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NUMERICAL SOLUTION OF THE OPTIMAL MODEL REDUCTION EQUATIONS*

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Abstract

An algorithm is proposed for numerically solving the optimal model reduction equations derived in [1]. These equations are in the form of a pair of modified Lyapunov equations coupled by an oblique projection that is a consequence of optimality and which determines the optimal reduced-order model. This form of the necessary conditions considerably simplifies previous results of Wilson and clearly demonstrates the suboptimality of the balancing method of Moore.

1. Introduction

The purpose of this paper is to propose a numerical algorithm for solving the optimal model reduction equations derived in [1]. These equations considerably simplify the necessary conditions for optimality first derived in [2] by exploiting the presence of a projection matrix which was not recognized in [2] and which is a direct consequence of optimality. An important benefit of these equations is that they immediately provide a rigorous optimality context for the "balancing" method of Moore [3] which is based on system-theoretic arguments as opposed to optimality criteria. As shown in [1], the balancing approach is always suboptimal and, when the "weak-subsystem" hypothesis is invoked, is nearly optimal.

The optimal model reduction equations are in the form of two n×n modified Lyapunov equations (see (2.11), (2.12)) coupled by an oblique (i.e., nonorthogonal) projection which determines the optimal reduced-order model. The highly structured form of these equations and the presence of the optimal projection are the motivation for devising new efficient numerical algorithms for computing optimal reduced-order models.

The algorithm proposed in this paper is iterative (which is necessitated by the coupling of the modified Lyapunov equations) and directly exploits the properties of the optimal projection matrix. The algorithm was successfully applied to an example considered in both [2] and [3].

2. Problem Statement and Main Result

The following notation and definitions will be needed:

n, m, ℓ, n_r	positive integers, $1 \le n_r \le n$
x, u, y, x_r, y_r	n, m, ℓ, n_r, ℓ-dimensional vectors
A, B, C	$n\times n$, $n\times m$, $\ell\times n$ matrices
A_r, B_r, C_r	$n_r\times n_r$, $n_r\times m$, $\ell\times n_r$ matrices
R, V	$\ell\times\ell$, $m\times m$ positive-definite matrices
I_r	$r\times r$ identity matrix
Z^T	transpose of vector or matrix Z
$\rho(Z)$	rank of matrix Z
$\mathbb{E}$	expected value
$\mathbb{R}$, $\mathbb{R}^{r\times s}$	real numbers, $r\times s$ real matrices
positive-semisimple matrix	matrix similar to a positive diagonal matrix

We consider the following optimal model-reduction problem. Given the system

$$\dot{x} = Ax + Bu, \tag{2.1}$$

$$y = Cx \tag{2.2}$$

find a reduced-order model

$$\dot{x}_r = A_r x_r + B_r u, \tag{2.3}$$

$$y_r = C_r x_r \tag{2.4}$$

which minimizes the model-reduction criterion

$$J(A_r,B_r,C_r) \triangleq \lim_{t\to\infty} \mathbb{E}\,[(y-y_r)^T R(y-y_r)].$$

The input u(t) is taken to be white noise with positive-definite intensity V. To guarantee that J is finite it is assumed that A is stable and we restrict our attention to the set of admissible reduced-order models

$$\mathcal{A} \triangleq \{(A_r,B_r,C_r)\colon A_r \text{ is stable}\}.$$

Since the value of J is independent of the internal realization of the transfer function corresponding to (2.3) and (2.4), we further restrict our attention to the set

$$\mathcal{A}_+ \triangleq \{(A_r,B_r,C_r) \in \mathcal{A}\colon (A_r,B_r) \text{ is controllable and } (A_r,C_r) \text{ is observable}\}.$$

The following lemma is needed for the statement of the main result.

<u>Lemma 2.1</u>. Suppose $\hat{Q},\hat{P} \in \mathbb{R}^{n\times n}$ are nonnegative definite. Then there exist $G,\Gamma \in \mathbb{R}^{n_r\times n}$ and positive-semisimple $M \in \mathbb{R}^{n_r\times n_r}$ such that

$$\hat{Q}\hat{P} = G^T M \Gamma, \tag{2.5}$$

$$\Gamma G^T = I_{n_r}. \tag{2.6}$$

We shall refer to $G,\Gamma \in \mathbb{R}^{n_r\times n}$ and positive-semisimple $M \in \mathbb{R}^{n_r\times n_r}$ satisfying (2.5) and (2.6) as a (G,M,Γ)-factorization of $\hat{Q}\hat{P}$.

