FEATURE

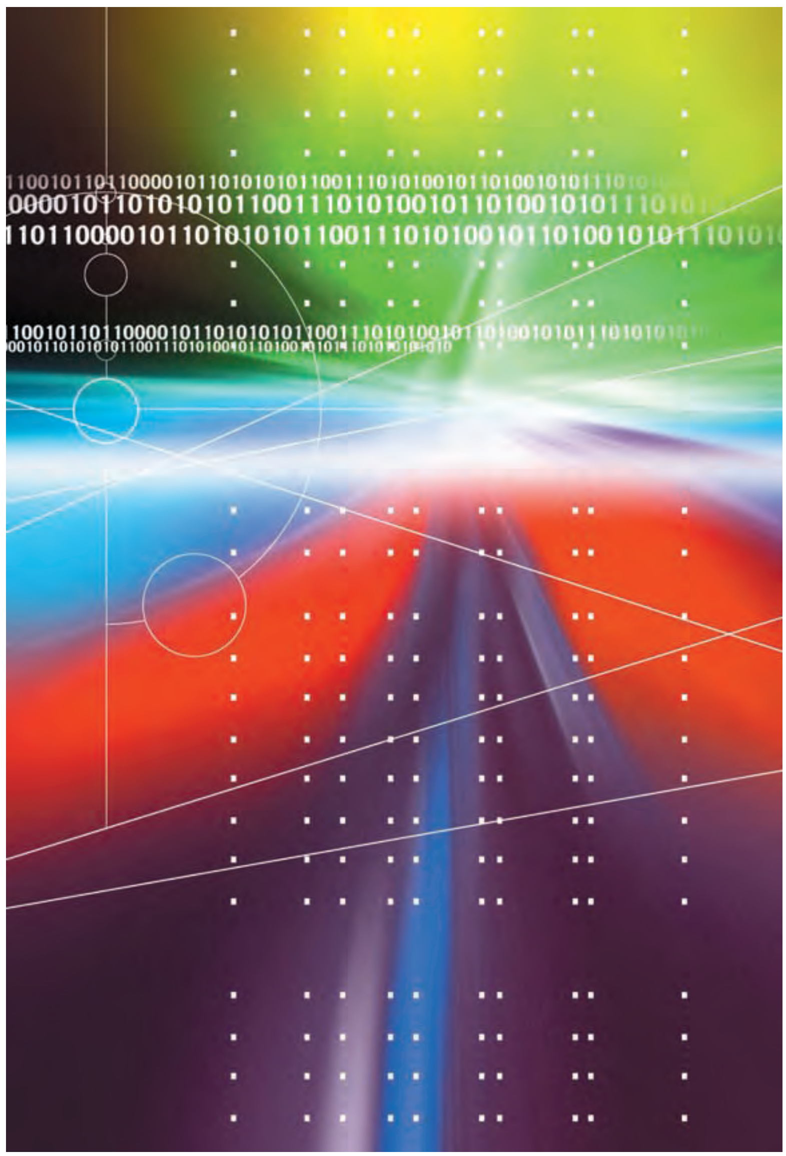
Naive Control of the Double Integrator

By Venkatesh G. Rao and Dennis S. Bernstein

In real-world applications, controllers must operate in the presence of an untold number of off-nominal conditions that either were not or could not be accounted for during formal synthesis. Engineers compensate for these real-world effects by extensive simulation and testing. Only after extensive evaluation and validation can a controller be accepted for critical applications.

In addition, it should be recognized that a controller designed with certain conditions in mind may be subjected to unexpected changes in the plant or environment and may be unable to withstand these changes. A controller that can withstand unexpected changes possesses a certain flexibility; a controller that cannot may be viewed as brittle.

In this article, we practice a form of controller evaluation that might be called "naive control." In naive control, a control algorithm derived under nominal, or ideal, conditions is evaluated by analytical or numerical means under off-nominal, or nonideal, conditions that were not assumed in the formal synthesis procedure. Under such nonideal conditions, the controller may or may not perform well; however, unexpectedly good performance may suggest hidden, serendipitous properties of the algorithm that were not considered in the original development of the controller. Note that this approach is distinct from robust control, which seeks to accom-



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modate off-nominal perturbations in the synthesis procedure.

We consider the double integrator plant, which is one of the most fundamental systems in control applications, representing single-degree-of-freedom translational and rotational motion. Applications of the double integrator include low-friction, free rigid-body motion, such as single-axis spacecraft rotation [1] and rotary crane motion [2]. Control of the double integrator has been of interest since the early days of control theory when it was used extensively to illustrate minimum-time and minimum-fuel controllers [3], [4].

The double integrator plant that we consider here includes a saturation nonlinearity on the control input. Hence the controlled plant is nonlinear. Saturated control of the double integrator has been studied in [5]-[8]. Many of the techniques developed for the control saturation problem [9] can be applied to this problem as well.

The controllers we have chosen comprise an eclectic set of algorithms, ranging from classical to nonlinear to adaptive. We consider the following controllers: minimum time, minimum energy, trap door, discontinuous sliding mode, continuous sliding mode, saturation, homogeneous, direct adaptive, and universal stabilizing. Most of these controllers require measurements of both position and velocity, whereas a few require only position. References for these controllers are given in the text of the article.

What is most striking about this collection of controllers is the way in which each is tuned. For example, some controllers require the inertia value, whereas others do not; some allow specification of the final convergence time, whereas others do not; and some allow specification of the peak control input, whereas others do not. While the tuning procedures differ significantly from controller to controller, what is clear is that every controller, whether it is fixed gain or adaptive, requires some kind of tuning.

Since the controllers are tuned by means of different procedures, and since each controller has different capabilities and features, it is challenging to construct criteria that provide a useful, if not uniformly fair, evaluation of controller capabilities. Our approach to this problem is as follows.

First, we specify nominal conditions in which the mass is given, and we tune each controller to minimize the worst-case time to the origin from a set of initial conditions located on a circle of specified radius in the position/velocity phase plane with a constraint on input amplitude. Next, we test each nominally tuned controller for stabilization under a set of off-nominal conditions. These conditions include inertia perturbation, real and imaginary pole perturbations, measurement delay, unmodeled dynamics, and input nonlinearities. For these off-nominal conditions, we also take note of the control input signal, specifically, the presence of chattering in the control signal, which may be undesirable in certain applications. Finally, we test the disturbance rejection and command-following abilities of each controller with steps and sinusoids.

In the next section we present the problem definition and nominal tuning procedure. Next, we briefly review the features of each controller considered here. Then, in the following three sections, we consider performance for off-nominal stabilization, disturbance rejection, and command following, respectively. Finally, we present some conclusions.

