{A}\nu_1 - \nu_2\hat{A})G^T & \Gamma\nu_{2\perp}Q_aV_2^{-1} & \Gamma\hat{A}C_2^* \\ \hline -R_2^{-1}P_a^T\nu_{1\perp}G^T & 0 & -R_2^{-1}P_a^TC_2^* \\ -B_2^*\hat{A}G^T & -B_2^*Q_aV_2^{-1} & D_{c22} \end{array}\right], \tag{3.1}$$

$$D_{c22} = -B_2^*(AQ + QA^T + V_1 - Q_aV_2^{-1}Q_a^T - B_1R_2^{-1}P_a^TQ\nu_1^T)U_1, \tag{3.2}$$

where U_1 is an arbitrary right inverse of C_2Q, and where

$$A_d = A + B_2D_{c22}C_2, \qquad \hat{A} = A_d - Q_aV_2^{-1}C_1 - B_1R_2^{-1}P_a,$$
$$Q_a = QC_1^T + V_{12}, \qquad P_a = B_1^TP + R_{12}^T,$$
$$C_2^* = QC_2^T(C_2QC_2^T)^{-1}, \quad \nu_1 = C_2^*C_2, \quad \nu_{1\perp} = I_n - \nu_1,$$
$$B_2^* = (B_2^TPB_2)^{-1}B_2^TP, \quad \nu_2 = B_2B_2^*, \quad \nu_{2\perp} = I_n - \nu_2.$$

Γ, G and τ are as defined in Lemma 2.1 and Q, P, $\hat{Q}$ and $\hat{P}$ are nonnegative definite matrices satisfying

$$(A_d - B_1R_2^{-1}P_a\nu_1 - \tau\hat{A}\nu_1)Q + Q(A_d - B_1R_2^{-1}P_a\nu_1 - \tau\hat{A}\nu_1)^T + V_1 - Q_aV_2^{-1}Q_a^T + \tau_\perp\nu_{2\perp}Q_aV_2^{-1}Q_a^T\nu_{2\perp}^T\tau_\perp^T = 0, \tag{3.3}$$

$$(A_d - \nu_2Q_aV_2^{-1}C_1 - \nu_2\hat{A}\tau)^TP + P(A_d - \nu_2Q_aV_2^{-1}C_1 - \nu_2\hat{A}\tau) + R_1 - P_a^TR_2^{-1}P_a + \tau_\perp^T\nu_{1\perp}^TP_a^TR_2^{-1}P_a\nu_{1\perp}\tau_\perp = 0, \tag{3.4}$$

$$\nu_{2\perp}(A_d - B_1R_2^{-1}P_a)\hat{Q} + \hat{Q}(A_d - B_1R_2^{-1}P_a)^T\nu_{2\perp}^T + \tau\hat{A}\nu_1Q + Q\nu_1^T\hat{A}^T\tau^T + \nu_{2\perp}Q_aV_2^{-1}Q_a\nu_{2\perp}^T - \tau_\perp\nu_{2\perp}Q_aV_2^{-1}Q_a^T\nu_{2\perp}^T\tau_\perp^T = 0, \tag{3.5}$$

$$\nu_{1\perp}^T(A_d - Q_aV_2^{-1}C_1)^T\hat{P} + \hat{P}(A_d - Q_aV_2^{-1}C_1)\nu_{1\perp} + \nu_2^TP\hat{A}\tau + \tau^T\hat{A}^TP\nu_2 + \nu_{1\perp}^TP_a^TR_2^{-1}P_a\nu_{1\perp}^T - \tau_\perp^T\nu_{1\perp}^TP_a^TR_2^{-1}P_a\nu_{1\perp}\tau_\perp = 0, \tag{3.6}$$

$$\text{rank}\,\hat{Q} = \text{rank}\,\hat{P} = \text{rank}\,\hat{Q}\hat{P} = n_c. \tag{3.7}$$

The optimal cost is given by

$$J = \text{tr}\,(\nu_{1\perp}QR_1) + \text{tr}\,\{(\nu_1Q + \hat{Q}) \times (R_1 - R_{12}R_2^{-1}R_{12}^T + PB_1R_2^{-1}B_1^TP)\}. \tag{3.8}$$

The proof is given in the Appendix.

