Proceedings of the 45th IEEE Conference on Decision & Control
Manchester Grand Hyatt Hotel
San Diego, CA, USA, December 13-15, 2006

Discrete-Time Adaptive Command Following and Disturbance Rejection with Unknown Exogenous Dynamics

Jesse B. Hoagg and Dennis S. Bernstein
Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109

1. Introduction

We present an adaptive controller that requires limited model information for stabilization, command following, and disturbance rejection for multi-input, multi-output minimum-phase discrete-time systems. Specifically, the controller requires knowledge of the open-loop system's relative degree and a bound on the first nonzero Markov parameter. Notably, the controller does not require knowledge of the command or disturbance spectrum as long as the command and disturbance signals are generated by Lyapunov-stable linear systems. Thus, the command and disturbance signals are combinations of discrete-time sinusoids and steps. In addition, the controller uses feedback action only and thus does not require a direct measurement of the command or disturbance signals. We prove global asymptotic convergence for command following and disturbance rejection.

2. Problem Formulation

Consider the multi-input, multi-output (MIMO) discrete-time system

$$x(k+1) = Ax(k) + Bu(k) + D_1 w(k), \tag{2.1}$$

$$y(k) = Cx(k) + D_2 w(k), \tag{2.2}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^l$, $u \in \mathbb{R}^l$, and $w \in \mathbb{R}^{l_w}$. Our goal is to design an adaptive output feedback controller under which the performance variable y converges to zero in the presence of the exogenous signal w. Note that w can represent either a command signal to be followed, an external disturbance to be rejected, or both. For example, if $D_1 = 0$ and $D_2 \neq 0$, then the objective is to have the output Cx follow the command signal $-D_2 w$. On the other hand, if $D_1 \neq 0$ and $D_2 = 0$, then the objective is to reject the disturbance w from the performance measurement Cx. The combined command following and disturbance rejection problem is considered when D_1 and D_2 are block matrices.

Next, define d to be the smallest positive integer such that the Markov parameter $\beta_d \triangleq CA^{d-1}B$ is nonzero. We make the following assumptions.

(A1) The triple (A, B, C) is controllable and observable.

(A2) If $\lambda \in \mathbb{C}$ and rank $\begin{bmatrix} A-\lambda I & B \\ C & 0 \end{bmatrix} < n+l$, then $|\lambda| < 1$.

(A3) d is known.

(A4) β_d is nonsingular.

(A5) There exists $\bar{\beta}_d \in \mathbb{R}^{l\times l}$ such that $\bar{\beta}_d\beta_d^{-1} + (\bar{\beta}_d\beta_d^{-1})^{\mathrm{T}} > I$ and $\bar{\beta}_d$ is known.

(A6) There exists an integer $\bar{n}$ such that $n \le \bar{n}$ and $\bar{n}$ is known.

(A7) The performance $y(k)$ is measured and available for feedback.

(A8) The exogenous signal $w(k)$ is generated by

$$x_w(k+1) = A_w x_w(k), \quad w(k) = C_w x_w(k), \tag{2.3}$$

where $x_w \in \mathbb{R}^{n_w}$, and, without loss of generality, A_w is cyclic.

(A9) For all $\lambda \in \operatorname{spec}(A_w)$, $|\lambda| = 1$ and λ is semisimple.

(A10) There exists an integer $\bar{n}_w$ such that $n_w \le \bar{n}_w$ and $\bar{n}_w$ is known.

(A11) A_w and C_w are not known.

(A12) The exogenous signal $w(k)$ is not measured.

Define the transfer function matrices $G_{yu}(z) \triangleq C(zI-A)^{-1}B$ and $G_{yw}(z) \triangleq C(zI-A)^{-1}D_1 + D_2$. Assumption (A1) implies that the McMillan degree of $G_{yu}(z)$ is n. In the single-input single-output (SISO) case, assumption (A1) prevents pole-zero cancellation when forming the transfer function $G_{yu}(z)$, which implies that the order of $G_{yu}(z)$ is n.