Miscellaneous Notation

The saturation and sign functions are defined as

$$\begin{aligned} \mathrm{sat}_\varepsilon(u) &= u, & |u| &< \varepsilon, \\ &= \varepsilon\,\mathrm{sign}(u), & |u| &\geq \varepsilon, \end{aligned}$$

and

$$\begin{aligned} \mathrm{sign}(u) &= -1, & u &< 0, \\ &= 0, & u &= 0, \\ &= 1, & u &> 0. \end{aligned}$$

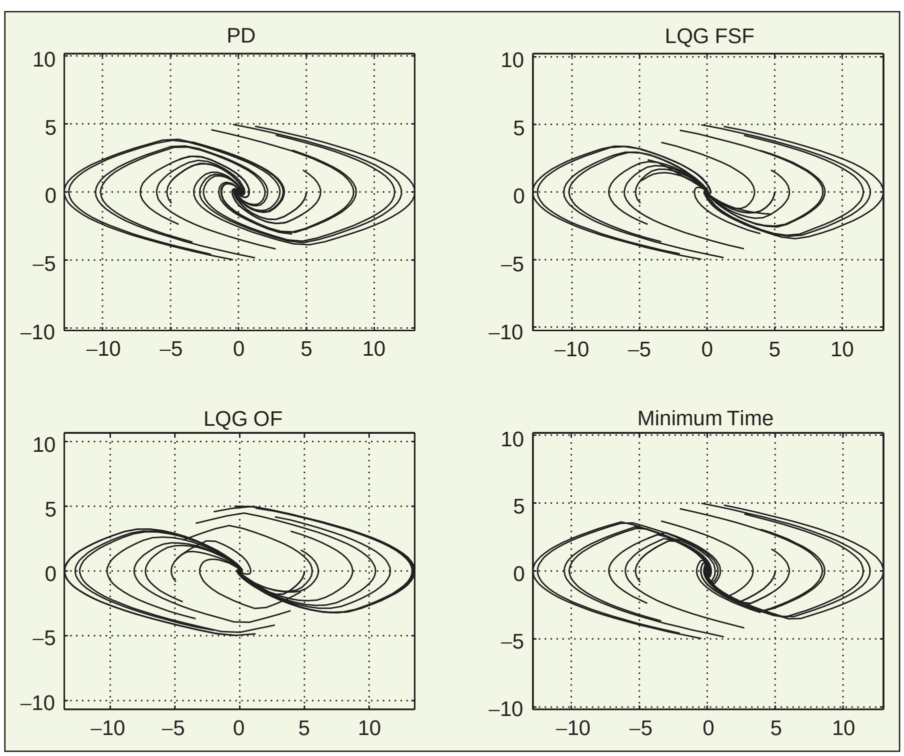

Figure 1. *Nominal phase portraits.*

Table 1. Controller properties.

Controller	Type	Meas. Reqd.	Order	Mass Reqd.?	NAST	TSB	TST Choose?	$u_{\max}$ Choose?	Max. Energy
PD	LTI, Cont.	q_1, q_2	0	N	23.33	∞	—	N	14.73
LQG Full State	LTI, Cont.	q_1, q_2	2	Y	20.30	∞	—	N	11.08
LQG Output Feedback	LTI, Cont.	q_1	2	Y	17.58	∞	—	N	10.31
Minimum Time	NLTI, Disc.	q_1, q_2	0	Y	13.89	Finite	N	Y	13.12
Minimum Energy Open Loop	NLTV, Disc.	q_{10}, q_{20}	0	Y	—	Finite	Y	N	—
Minimum Energy Closed Loop	NLTV, Disc.	q_1, q_2	0	Y	24.85	Finite	Y	N	20.46
Saturation	NLTI, Cont.	q_1, q_2	0	Y	36.77	∞	—	Y	5.03
Trap Door	LTV, Cont.	q_1	2	Y	20.63	Finite	Y	N	14.29
Discontinuous Sliding Mode	NLTI, Disc.	q_1, q_2	0	Y	62.60	∞	—	N	4.85
Continuous Sliding Mode	NLTI, Cont.	q_1, q_2	0	Y	36.77	Finite	N	Y	5.09
Homogeneous	NLTI, Cont.	q_1, q_2	0	N	25.45	Finite	N	N	14.29
Direct Adaptive	NLTI, Cont.	q_1, q_2	2	N	48.50	∞	—	N	13.00
Universal Stabilizing	NLTI, Cont.	q_1, q_2	1	N	—	∞	—	N	—

LTI: linear time invariant; LTV: linear time varying; NLTI: nonlinear time invariant; NLTV: nonlinear time varying; Cont./Disc.: continuous/discontinuous control signal

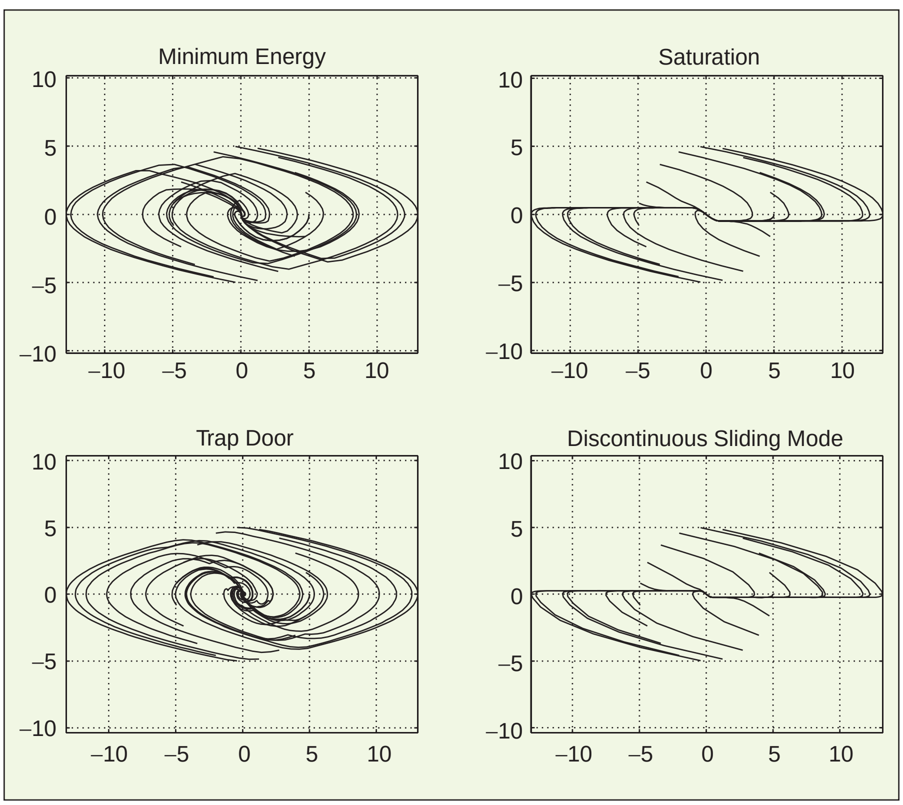

Figure 2. *Nominal phase portraits.*

Double Integrator Plant and Nominal Tuning

The equations of the double integrator are given by

$$\dot{x} = Ax + B\text{sat}_1(u),$$
$$y = Cx, \tag{1}$$

where

$$x = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \tag{2}$$

where q_1 and q_2 are the position and velocity, respectively, of a body having mass *m*,

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for full state feedback (FSF) and

$$C = [1 \quad 0]$$

for output feedback (OF). We assume that control begins at time $t = 0$ with initial conditions

$q_1(0) = q_{10}$ and $q_2(0) = q_{20}$. The saturation function, which enforces a unity constraint on the allowable input amplitude, provides a practically meaningful bound on control authority common to all controllers.

The nominal value of the mass is $m = 1$. For this nominal value, each controller is tuned to have good response with a unity saturation constraint enforced. Some of the controllers are "overtuned" in the sense that the control signal u (prior to saturation) may have a magnitude greater than unity, in which case the saturation constraint is active. The tuning is performed for a collection of 20 initial conditions equally spaced on a circle of radius 5 about the origin in the q_1, q_2 phase plane. The achieved settling time (AST) is the maximum time for the trajectory to reach and remain within a circle of radius 0.01 about the origin for the given collection of initial conditions. The nominal tuning objective is to minimize the nominal achieved settling time (NAST), which is the achieved settling time under nominal conditions.