It is seen that the solution is given by a set of two algebraic Riccati equations and two Lyapunov equations coupled by three oblique projections τ, ν_1 and ν_2. The dynamic projection τ arises as a direct consequence of the fixed structure and appeared originally in Hyland and Bernstein (1984). The static projections ν_1 and ν_2 are consequences of the singularity of the measurement noise and the control weight, respectively (Bernstein, 1987; Haddad and Bernstein, 1987; Halevi, 1989). Similar projections appear in weighted least squares problems. The three projections are disjoint as can be seen from the following relations

$$\tau\nu_1 = 0, \quad \nu_2\tau = 0, \quad \nu_2\nu_1 = 0, \tag{3.9}$$

which are derived in the proof of Theorem 3.1. The first two relations immediately imply that $n_c \leq n - \max(r_2, m_2)$. However, taking into account the fact that these are projection matrices we obtain a tighter upper bound on n_c. Specifically, since

$$\begin{bmatrix} \Gamma\nu_{2\perp} \\ C_2\nu_{2\perp} \\ B_2^* \end{bmatrix} [\nu_{1\perp}G^T \quad C_2^* \quad B_2] = I_{n_c + r_2 + m_2}, \tag{3.10}$$

it follows from Sylvester's inequality that $n_c + r_2 + m_2 \leq n$. Furthermore, multiplying the matrices in equation (3.10) in reverse order we can define a new $n \times n$ projection θ with rank $n_c + r_2 + m_2$, as

$$\theta = \nu_{1\perp}\tau\nu_{2\perp} + \nu_1\nu_{2\perp} + \nu_2 = I_n - \nu_{1\perp}\tau_\perp\nu_{2\perp}, \tag{3.11}$$

which means that $\theta_\perp = \nu_{1\perp}\tau_\perp\nu_{2\perp}$ is projection as well.

The results of Theorem 3.1 are the most general for a time-invariant, fixed-order proper compensator and as such include all previously solved cases. Specifically,

(i) the case of singular measurement noise and nonsingular control weighting (Halevi *et al.*, 1989) is recovered by setting $v_2 = 0$ and $A_d = A$. C_{c2}, D_{c21} and D_{c22} vanish. In the dual case of nonsingular measurement and singular control weighting we set $v_1 = 0$, $A_d = A$ and B_{c2}, D_{c12} and D_{c22} vanish,

(ii) The nonsingular case (Hyland and Bernstein, 1984) is obtained by setting both $v_1 = 0$, $v_2 = 0$. B_{c2}, C_{c2} and D_2 vanish and $A_d = A$. If in addition $n_c = n$, i.e. the full-order case then set $\tau = I_n$ and equations (3.5)–(3.6) are superfluous. Equations (3.3)–(3.4) become the standard Riccati equations and separation holds;

(iii) static optimal output feedback is achieved by setting $\tau = 0$ in equations (3.3)–(3.4). A_c, B_c, C_c and equations (3.5)–(3.6) all vanish.

It is shown in Hyland and Bernstein (1984) that in reduced order LQG compensation the separation principle is no longer valid. However the singular compensator may be expressed in a form reminiscent of the familiar observer-state feedback form by defining

$$\hat{x} = v_{1\perp} G^T x_c + C_2^* y_2, \tag{3.12}$$

which is in accordance with known results in singular estimation (e.g. Friedland, 1971; Halevi, 1988). Note also that $B_2^* \hat{x} = 0$, another known property of solutions to the "cheap control" problem (Friedland, 1971). This means that $\hat{x}$ belongs to the nullspace of v_2. With this definition we can rearrange the dynamic equation of the optimal compensator and obtain

$$G^T \dot{x}_c = \tau[A\hat{x} + B_1 u_1 + B_2 u_2 + Q_a V_2^{-1}(y_1 - C_1 \hat{x})], \tag{3.13}$$

which may be interpreted as the optimal projection applied to the optimal state estimator, i.e. the Kalman Filter. Notice that $G^T x_c = \tau \hat{x}$. The static output equations of the compensator are given by

$$u_1 = -R_2^{-1} P_a \hat{x}, \tag{3.14}$$

$$u_2 = -B_2^*[(A - B_1 R_2^{-1} P_a)\tau\hat{x} + Q_a V_2^{-1}(y_1 - C_1 \tau \hat{x})] + D_{c22}(y_2 - C_2 \tau \hat{x}). \tag{3.15}$$

Although equation (3.14) is very clear, it is difficult to interpret the structure of u_2 as well as the expression for D_{c22}. It should be noted however that in case the measurement is nonsingular, D_{c22} vanishes, $\tau\hat{x} = \hat{x}$ and equation (3.15) reduces to a familiar form (Friedland, 1971).