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Main Theorem. Suppose $(A_r,B_r,C_r) \in \mathcal{A}_+$ solves the optimal model-reduction problem. Then there exist nonnegative-definite matrices $\hat{Q},\hat{P} \in \mathbb{R}^{n\times n}$ such that, for some (G,M,Γ)-factorization of $\hat{Q}\hat{P}$, A_r, B_r and C_r are given by

$$A_r = \Gamma A G^T, \tag{2.7}$$

$$B_r = \Gamma B, \tag{2.8}$$

$$C_r = CG^T, \tag{2.9}$$

and such that, with $\tau \triangleq G^T\Gamma$, and $\tau_\perp \triangleq I_n - \tau$, the following conditions are satisfied:

$$\rho(\hat{Q}) = \rho(\hat{P}) = \rho(\hat{Q}\hat{P}) = n_r, \tag{2.10}$$

$$0 = A\hat{Q} + \hat{Q}A^T + BVB^T - \tau_\perp BVB^T\tau_\perp^T, \tag{2.11}$$

$$0 = A^T\hat{P} + \hat{P}A + C^TRC - \tau_\perp^T C^TRC\tau_\perp. \tag{2.12}$$

The Main Theorem consists of necessary conditions in the form of two modified Lyapunov equations (2.11) and (2.12) plus rank conditions (2.10) which must be satisfied when an optimal reduced-order model exists. The modified Lyapunov equations are coupled by the n×n matrix τ which is a projection (idempotent matrix) since $\tau^2 = G^T\Gamma G^T\Gamma = G^T I_{n_r}\Gamma = \tau$. Note that in general τ is an oblique projection and not necessarily an orthogonal projection since it may not be symmetric. We shall refer to the projection τ corresponding to the solution of the optimal model-reduction problem as the "optimal projection". It should be stressed that the form of the optimal reduced-order model (2.7)-(2.9) is a direct consequence of optimality and not the result of an a priori assumption on the structure of the reduced-order model.

Observe that if x_r is replaced by Sx_r then an "equivalent" reduced-order model is obtained with (A_r,B_r,C_r) replaced by $(SA_rS^{-1},SB_r,C_rS^{-1})$. Clearly, $J(A_r,B_r,C_r) = J(SA_rS^{-1},SB_r,C_rS^{-1})$. It can be shown (see [1]) that this transformation corresponds to the alternative factorization $\hat{Q}\hat{P} = (S^{-T}G)^T(SMS^{-1})(S\Gamma)$ and, moreover, all (G,M,Γ)-factorizations of $\hat{Q}\hat{P}$ are related by a nonsingular transformation.

The following result shows that there exists a similarity transformation which simultaneously diagonalizes $\hat{Q}\hat{P}$ and τ.

Proposition 2.1. There exists invertible $\Phi \in \mathbb{R}^{n\times n}$ such that

$$\hat{Q} = \Phi^{-1}\begin{bmatrix}\Lambda_{\hat{Q}} & 0\\ 0 & 0\end{bmatrix}\Phi^{-T}, \quad \hat{P} = \Phi^{T}\begin{bmatrix}\Lambda_{\hat{P}} & 0\\ 0 & 0\end{bmatrix}\Phi, \tag{2.13}$$

$$\hat{Q}\hat{P} = \Phi^{-1}\begin{bmatrix}\Lambda & 0\\ 0 & 0\end{bmatrix}\Phi, \quad \tau = \Phi^{-1}\begin{bmatrix}I_{n_r} & 0\\ 0 & 0\end{bmatrix}\Phi, \tag{2.14a,b}$$

where $\Lambda_{\hat{Q}},\Lambda_{\hat{P}} \in \mathbb{R}^{n_r\times n_r}$ are positive diagonal, $\Lambda \triangleq \Lambda_{\hat{Q}}\Lambda_{\hat{P}}$ and the diagonal elements of Λ are the eigenvalues of M.

3. Proposed Algorithm

The following algorithm is proposed for solving the optimal model-reduction equations:

STEP 1: Initialize $\tau^{(0)} = I_n$;

STEP 2: Solve for $\hat{Q}^{(k)}$, $\hat{P}^{(k)}$:

$$0 = (A - \tau^{(k)}A\tau_\perp^{(k)})\hat{Q}^{(k)} + \hat{Q}^{(k)}(A - \tau^{(k)}A\tau_\perp^{(k)})^T + BVB^T, \tag{3.1}$$

$$0 = (A - \tau_\perp^{(k)}A\tau^{(k)})^T\hat{P}^{(k)} + \hat{P}^{(k)}(A - \tau_\perp^{(k)}A\tau^{(k)}) + C^TRC; \tag{3.2}$$

STEP 3: Factor:

$$\hat{Q}^{(k)}\hat{P}^{(k)} = S^{(k)}(\Sigma^{(k)})^2(S^{(k)})^{-1}, \tag{3.3}$$

$$\Sigma^{(k)} = \mathrm{diag}(\sigma_1^{(k)},\ldots,\sigma_n^{(k)}),$$

$$\sigma_1^{(k)} \ge \sigma_2^{(k)} \ge \ldots \ge \sigma_n^{(k)} \ge 0;$$

STEP 4: Update:

$$\tau^{(k+1)} = S^{(k)}\begin{bmatrix}I_{n_r} & 0\\ 0 & 0\end{bmatrix}(S^{(k)})^{-1}; \tag{3.4}$$

STEP 5: Check for convergence; if not, increment k and return to STEP 2;