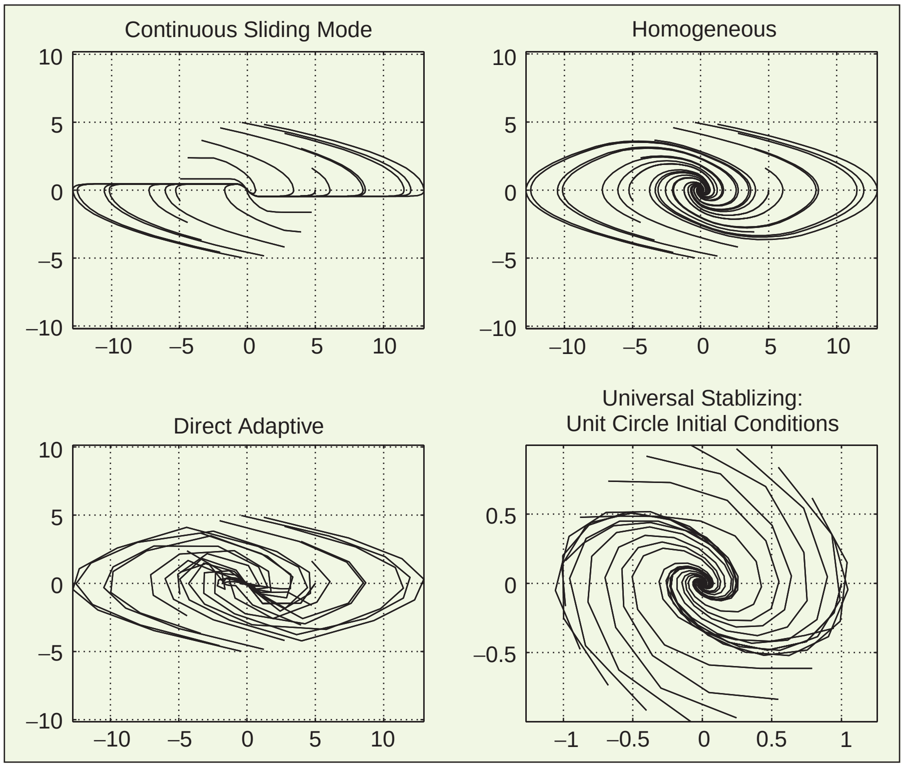

Figure 3. *Nominal phase portraits.*

We note that the mass m and the control saturation $u_{\max}$ may be viewed as additional tuning parameters. For the purposes of this study, however, we have chosen to fix these two parameters.

Description of the Controllers

Here we briefly describe the controllers that will be compared. Table 1 summarizes their properties. The theoretical settling time (TST) is the time to reach the origin under nominal conditions. The term *theoretical settling behavior* (TSB) indicates whether the theoretical settling time is finite or infinite. The TST, when finite, may or may not be assignable. To simplify the notation, the dependence of variables on time t is shown only for the time-varying controllers. Table 1 also lists the maximum energy used by each controller to achieve the settling specification under nominal conditions and over the set of initial conditions. However, energy was not considered in the tuning process.

Linear Time-Invariant Controllers

The proportional-derivative (PD) controller is given by

$$u = -k_1 q_1 - k_2 q_2. \tag{3}$$

As shown in [8], [10], and [11], the saturated closed-loop system (1), (3) is globally asymptotically stable for all positive k_1, k_2. Nominal tuning was performed empirically by setting $k_1 = 1, k_2 = 1.25$. This controller is zeroth order and has infinite theoretical settling behavior.

Linear quadratic Gaussian (LQG) was used to obtain two linear time-invariant (LTI) controllers with full state feedback and output feedback, respectively. The controller is given by

$$\dot{x}_c = A_c x_c + B_c y, \tag{4}$$

$$u = C_c x_c, \tag{5}$$

where $x_c \in \mathrm{R}^2$, and the matrices A_c, B_c, C_c comprise the state space compensator model. Nominal tuning for the LQG controllers was performed empirically by choosing the weighting matrices

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1.7, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W = 15 \tag{6}$$

for full state feedback and

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 4.5, \quad V = 1, \quad W = 15 \tag{7}$$

for output feedback, where Q, R, V, and W are the state weighting, control weighting, measurement noise intensity, and disturbance noise intensity matrices, respectively. The LQG controllers are second order and have infinite theoretical settling behavior. We note that an LQG controller with full state feedback is different from a static linear quadratic regulator (LQR) controller.

Table 2. Performance summary for stabilization.					
Controller	Mass Variation	Pole Perturbation	Measurement Delay	Unmodeled Dynamics	Input Nonlinear
PD	G	G	G	G	F
LQG with Full State Feedback	G	G	E	G	F
LQG with Output Feedback	P	VP	E	P	P
Minimum Time	F	F	VP	G	VP
Minimum Energy Closed Loop	F	F	P	P	F
Saturation	G	G	G	G	G
Trap Door	P	P	VP	VP	P
Discontinuous Sliding Mode	F	G	VP	F	F
Continuous Sliding Mode	G	G	F	F	G
Homogeneous	G	G	G	F	G
Direct Adaptive	P	VP	P	VP	F
E: excellent; G: good; F: fair; P: poor; VP: very poor					

Minimum Time Controller

The minimum time controller [3] is given by

$$\begin{aligned} u &= -u_{\max}\operatorname{sign}\left(q_2+\operatorname{sign}(q_1)\sqrt{2|q_1|u_{\max}/m}\right), \\ &\qquad\qquad q_2+\operatorname{sign}(q_1)\sqrt{2|q_1|u_{\max}/m}\neq 0, \\ &= -u_{\max}\operatorname{sign}(q_1), \qquad q_2+\operatorname{sign}(q_1)\sqrt{2|q_1|u_{\max}/m}=0. \end{aligned} \tag{8}$$

The controller is zeroth order and has finite theoretical settling behavior. Since the control satisfies $|u|\le u_{\max}$, nominal tuning was performed by setting $u_{\max}=1$. To reduce chattering, the sign function was replaced by the hyperbolic tangent function with high finite slope. A similar technique is used in [12]. This modification is not appropriate, however, if the actuator has on-off action.

Minimum Energy Controller

The minimum energy controller [3] in open-loop form is given by

$$u(t)=m\left(\frac{12q_{10}}{t_f^3}+\frac{6q_{20}}{t_f^2}\right)t-m\left(\frac{6q_{10}}{t_f^2}+\frac{4q_{20}}{t_f}\right), \qquad t\in[0,t_f], \tag{9}$$

$$u(t)=0, \qquad t>t_f, \tag{10}$$

and in closed-loop form by

$$u(t)=\frac{-6m}{(t_f-t)^2}q_1(t)-\frac{4m}{t_f-t}q_2(t), \qquad t\in[0,t_f], \tag{11}$$

$$u(t)=0, \qquad t>t_f, \tag{12}$$

where t_f is the theoretical settling time. The controller is zeroth order and has finite theoretical settling behavior. Nominal tuning of the controller was performed empirically by setting $t_f = 4.17$.

Saturation Controller

A bounded controller for a chain of integrators was proposed in [6]. For the double integrator, a slightly more general form than the one in [6] is given by

$$u=-\operatorname{sat}_{u_{\max}}\left[bmq_2+\operatorname{sat}_\varepsilon\left(amq_1+(a/b)mq_2\right)\right]. \tag{13}$$

A slight extension of the proof in [6] shows that if $\varepsilon<(1/2)\,u_{\max}$, then the zero solution of the closed-loop system is attractive for all $a,b>0$. The controller is zeroth order and has infinite theoretical settling behavior. Nominal tuning was performed by choosing $\varepsilon=0.49$ and $u_{\max}=1$. For nominal tuning, the parameters a, b were both set to 1 and the mass was set to its nominal value of $m=1$, which yields the original expression in [6].