4. *The quasi full-order case*

In this section we discuss the optimal fixed-structure compensator under the following assumptions:

(i) The control weighting is nonsingular, i.e. $m_2 = 0$;

(ii) $C_2(V_1 - V_{12} V_2^{-1} V_{12}) C_2^T > 0$;

(iii) $n_c = n - r_2$.

The unconstrained optimal compensator in this case (Assumptions (i)–(ii)) has the assumed structure with $n_c = n - r_2$ (Bryson and Johansen, 1965). Therefore we would like to see how these results are recovered from Theorem 3.1. If the matrix in (ii) is singular then the unconstrained optimal compensator is improper (Halevi and Palmor, 1986) and no comparison can be made with our compensator which is fixed to be proper. The key point of the analysis is given in the following lemma.

Lemma 4.1. Suppose Assumptions (i)–(iii) hold. Then

$$v_{1\perp} \tau_\perp = 0, \tag{4.1}$$

$$v_1 = (v_{1\perp} Q \bar{A}^T + \bar{V}_1) C_2^T (C_2 \bar{V}_1 C_2^T)^{-1} C_2, \tag{4.2}$$

where

$$\bar{A} = A - V_{12} V_2^{-1} C_1, \quad \bar{V}_1 = V_1 - V_{12} V_2^{-1} V_{12}^T.$$

Proof. The combined projection θ in equation (3.11) has in this case rank n, therefore $\theta = I_n$, and $v_{1\perp}\tau_{1\perp}v_{2\perp} = 0$. Equation (4.1) is obtained by substituting $v_2 = 0$. Notice that equation (4.1) means that $v_{1\perp}\tau = v_{1\perp}$. Premultiplying equation (3.3) by τ and postmultiplying it by C_2^T we have, using $\tau v_1 = 0$, $Q v_1^T = v_1 Q$, $v_1 Q C_2^T = Q C_2^T$ and the definitions of Q_a, $\bar{A}$ and $\bar{V}_1$,

$$\tau(Q\bar{A}^T + \bar{V}_1) C_2^T = 0. \tag{4.3}$$

Premultiplying equation (4.3) by $v_{1\perp}$, we obtain

$$(v_{1\perp} Q \bar{A}^T + \bar{V}_1) C_2^T - Q C_2^T (C_2 Q C_2^T)^{-1} C_2 \bar{V} C_2^T = 0. \tag{4.4}$$

Postmultiplying by $(C_2 \bar{V}_1 C_2^T)^{-1} C_2$ yields equation (4.2). This completes the proof.

As v appears on both sides of equation (4.2) its usefulness seems limited at first sight. However, v_1 is completely determined by $v_{1\perp}Q$, which is not true in the general case. We define

$$Q_0 \triangleq v_{1\perp} Q = Q v_{1\perp}^T, \quad \pi \triangleq \bar{V} C_2^T (C_2 \bar{V}_1 C_2^T)^{-1} C_2.$$

The matrix π is an oblique projection which is determined from the problem data. With these definitions we have

$$v_\perp = \pi_\perp - Q_0 \bar{A}^T C_2^T (C_2 \bar{V}_1 C_2^T)^{-1} C_2. \tag{4.5}$$

Premultiplying and postmultiplying equation (3.3) by $v_{1\perp}$ and $v_{1\perp}^T$, respectively, we obtain

$$v_{1\perp} \bar{A} Q_0 + Q_0 \bar{A}^T v_{1\perp}^T + v_{1\perp} \bar{V}_1 v_{1\perp}^T - Q_0 C_1^T V_2^{-1} C_1 Q_0 = 0. \tag{4.6}$$

Substituting equation (4.5), using $C_2 \pi_\perp = 0$ and $\bar{V}_1 \pi_\perp^T = \pi_\perp \bar{V}_1$ yields

$$\pi_\perp \bar{A} Q_0 + Q_0 \bar{A}^T \pi_\perp^T - Q_0 [\bar{A}^T C_2^T (C_2 \bar{V}_1 C_2^T)^{-1} C_2 \bar{A} + C_1^T V_2^{-1} C_1] Q_0 + \pi_\perp \bar{V}_1 = 0. \tag{4.7}$$