STEP 6: Set:

$$\hat{Q} = \tau^{(\infty)}\hat{Q}(\tau^{(\infty)})^T, \quad \hat{P} = (\tau^{(\infty)})^T\hat{P}\tau^{(\infty)}. \tag{3.5}$$

To show that the proposed algorithm leads to the solution of the optimal model-reduction equations, expand (2.11) and (2.12) in the "optimal projection realization" (2.13), (2.14) to obtain

$$0 = \hat{A}_r\Lambda_{\hat{Q}} + \Lambda_{\hat{Q}}\hat{A}_r^T + \hat{B}_rV\hat{B}_r^T, \tag{3.6}$$

$$0 = \hat{A}_{2r}\Lambda_{\hat{Q}} + \hat{B}_2V\hat{B}_r^T, \tag{3.7}$$

$$0 = \hat{A}_r^T\Lambda_{\hat{P}} + \Lambda_{\hat{P}}\hat{A}_r + \hat{C}_r^TR\hat{C}_r, \tag{3.8}$$

$$0 = \Lambda_{\hat{P}}\hat{A}_{r2} + \hat{C}_r^TR\hat{C}_2, \tag{3.9}$$

where

$$\Phi A\Phi^{-1} = \begin{bmatrix}\hat{A}_r & \hat{A}_{r2}\\ \hat{A}_{2r} & \hat{A}_{22}\end{bmatrix}, \quad \Phi B = \begin{bmatrix}\hat{B}_r\\ \hat{B}_2\end{bmatrix}, \quad C\Phi^{-1} = [\hat{C}_r \;\; \hat{C}_2].$$

It can readily be seen that (3.5) yields the solution of (3.6)-(3.9) if convergence has occurred in (3.1) and (3.2). Note that in the special case $R = I_m$ and $V = I_\ell$, the first iteration of the proposed algorithm leads to $\hat{Q}^{(0)} = W_c$, $\hat{P}^{(0)} = W_o$ (i.e., the controllability and observability gramians) and the projection $\tau^{(1)}$ defined by (3.4) is equivalent to the approach of [3]. Note that (3.1) and (3.2) have the form of "standard" Lyapunov equations.

Unfortunately, convergence of the proposed algorithm has not been proven and it has not been shown that $A - \tau^{(k)}A\tau_\perp^{(k)}$ and $A - \tau_\perp^{(k)}A\tau^{(k)}$ remain stable at each step. The proposed algorithm was, however, successfully applied to the following model reduction problem considered in [2] and [3]:

$$A = \begin{bmatrix} 0 & 0 & 0 & -150 \\ 1 & 0 & 0 & -245 \\ 0 & 1 & 0 & -113 \\ 0 & 0 & 1 & -19 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$C = [0 \quad 0 \quad 0 \quad 1].$$

Table 1 summarizes the results obtained for $n_r = 3,2,1$. In each case the proposed algorithm converged linearly in less than eight interactions. As pointed out in [3], Wilson's result seems to imply a lack of final convergence. The fact that the second-order modes of this example are rather widely spaced ($\sigma_1 = .126$, $\sigma_2 = .0521$, $\sigma_3 = .0113$, $\sigma_4 = .0028$) appears to account for the near optimality of the results of [3] (see [1]).

4. Relevance to Fixed-Order Dynamic Compensator Design

We now briefly discuss the relevance of the Main Theorem to the problem of fixed-order dynamic compensator design. Given the control system

$$\dot{x} = Ax + Bu + w_1,$$

$$y = Cx + w_2$$

design a fixed-order dynamic compensator

$$\dot{x}_c = A_c x_c + B_c y,$$

$$u = C_c x_c$$

which minimizes the performance criterion

$$J(A_c, B_c, C_c) \triangleq \lim_{t\to\infty} \mathbb{E}[x^T R_1 x + u^T R_2 u].$$

Here $u \in \mathbb{R}^m$, $x_c \in \mathbb{R}^{n_c}$, $n_c \le n$, w_1 is white disturbance noise, w_2 is nonsingular white observation noise, R_1 is nonnegative definite and R_2 is positive definite. Necessary conditions characterizing optimal (A_c, B_c, C_c) have been developed in [4-8] along the same lines as the Main Theorem. These conditions, called the optimal projection equations for fixed-order dynamic compensation, consist of four matrix equations (two modified Riccati equations and two modified Lyapunov equations) coupled by a projection. The modified Riccati equations, not surprisingly, are similar in form to the covariance and cost Riccati equations of LQG theory and the modified Lyapunov equations are similar to the optimal model-reduction equations (2.11) and (2.12). Hence, while the modified Riccati equations govern optimal estimation and optimal control, the additional modified Lyapunov equations characterize "optimal reduction". The important fact that all four equations are coupled supports the view that optimal fixed-order dynamic-compensators cannot in general be designed by means of a stepwise procedure, e.g., by either open-loop model reduction followed by LQG or LQG followed by closed-loop model reduction.

Table 1

Relative Impulse Response Error ([3])

Order n_r	Wilson [2]	Moore [3]	Optimal Model-Reduction Equations
3	-	.001311	.001306
2	.04097	.03938	.03929
1	-	.4321	.4268

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