Trap Door Controller

The trap door controller [13] has the form

$$\dot{x}_c(t)=A_c x_c(t)+B_c q_1(t), \tag{14}$$

$$u(t)=C_c(t)x_c(t)+D_c(t)q_1(t), \tag{15}$$

where

$$A_c=\begin{bmatrix}0 & 1\\ -k/m_0 & 0\end{bmatrix}, \qquad B_c=\begin{bmatrix}0\\ k/m\end{bmatrix}, \tag{16}$$

$$C_c(t) = [k \quad 0], \quad t \in [0,t_f), \quad D_c(t) = -K - k, \; t \in [0,t_f),$$
$$= [0 \quad 0], \quad t \geq t_f, \qquad\qquad = 0, \qquad t \geq t_f, \tag{17}$$

and where $K, k, m_0 > 0$. The trap door controller emulates the dynamics of a virtual spring-mass-damper system, where K, k, and m_0 are the virtual dynamic parameters. This controller is second order, requires output feedback, and has finite theoretical settling behavior. Tuning of the controller is performed by choosing nonnegative integers n, p and positive real numbers K, k, m_0 that satisfy the relation

$$\frac{k}{K} = \frac{m_0}{m} = \frac{4(2(p-n)+1)^2}{(4n+1)(4p+3)}, \tag{18}$$

which yields the theoretical settling time

$$t_f = \frac{\pi}{2}\sqrt{\frac{(4n+1)(4p+3)m}{K}}. \tag{19}$$

The control is bounded by

$$|u(t)| \leq \sqrt{(K+k)(Kq_{10}^2 + mq_{20}^2)}, \qquad t \geq 0. \tag{20}$$

Since this bound was found to be nonconservative, nominal tuning was performed by setting $n = p = 0$ and $K = 3/7$, which yields $k = 4/7$, $m_0 = 4/3$, $t_f = 4.16$, and $|u(t)| \leq 1$.

Discontinuous Sliding Mode Controller

A discontinuous sliding mode controller [14] for the double integrator is given by

$$u = -\lambda mq_2 - k\,\mathrm{sign}(mq_1 + \lambda mq_2), \tag{21}$$

where $k > 0$ and $\lambda > 0$. This controller is zeroth order and has infinite theoretical settling behavior. Nominal tuning was performed empirically by setting $\lambda = 0.8$ and $k = 0.2$.

Continuous Sliding Mode Controller

The continuous sliding mode controller [7] is given by

$$u = -\mathrm{sat}_\varepsilon\left(\mathrm{sign}(mq_2)|mq_2|^\alpha\right) - \mathrm{sat}_\varepsilon\left(\mathrm{sign}(\phi)|\phi|^{\frac{\alpha}{2-\alpha}}\right), \tag{22}$$

where

$$\phi = mq_1 + \frac{1}{2-\alpha}\mathrm{sign}(mq_2)|mq_2|^{2-\alpha} \tag{23}$$

for $\alpha \in (0,1)$. This controller is zeroth order, has finite theoretical settling behavior, and satisfies $|u| \leq 2\varepsilon$. Nominal tuning was performed empirically by setting $\varepsilon = 0.5$ and $\alpha = 0.9$. Smooth sliding mode controllers derived from the minimum time controller and discontinuous sliding mode controllers are given in [15] and [16].

Homogeneous Controller

A smooth finite-time controller yielding homogeneous closed-loop dynamics for a chain of integrators was given in [17]. For the double integrator, this controller is given by

$$u = -k_1\mathrm{sign}(q_1)|q_1|^{\frac{\alpha}{2-\alpha}} - k_2\mathrm{sign}(q_2)|q_2|^\alpha, \tag{24}$$

where k_1 and k_2 are positive real numbers and $\alpha \in (0,1)$. This controller is zeroth order and has finite theoretical settling behavior. Note that the mass m does not appear in the control law, which implies stability robustness to mass variation, though the inertia value must be positive. Nominal tuning was performed empirically by setting $k_1 = 0.7$, $k_2 = 0.7$, and $\alpha = 0.8$.

Direct Adaptive Controller

A direct adaptive controller [18], [19] for second-order systems in companion form with full state feedback is given by

$$\dot{k}_1 = -\lambda_1 pq_1^2 - (\lambda_1 + p\lambda_{12})q_1q_2 - \lambda_{12}q_2^2, \tag{25}$$

$$\dot{k}_2 = -\lambda_{12} pq_1^2 - (\lambda_{12} + \lambda_2 p)q_1q_2 - \lambda_2 q_2^2, \tag{26}$$

$$u = -k_1q_1 - k_2q_2, \tag{27}$$

where

$$\begin{bmatrix} \lambda_1 & \lambda_{12} \\ \lambda_{12} & \lambda_2 \end{bmatrix}$$

is positive definite, $p > 0$, and k_1, k_2 are adaptive gains. This controller is second order and has infinite theoretical settling behavior. The mass m does not appear in the control law, which implies stability robustness to mass variation, though the inertia value must be positive. Nominal tuning was performed empirically by setting $\lambda_1 = 0.25, \lambda_2 = 1, p = 0.3$, and $\lambda_{12} = 0$.

Universal Stabilizing Controller

A universal stabilizing controller [20] for the double integrator is given by

$$\dot{k} = (c_1q_1 + c_2q_2)^2, \tag{28}$$

$$u = -(\log k)\left(\cos\sqrt{\log k}\right)(c_1q_1 + c_2q_2), \tag{29}$$

where c_1, c_2 are positive real numbers and k is an adaptive gain such that $k(0) \geq 1$. This controller is first order and has infinite theoretical settling behavior. The mass m does not appear in the control law, implying stability robustness to mass variation. In addition, this controller can be shown to be stable for negative values of inertia. A "negative" value of mass might arise as a result of a software or wiring error. Without saturation, the controller demonstrated its ability to stabilize the plant irrespective of the sign of the mass. With saturation and initial conditions on the circle of radius 5, however, the controller could not be tuned to achieve nominal stabilization. Therefore, this controller was not considered further.

Nominal Performance

The phase portraits for the nominally tuned controllers are shown in Figs. 1-3. Table 1 lists the maximum nominal achieved settling times, calculated by taking the maximum NAST from the 20 trajectories with uniformly spaced initial conditions on the circle of radius 5.

Off-Nominal Stabilization

The controllers were tested for their performance in stabilization, disturbance rejection, and command following. In this section we consider stabilization. Disturbance rejection and command following results are described in the following sections.

For performance analysis with regard to stabilization, achieved settling time was used for comparison. For each test, qualitative attributes such as the nature of control action and transient response are used to judge the performance of each controller. Table 2 summarizes the results of the tests. The results of each of the tests are shown in terms of the degradation of achieved settling time. The range of variation for the parameter under consideration is noted in the text. Where a particular controller does not settle, or goes unstable, the graph for that controller does not extend to the full range. For the trap door and minimum energy closed-loop controllers, the theoretical setting time t_f is explicitly chosen, with the control shut off at $t = t_f$. If, for off-nominal operation, however, the settling requirement is not achieved, then these controllers are restarted at the shutoff time with new initial conditions.