Equation (4.7) is a standard Riccati equation for Q_0 and is identical with the results obtained in Halevi (1988). In Halevi (1989) it was shown that Q_0 is the error covariance of the optimal reduced order estimator for a singular measurement system. Using equation (4.1), (3.4) reduces to the standard control Riccati equation

$$A^T P + PA + R_1 - P_a^T R_2^{-1} P_a = 0. \tag{4.8}$$

Notice that since equations (4.7)–(4.8) are independent of each other and of the projection τ, separation holds in this case. It remains to be shown that the compensator matrices are determined by Q_0 and P. These matrices are given by

$$\left[\begin{array}{c|cc} A_c & B_{c1} & B_{c2} \\ \hline C_{c1} & D_{c11} & D_{c12} \end{array}\right] = \left[\begin{array}{c|cc} \Gamma \hat{A} v_{1\perp} V_2^{-1} & \Gamma Q_a V_2^{-1} & \Gamma \hat{A} C_2^* \\ \hline -R_2^{-1} P_a v_{1\perp} G^T & 0 & -R_2^{-1} P_a C_2^* \end{array}\right], \tag{4.9}$$

where in this case $\hat{A} = A - Q_a V_2^{-1} C_1 - B_1 R_2^{-1} P_a$. First notice that in all expressions Q_a is premultiplied by Γ. Since $\Gamma v_1 = 0$ we have

$$\Gamma Q_a = \Gamma v_{1\perp} Q_a = \Gamma Q_0 C_1^T - \Gamma(\pi_\perp - Q_0 \bar{A}^T C_2^T (C_2 \bar{V}_1 C_2^T)^{-1} C_2) V_{12}.$$

Hence Q is not required explicitly. Next, the solution is given in terms of Γ and $v_{1\perp} G^T$, which are a factorization of the projection $v_{1\perp}\tau$. Certainly the optimality of the compensator is independent of its internal realization, and this degree of freedom is equivalent to using *any* such factorization (Hyland and Bernstein, 1984). But from Lemma 4.1 it follows that we can use any factorization of $v_{1\perp}$ which is known.

5. *Conclusions*

The optimal fixed-order dynamic compensator for the singular LQG problem was fully characterized by necessary conditions. The compensator is prefixed to be proper hence the problem of output, and possibly white noise, differentiation is avoided. If certain level of differentiation is allowed it can be carried out first and then by redefining the output the results of this paper can be used. The solution is given by a set of two algebraic Riccati equations and two Lyapunov equations coupled by three projection matrices. One is the "dynamic" projection which appears in all optimal L_2 order reduction problems. The other two are the control and estimation versions of the "static" projection which appears whenever the problem is singular. The results of this paper are general and various previously solved cases can be viewed as special cases. The derivation in Section 4 shows that when the unconstrained optimal compensator has the prefixed structure, i.e. it does not contain differentation, our results agree with those obtained by other techniques such as the limiting method.

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Appendix. Proof of Theorem 3.1

First, for notational convenience we define

$$\bar{R}_{12} = E_1^T E_2 = [R_{12} \quad 0_{n\times m_2}], \quad \bar{R}_2 = E_2^T E_2 = \begin{bmatrix} R_2 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\bar{V}_{12} = D_1 D_2^T = [V_{12} \quad 0_{n\times r_2}], \quad \bar{V}_2 = D_2 D_2^T = \begin{bmatrix} V_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\bar{C}^T\bar{C} = \begin{bmatrix} R_1 + \bar{R}_{12}D_cC + C^TD_c^T\bar{R}_{12}^T + C^TD_c^T\bar{R}_2D_cC & \bar{R}_{12}C_c + C^TD_c^T\bar{R}_2C_c \\ C_c^T\bar{R}_2D_cC + C_c^T\bar{R}_{12}^T & C_c^T\bar{R}_2C_c \end{bmatrix}, \quad (A.1)$$

$$\bar{B}\bar{B}^T = \begin{bmatrix} V_1 + \bar{V}_{12}D_c^TB^T + BD_c\bar{V}_{12} + BD_c\bar{V}_2D_c^TB^T & \bar{V}_{12}B_c^T + BD_c\bar{V}_2B_c^T \\ B_c\bar{V}_2D_c^TB^T + B_c\bar{V}_{12}^T & B_c\bar{V}_2B_c^T \end{bmatrix}. \quad (A.2)$$