Let $G_{yu}(z)$ have a left coprime matrix-fraction description $G_{yu}(z) = \mu(z)^{-1}\nu(z)$, where $\mu(z)$ and $\nu(z)$ are $l\times l$ polynomial matrices. Without loss of generality, we assume that $\mu(z)$ is in column-Hermite form, that is, $\mu(z)$ is upper triangular where each diagonal entry is a monic polynomial whose degree is higher than the degree of all of the remaining entries in its column [1, Theorem 6.3-2]. Thus, we can write $\mu(z) = z^m\mu_0 + z^{m-1}\mu_1 + \cdots + z\mu_{m-1} + \mu_m$, where $m \le n$ and $\mu_0, \ldots, \mu_m \in \mathbb{R}^{l\times l}$ are upper triangular. Note that the leading coefficient matrix μ_0 is not necessarily I_l. However, it can be seen that there exists an $l \times l$ upper-triangular polynomial matrix

$$Q(z) \triangleq \begin{bmatrix} z^{h_{11}} & q_{12}z^{h_{12}} & \cdots & q_{1l}z^{h_{1l}} \\ & z^{h_{22}} & \cdots & q_{2l}z^{h_{2l}} \\ & & \ddots & \vdots \\ & & & z^{h_{ll}} \end{bmatrix}, \tag{2.4}$$

such that the leading term of $\alpha(z) \triangleq Q(z)\mu(z)$ is $z^m I_l$. Thus, we can write $\alpha(z) = z^m I_l + z^{m-1}\alpha_1 + z^{m-2}\alpha_2 + \cdots + z\alpha_{m-1} + \alpha_m$, where, for all $i = 1, \ldots, m$, $\alpha_i \in \mathbb{R}^{l\times l}$. Furthermore, $G_{yu}(z)$ has the matrix-fraction description $G_{yu}(z) = \alpha(z)^{-1}\beta(z)$, where $\beta(z) \triangleq Q(z)\nu(z)$, and we can write $\beta(z) = z^{m-d}\beta_d + z^{m-d-1}\beta_{d+1} + \cdots + z\beta_{m-1} + \beta_m$, where, for all $i = d, \ldots, m$, $\beta_i \in \mathbb{R}^{l\times l}$. Note that $\alpha(z)$ and $\beta(z)$ are not necessarily left coprime. However, since $\mu(z)$ and $\nu(z)$ are left coprime, it follows that $Q(z)$ is the greatest common left divisor of $\alpha(z)$ and $\beta(z)$. Furthermore, since $\det Q(z) = z^{h_{11}+\cdots+h_{ll}}$, the pole-zero cancellation that occurs when forming the transfer function $G_{yu}(z) = \alpha(z)^{-1}\beta(z)$ occurs only at $z = 0$.

Now, assuming that G_{yw} has a matrix-fraction description of the form $G_{yw} = \alpha(z)^{-1}\gamma(z)$, which is not necessarily left coprime, we can write $\gamma(z) = z^m\gamma_0 + z^{m-1}\gamma_1 + \cdots + z\gamma_{m-1} + \gamma_m$, where, for all $i = 0, \ldots, m$, $\gamma_i \in \mathbb{R}^{l\times l_w}$. Therefore, the state-space system (2.1), (2.2) has the time-series representation

$$y(k) = \sum_{i=1}^{m} -\alpha_i y(k-i) + \sum_{i=d}^{m} \beta_i u(k-i) + \sum_{i=0}^{m} \gamma_i w(k-i). \tag{2.5}$$

Definition 2.1. *Let G be a strictly proper transfer function matrix. Then the* normal rank *of G is rank G = rank $G(\lambda)$ for* *almost all* $\lambda \in \mathbb{C}$.

Assumption (A4) implies that, for all sufficiently large $\lambda \in \mathbb{C}$, rank $G_{yu}(\lambda) = l$. To see this, note that $G_{yu}(z) = z^{-d}\beta_d + \sum_{i=d+1}^{\infty} z^{-i} H_i$, where $H_i \triangleq CA^{i-1}B$ is the ith Markov parameter and β_d is nonsingular. Thus, $G_{yu}(z)$ has full normal rank, that is, rank $G_{yu} = l$. Furthermore, rank $G_{yu} = l$ implies that rank $\nu = l$.