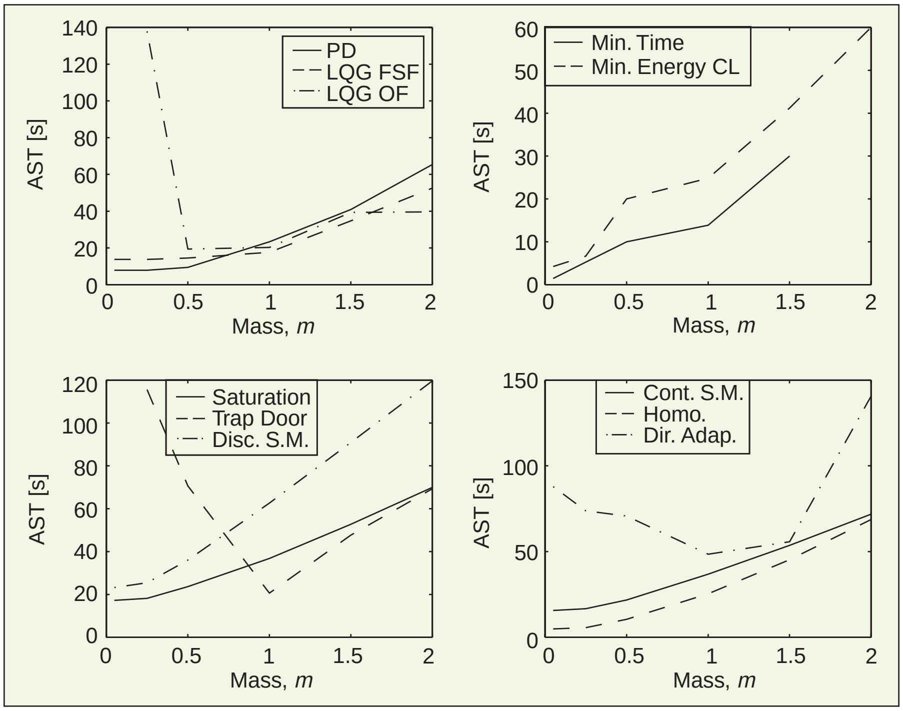

Figure 4. *Performance comparison with mass variation.*

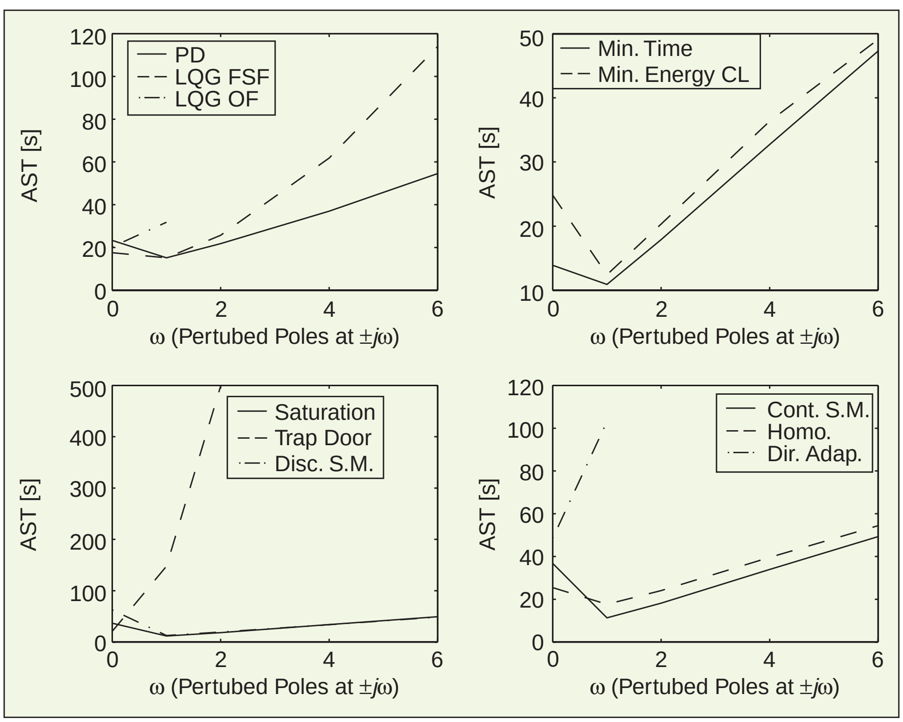

Figure 5. *Performance comparison with imaginary axis pole perturbation.*

Robustness to Mass Variation

Tests were conducted for true mass values $m \in \{0.05, 0.25, 0.5, 1, 1.5, 2\}$. The value of m, which appears as a parameter in certain controllers, was set to 1. The results are given in Fig. 4, which shows the degradation in achieved settling time with mass variation. Significant chattering was observed in the minimum time and discontinuous sliding mode controllers.

Pole Location

The poles of the double integrator were moved apart on the imaginary axis such that the plant becomes $1/(s^2+\omega_n^2)$, with $\omega_n \in [1,6]$. The degradation in settling times with respect to ω_n is shown in Fig. 5. The trap door, LQG output feedback, and direct adaptive controllers fared poorly, while the LTI and saturation controllers fared better than the rest. In addition, both poles were moved along the real axis such that the plant becomes $1/(s+a)^2$, with $a \in [-0.2, 0.2]$. The degradation in settling times with respect to the pole position on the real axis is shown in Fig. 6. Beyond $a = -0.2$ on the real axis, none of the controllers was able to stabilize the plant. The saturation and sliding mode controllers performed better than the rest.

Measurement Delay

An output delay $e^{-\tau s}$ in the range $\tau \in [0.1, 0.8]$ s was inserted in the measurement feedback path. The minimum time controller failed to settle and the discontinuous sliding mode, trap door, and direct adaptive controllers were destabilized by a delay of less than 0.2 s The LTI, saturation, and homogeneous controllers performed best overall. The degradation in settling times is shown in Fig. 7. Note that the minimum energy controller rapidly goes unstable beyond a delay value of $\tau = 0.2$ s, with a better settling performance than nominal for a small delay of 0.1 s. This anomaly appears because of the limited set of initial conditions considered coupled with the rapid transition to instability in the presence of delay.

Unmodeled Dynamics

Unmodeled dynamics due to a flexible appendage were considered. Two appendage mass values, $m_0 = 0.2$ and $m_0 = 0.5$, were considered, and stiffness values k were considered in the range $k \in [0.4, 1.4]$. The LQG output feedback, minimum energy closed-loop, and trap door controllers performed poorly. Of the successful controllers, the PD and LQG full state feedback controllers performed the best. Fig. 8 shows the degradation in settling times for the case $m_0 = 0.2$. The performance was found to be very sensitive to initial conditions.

Measurement Noise

Varying degrees of measurement noise were added to the feedback signal. The results of these tests were inconclusive.

Input Nonlinearities

A variety of input nonlinearities were added to the system. The key observations are summarized below.

Rate Limit

Increasingly severe rate limits were imposed on the control signal. The minimum-time and discontinuous sliding mode controllers failed for all but the highest rate limits, whereas the saturation controller fared exceptionally well, surviving extremely severe rate limits. The direct adaptive controller fared poorly.