We construct the Lagrangian

$$H = \mathrm{tr}\,[\tilde{Q}\bar{C}^T\bar{C} + (\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \bar{B}\bar{B}^T)\tilde{P}]. \quad (A.3)$$

The necessary conditions for optimality are:

$$\partial H/\partial \tilde{P} = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \bar{B}\bar{B}^T = 0, \quad (A.4)$$

$$\partial H/\partial Q = \tilde{P}\tilde{A} + \tilde{A}^T\tilde{P} + \bar{C}^T\bar{C} = 0, \quad (A.5)$$

$$\partial H/\partial A_c = P_{12}^TQ_{12} + P_2Q_2 = 0, \quad (A.6)$$

$$\partial H/\partial B_c = P_{12}^TQ_1C^T + P_2Q_{12}^TC^T + P_{12}^T\bar{V}_{12} + P_{12}^TBD_c\bar{V}_2 + P_2B_c\bar{V}_2 = 0, \quad (A.7)$$

$$\partial H/\partial C_c = B^TP_1Q_{12} + B^TP_{12}Q_2 + \bar{R}_{12}^TQ_{12} + \bar{R}_2D_cCQ_{12} + \bar{R}_2C_cQ_2 = 0, \quad (A.8)$$

$$\partial H/\partial D_c = B^TP_1Q_1C^T + B^TP_{12}Q_{12}^TC^T + \bar{R}_{12}^TQ_1C^T + \bar{R}_2D_cCQ_1C^T + \bar{R}_2C_cQ_{12}^TC^T + B^TP_1\bar{V}_{12} + B^TP_1BD_c\bar{V}_2 + B^TP_{12}B_c\bar{V}_2 = 0. \quad (A.9)$$

Equation (A.9) is a compact form for the partial derivatives with respect to D_{c12}, D_{c21} and D_{c22}. Its upper-left subblock is meaningless. We define now

$$G^T = Q_{12}Q_2^{-1}, \quad \hat{Q} = Q_{12}Q_2^{-1}Q_{12}^T,$$
$$Q = Q_1 - \hat{Q}, \quad \Gamma = -P_2^{-1}P_{12}^T,$$
$$\hat{P} = P_{12}P_2^{-1}P_{12}^T, \quad P = P_1 - \hat{P},$$

where the partitions are as in Section 2. From (A.6) we have $\Gamma G^T = I_{n_c}$, hence $\tau = G^T\Gamma$ is a projection matrix. (A.9)–(A.8) $GC^T + B^T\Gamma^T$ (A.7) yields

$$B^TPQC^T + \bar{R}_{12}^TQC^T + \bar{R}_2D_cCQC^T + B^TP\bar{V}_{12} + B^TPBD_c\bar{V}_2 = 0. \quad (A.10)$$

The subblocks of this equation give the expressions for $D_{c_{12}}$ and $D_{c_{21}}$ in (3.1) and

$$B_2^TPQC_2^T = 0. \quad (A.11)$$

From equation (A.7) we obtain the expression for B_{c1} and

$$\Gamma Q C_2^T = 0. \quad (A.12)$$

Equation (A.8) gives C_{c1} and

$$B_2^T P G^T = 0. \quad (A.13)$$

Equations (A.11)–(A.13) imply the disjointness relationships

$$\nu_2 \nu_1 = 0, \quad \tau \nu_1 = 0, \quad \nu_2 \tau = 0. \quad (A.14)–(A.16)$$

We define now the transformation matrices

$$T_1 = \begin{bmatrix} I_n & -G^T \\ 0 & I_{n_c} \end{bmatrix}, \quad T_2 = \begin{bmatrix} I_n & \Gamma^T \\ 0 & I_{n_c} \end{bmatrix}. \quad (A.17)–(A.18)$$

Although the use of those transformations is not necessary, it simplifies the derivation considerably. Notice that $T_1 \bar{Q} T_1^T = \text{diag}\{Q, Q_2\}$, $T_2 \bar{P} T_2^T = \text{diag}\{P, P_2\}$. Subblocks of $T_1(A.4)T_1^T$ and $T_2(A.5)T_2^T$ give

$$(A + BD_cC - G^TB_cC)Q + Q(A + BD_cC - G^TB_cC)^T + V_1 + (BD_c - G^TB_c)\bar{V}_{12}^T + \bar{V}_{12}(BD_c - G^TB_c)^T + (BD_c - G^TB_c)\bar{V}_2(BD_c - G^TB_c)^T = 0, \quad (A.19)$$

$$(A + BD_cC - G^TB_cC)Q_{12} + (BC_c - G^TA_c)Q_2 + QC^TB_c^T + \bar{V}_{12}B_c^T + (BD_c - G^TB_c)\bar{V}_2B_c^T = 0, \quad (A.20)$$

$$A_cQ_2 + B_cCQ_{12} + Q_{12}^TC^TB_c^T + Q_2A_c^T + B_c\bar{V}_2B_c^T = 0, \quad (A.21)$$

$$P(A + BD_cC + BC_c\Gamma) + (A + BD_cC + BC_c\Gamma)^TP + R_1 + \bar{R}_{12}(D_cC + C_c\Gamma) + (D_cC + C_c\Gamma)^T\bar{R}_{12}^T + (D_cC + C_c\Gamma)^T\bar{R}_2(D_cC + C_c\Gamma) = 0, \quad (A.22)$$

$$(A + BD_cC + BC_c\Gamma)^TP_{12} + (B_cC + A_c\Gamma)^TP_2 + PBC_c + \bar{R}_{12}C_c + (D_cC + C_c\Gamma)^T\bar{R}_2C_c = 0, \quad (A.23)$$

$$P_2A_c + P_{12}^TBC_c + A_c^TP_2 + C_c^TB^TP_{12} + C_c^T\bar{R}_2C_c = 0. \quad (A.24)$$

Equation (A.21) can be written as Hyland and Bernstein (1984)

$$(A_c + B_cCQ_{12}Q_2^+)Q_2 + Q_2(A_c + B_cCQ_{12}Q_2^+)^T + B_c\bar{V}_2B_c^T = 0, \quad (A.25)$$

where $(\ )^+$ denotes the generalized inverse. In case $\bar{V}_2$ is nonsingular, i.e. the measurement noise is nonsingular, the controllability of (A_c, B_c) guarantees that $Q_2 > 0$. Similar arguments can be applied to P_2, using equation (A.24), $\bar{R}_2$ and the observability of (A_c, C_c). In the general case $\bar{V}_2$ and $\bar{R}_2$ are singular, therefore Assumption 3.

$$P_2^{-1}(A.23)^TQC_2^T - C_c^T(A.10)[0 \quad I_{r2}]^T$$

yields

$$\Gamma(A + BD_cC - G^TB_cC)QC_2^T = 0. \quad (A.26)$$

Substituting B_{c1}, C_{c12} and D_{c21} we get B_{c2}. Similarly, $B_2^T(A.20)Q_2^{-1} - [0 \quad I_{m2}](A.10)B_c^T$ gives

$$B_2^TP(A + BD_cC + BC_c\Gamma)G^T = 0. \quad (A.27)$$

Substituting C_{c1}, D_{c12} and D_{c21} we obtain C_{c2}. $B_2^TP(A.19)U_1 - (A.14)(D_{c12}^TB_1^T + D_{c22}^TB_2^T - B_{c2}^TG)U_1$ gives

$$D_{c22} = -B_2^*(AQ + QA^T + V_1 - Q_aV_2^{-1}Q_a^T) \times U_1 + B_2^*B_1R_2^{-1}P_aC_2^*. \quad (A.28)$$

Multiplying the second term by $C_2QU_1 = I_{r2}$ yields equation (3.2). A dual expression, containing a left inverse of PB_2 can be obtained from equation (A.22). A_c follows from $\Gamma(A.20)Q_2^{-1}$. Equations (3.3)–(3.6) are obtained by substituting (A_c, B_c, C_c, D_c) into (A.19), $(A.20)G + G^T(A.20)^T + G^T(A.21)G^T$, (A.22) and $(A.23)\Gamma + \Gamma^T(A.23)^T + \Gamma^T(A.24)\Gamma$, respectively. Finally direct substitution of D_c and C_c gives the optimal cost (3.8).