Backlash

Backlash, or hysteresis, had the effect of producing sustained oscillations in most of the controllers. Both sliding

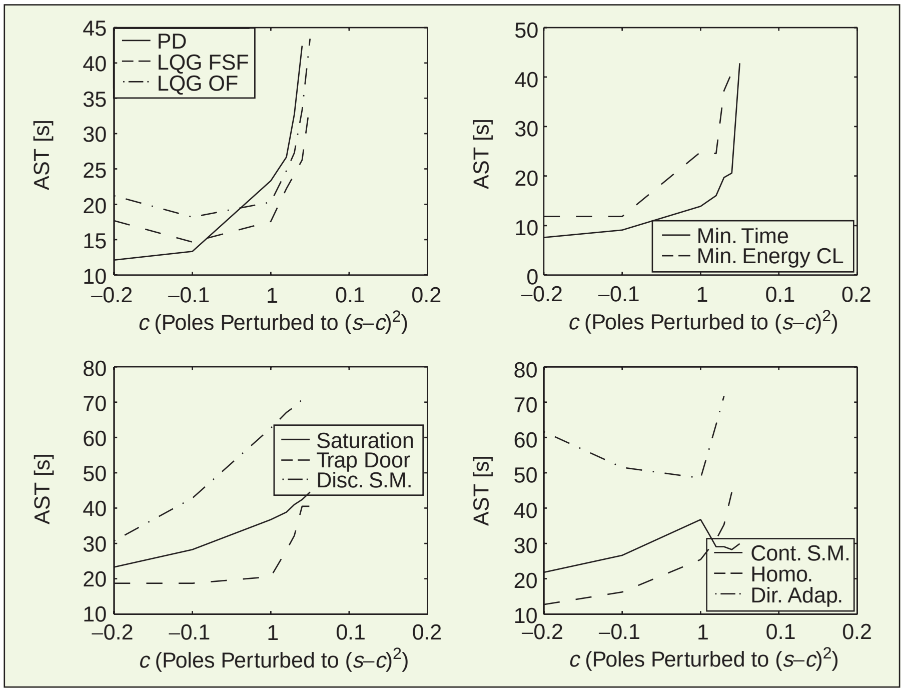

Figure 6. *Performance comparison with real axis pole perturbation.*

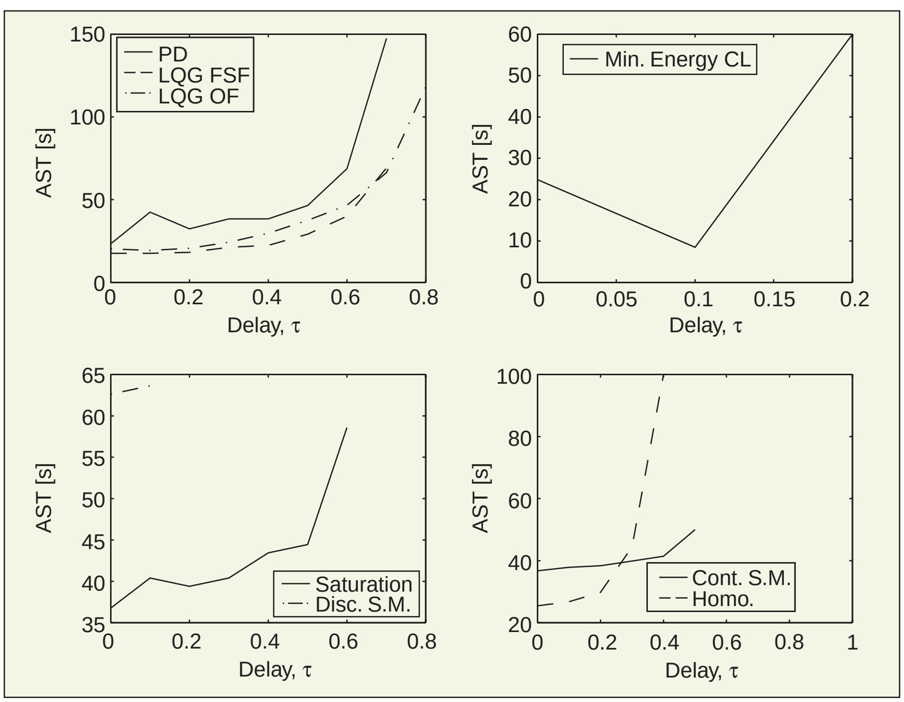

Figure 7. *Performance comparison with measurement delay.*

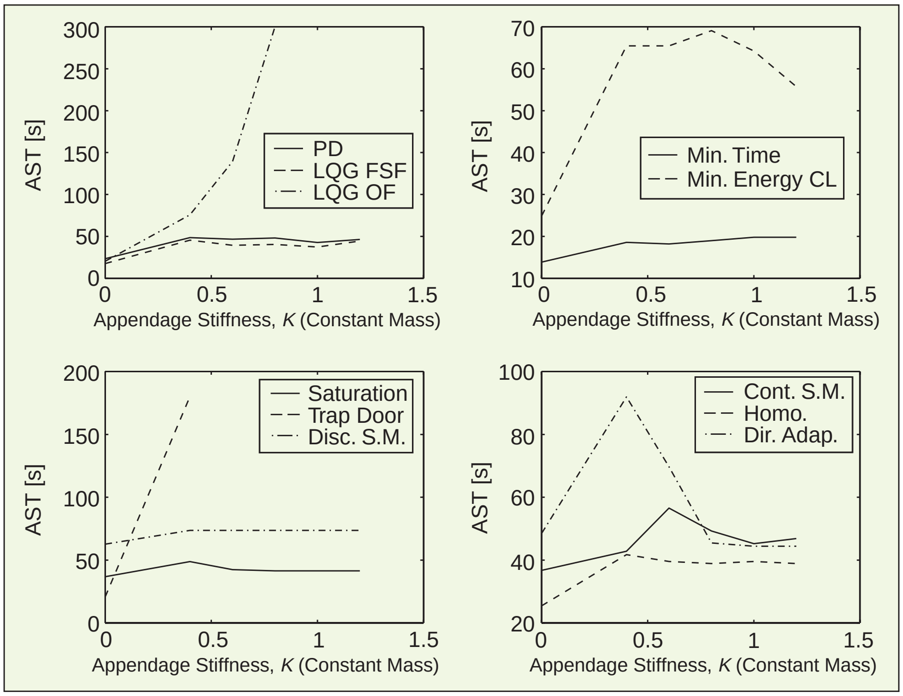

Figure 8. *Performance comparison with flexible appendage.*

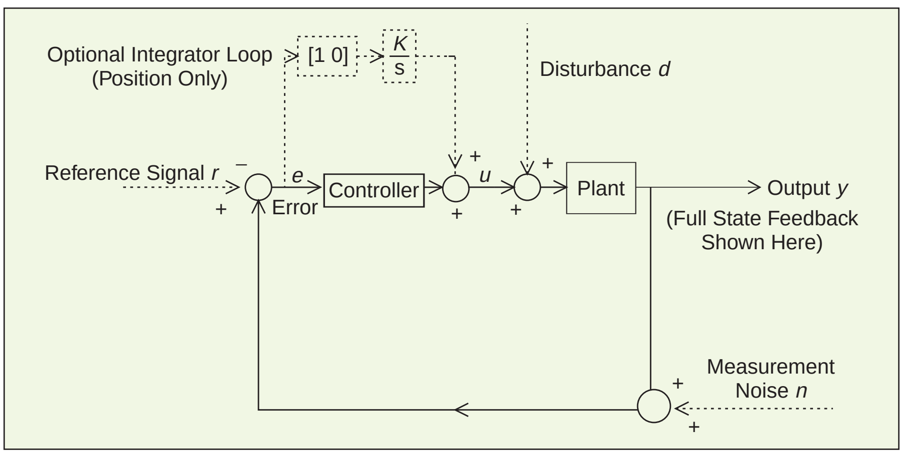

Figure 9. *Block diagram for disturbance rejection and tracking. (Note: Dashed lines represent elements that may be present in certain tests.)*

mode controllers did exceptionally well. The performance of the remaining controllers was comparable.

Relay and Dead Zone

Relay and dead zone input nonlinearities were considered. For the minimum time controller, the relay nonlinearity had no effect, which is to be expected because the control action is bang-bang. The results were inconclusive for the remaining tests.

Disturbance Rejection

Fig. 9 shows the relevant details for the introduction of disturbance signals in the tests. Two types of disturbances were applied to the plant, specifically, steps of magnitude 0.25, 0.5, 0.75 and a sinusoid of frequency 0.1 Hz. The results are shown in Table 3. The minimum time controller fared best, canceling all of the disturbances completely, whereas the discontinuous sliding mode controller fared worst, going unstable with the 0.25 step. The direct adaptive controller canceled step disturbances for many of the initial conditions and reduced the steady-state error for the rest. The saturation controller survived the 0.25 step but failed the others. Only the minimum time controller was able to cancel the sinusoid, whereas all others failed to do so. In terms of the magnitude of steady-state oscillations, the direct adaptive and continuous sliding mode controllers fared better than average, while the homogeneous controller fared worse than average. Table 3 shows the qualitative rating of the controllers for the disturbance rejection tests.

Integral Control

The homogeneous and saturation controllers are both derived from families of controllers for arbitrarily long integrator chains. We exploited this feature by using the corresponding controller for the triple integrator plant with the full state augmented with the integral of position, as shown in Fig. 10. The resulting triple integrator plant can be written as $\dot{q}_1 = q_2$, $\dot{q}_2 = q_3$, $\dot{q}_3 = u$. For the homogeneous triple integrator controller [17], the control is given by

$$u = -k_1 \text{sign}(q_1)|q_1|^{\frac{\alpha}{3-2\alpha}} - k_2 \text{sign}(q_2)|q_2|^{\frac{\alpha}{2-\alpha}} - k_3 \text{sign}(q_3)|q_3|^{\alpha}, \tag{30}$$

where $s^3 + k_3 s^2 + k_2 s + k_1$ is Hurwitz and $0 \le \alpha < 1$. Here the state q_1 represents the integral of position. For the triple integrator saturation controller [6], the control is given by

$$u = -\text{sat}_{u_{\max}}\left(mq_1 + \text{sat}_{\varepsilon_2}\left(mq_2 + \text{sat}_{\varepsilon_1}(mq_3)\right)\right), \tag{31}$$

where $\varepsilon_1 < (\varepsilon_2/2) < (u_{\max}/4)$. With this enhancement, the homogeneous and saturation controllers exhibited excellent step disturbance rejection. The direct adaptive controller was augmented for integral control using [18], [19]

$$u = -k_1 q_1 - k_2 q_2 - \phi, \tag{32}$$

where ϕ is the integral control term given by

$$\dot{\phi} = -\lambda_3 p q_1 - \lambda_3 q_2, \qquad \lambda_3 > 0. \tag{33}$$

From Fig. 9, for LTI controllers, the closed-loop disturbance transfer function from disturbance to output is $G(s)/(1+G(s)H(s))$, where $G(s)=1/s^2$ and $H(s)$ is the controller transfer function. For steps, it follows from the final value theorem that the disturbance transfer function is zero at steady state if and only if the closed-loop system is stable and the controller has an integrator. For the PD, an integral term was added to yield a proportional-integral-derivative (PID) controller. The LQG controllers were modified to include integrators by augmenting the plant states with the integral of the position error and deriving LQG controllers for the augmented plants.

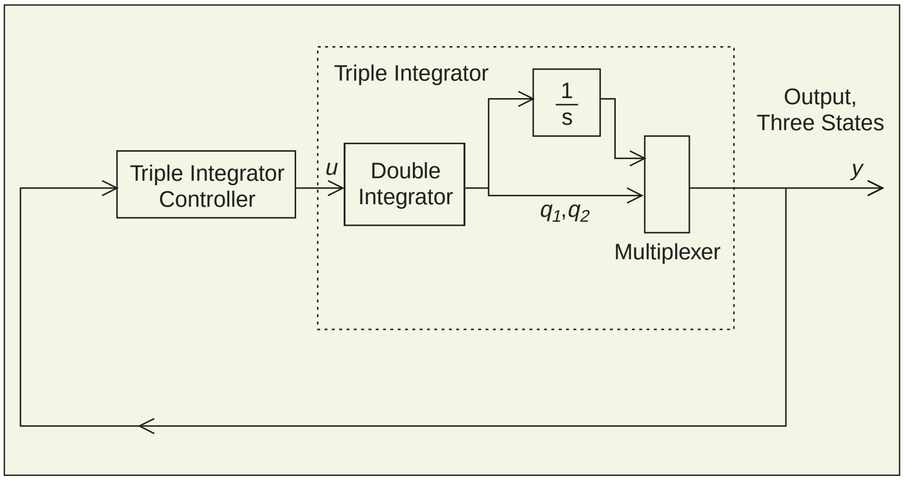

Figure 10. *Block diagram for triple integrator controllers. (Reference, noise, disturbance introduced as in Fig. 9.)*

The modified LTI controllers, as well as the direct adaptive controller modified by (32) and (33), showed marked improvement. Steps were canceled completely, and, for sinusoids, the magnitude of oscillations was considerably reduced. For the remaining controllers, an ad hoc integrator was added, as shown in Fig. 9. Although this construction is not theoretically substantiated, it does provide some improvement if the integrator gain is carefully chosen.

Finally, although an internal model could be used to improve disturbance rejection for sinusoidal disturbances, this approach requires knowledge of the disturbance frequency, which would be inconsistent with our naive control philosophy.

Command Following

The controllers were subjected to command following tests, even though most of them are designed for stabilization only. Two types of inputs were considered, namely, steps and sinusoids. Step following was satisfactory for all but the direct adaptive and trap door controllers, which went unstable. The remaining controllers exhibited zero steady-state error for the nominal plant. For the LTI controllers, we can explain this phenomenon as follows. From Fig. 9, the command following transfer function from the reference to the output is $L(s)/(1+L(s))$, where $L(s)=G(s)H(s)$ is the loop transfer function, $G(s)=1/s^2$, and $H(s)$ is the controller transfer function. The two integrators in the plant ensure that the command following transfer function is unity at dc. For sinusoids, none of the controllers performed well. Step responses are shown in Figs. 11-14, and a qualitative rating is shown in Table 3.

Integrators for Command Following

Integral control, used as described in the previous section, improved the performance of the LTI, homogeneous, and saturation controllers. With the addition of an integrator, as in (32) and (33), the direct adaptive controller was able to

Table 3. Performance summary for disturbance rejection and command following.

Controller	Reject Step	Reject Sine	Follow Step	Integrator Added
PD	G	F	F	Improves
LQG with Full State Feedback	F	F	F	Improves
LQG with Output Feedback	P	P	P	Improves
Minimum Time	E	E	E	Not required
Minimum Energy Closed Loop	F	P	G	Improves but robustness degrades
Saturation	VP	P	F	Improves
Trap Door	P	VP	VP	Goes unstable
Discontinuous Sliding Mode	VP	F	F	Improves but robustness degrades
Continuous Sliding Mode	P	F	G	Improves but robustness degrades
Homogeneous	F	F	F	Improves
Direct Adaptive	E	E	VP	Improves
E: excellent; G: good; F: fair; P: poor; VP: very poor				

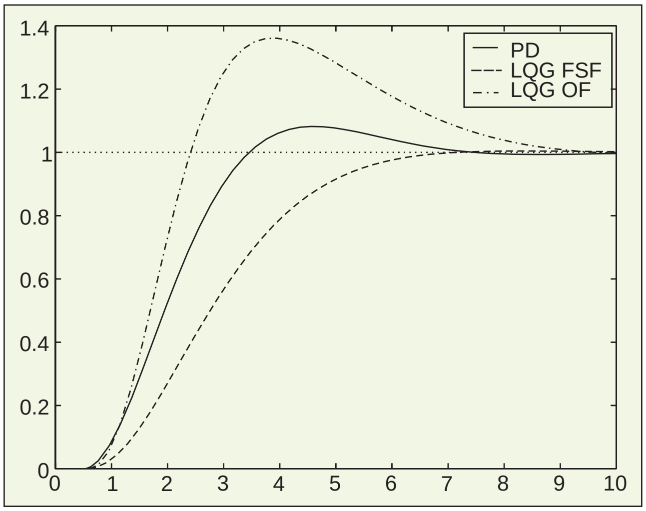

Figure 11. *Unit step responses (PD, LQG with full state feedback, LQG with output feedback).*

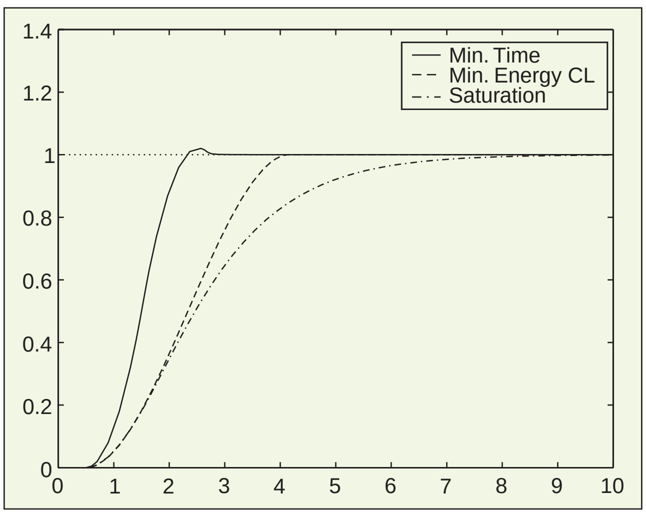

Figure 12. *Unit step responses (minimum time, minimum energy, saturation).*

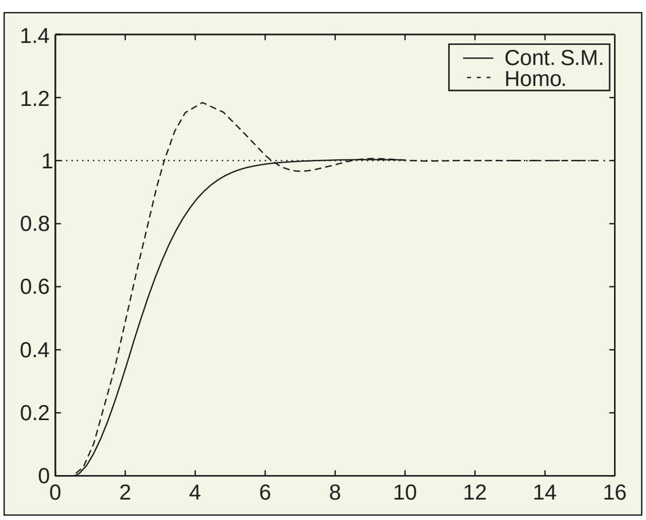

Figure 13. *Unit step responses (continuous sliding mode, homogeneous).*

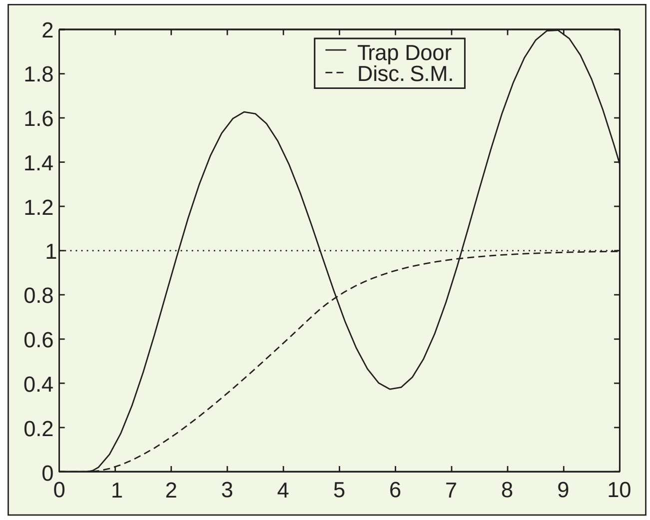

Figure 14. *Unit step responses (trap door, discontinuous sliding mode).*

follow steps. Integrators did not improve the sinusoidal command following performance for any of the controllers.

Discussion and Conclusions

In this article we compared a diverse set of controllers. It is important to stress that comparisons cannot be made completely fair simply because it is impossible to test for all possible qualities of every controller. However, certain general conclusions can be drawn. First, the controllers we considered fall into three fairly well-defined groups. The controllers in the first group, comprising the saturation [6], continuous sliding mode [7], PD, LQG full state feedback, and homogeneous [17] controllers, are noticeably superior to the rest. They exhibit good robustness and graceful performance degradation. The saturation controller performs especially well but suffers from very poor disturbance rejection properties. However, the saturation and homogeneous controllers exhibit excellent disturbance rejection with the use of their augmented, triple integrator versions.

The controllers in the second group, comprising the minimum time, the minimum energy closed-loop, and the discontinuous sliding mode [14] controllers, have obvious weaknesses, which may not be too critical for certain applications. The performance of the controllers in this group is noticeably poorer than those in the first group. The weakness of the minimum time and discontinuous sliding mode controllers is in their discontinuous control action, which caused them to chatter. Both controllers failed the rate limit test for reasonable rate limits and performed very poorly with delays. The discontinuous control action is not a problem, however, for applications involving on-off actuators. The minimum energy closed-loop controller is poor at handling delays, disturbances, and unmodeled dynamics. One surprising feature of the minimum time controller is its excellent disturbance rejection capability, without the need for an integrator. Specifically, the minimum time controller is able to cancel steps and sinusoids completely within the range of its control amplitude. This property is clearly due to its high gain character; however, this remains to be demonstrated analytically. Performance does not degrade gracefully in this group of controllers, however, which detracts from their interesting features.

The controllers in the last group, comprising the LQG output feedback, trap door, minimum energy open-loop, universal stabilizing, and direct adaptive controllers, have serious performance problems. In defense of these controllers, we note that they either require less feedback (the first two need position only, whereas the third requires only occasional measurements for periodic resetting) or use less modeling information. These features make these controllers attractive in certain applications. Of this group, the universal stabilizing controller and minimum energy open-loop controller could not stabilize the nominal plant for all the initial conditions considered and were therefore not tested further. The direct adaptive controller is very sensitive to unmodeled dynamics and measurement delays but exhibits very good disturbance rejection properties, which are further improved upon adding integral control. The LQG output feedback and trap door controllers both use only position feedback, and, of these two, the LQG output feedback controller performs better. Both of these controllers have poor disturbance rejection properties.

Finally, the controllers we considered have been incorporated in a Simulink-based toolbox for real-time implementation on a dSPACE, Inc., 1103 system. These controllers will be compared experimentally on a rotational air spindle platform with double integrator dynamics and reaction wheel torque actuators [21]. The results of this comparison will be reported in a future article.

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