Definition 2.2. *Let G be a strictly proper $s \times t$ transfer function matrix with the Smith-McMillan form*

$$U_1(z) \begin{bmatrix} {\scriptstyle q_1(z)/p_1(z)} & & & \\ & \ddots & & \\ & & {\scriptstyle q_r(z)/p_r(z)} & \\ & & & {\scriptstyle 0_{(s-r)\times(t-r)}} \end{bmatrix} U_2(z),$$

where r = rank G, U_1 and U_2 are unimodular matrices, and $q_1, \ldots, q_r, p_1, \ldots, p_r$ are monic polynomials such that, for all $i = 1, \ldots, r$, q_i and p_i are coprime and, for all $i = 1, \ldots, r-1$, p_i divides p_{i+1} and q_{i+1} divides q_i. Then the poles *of G are the roots of p_1, and the* transmission zeros *of G are the roots of q_r.*

Lemma 2.1. *[2, Theorem 2.4] Let G be a strictly proper $s \times t$ transfer function matrix with a left coprime matrix-fraction description $G(z) = P(z)^{-1}Z(z)$. Then $\lambda \in \mathbb{C}$ is a transmission zero of G if and only if rank $Z(\lambda) <$ rank Z. Furthermore, $p \in \mathbb{C}$ is a pole of G if and only if det $P(p) = 0$.*

Assumption (A2) states that the invariant zeros of (A, B, C) are contained in the open unit circle. Since (A, B, C) is a minimal realization of $G_{yu}(z)$, it follows that the invariant zeros of (A, B, C) are exactly the transmission zeros of $G_{yu}(z)$ [3, Theorem 12.10.8]. Therefore, assumption (A2) is equivalent to the assumption that the transmission zeros of $G_{yu}(z)$ are contained in the open unit circle. Since $\mu(z)$ and $\nu(z)$ are left coprime, it follows from Lemma 2.1 that assumption (A2) is equivalent to the assumption that, if $\lambda \in \mathbb{C}$ and rank $\nu(\lambda) <$ rank ν, then $|\lambda| < 1$. Furthermore, since rank $\nu = l$ by assumption (A4), it follows that assumption (A2) implies that, if $\lambda \in \mathbb{C}$ and det $\nu(\lambda) = 0$, then $|\lambda| < 1$. Consequently, since det $\beta(\lambda) =$ det $Q(\lambda)$det $\nu(\lambda) = z^{h_{11}+\cdots+h_{ll}}$det $\nu(\lambda)$, it follows that, if $\lambda \in \mathbb{C}$ and det $\beta(\lambda) = 0$, then $|\lambda| < 1$.

For SISO systems, assumption (A5) specializes to the assumption that sgn β_d is known and an upper bound on the magnitude $|\beta_d|/2$ is known. For MIMO systems, assumption (A5) is a generalization of this SISO assumption. In particular, if β_d is positive definite, then assumption (A5) specializes to the assumption that an upper bound on the magnitude of $\lambda_{\max}(\beta_d)/2$ is known. Similarly, if β_d is negative definite, then assumption (A5) specializes to the assumption that an upper bound on the magnitude of $|\lambda_{\min}(\beta_d)|/2$ is known. More precisely, if β_d is positive definite, then assumption (A5) is satisfied with $\bar{\beta}_d > \frac{\lambda_{\max}(\beta_d)}{2} I_l$, while if β_d is negative definite, then assumption (A5) is satisfied with $\bar{\beta}_d > \frac{|\lambda_{\min}(\beta_d)|}{2} I_l$.

3. Nonminimal State Space Realization

We use a nonminimal state-space realization of the time-series system (2.5) whose state consists entirely of measured information. More specifically, the state consists of past values of the performance $y(k)$ and the control $u(k)$. To construct the nonminimal state-space realization of the time series system (2.5), we introduce the following notation. For a positive integer p, define the nilpotent matrix

$$\mathcal{N}_p \triangleq \begin{bmatrix} 0_{l\times l} & \cdots & 0_{l\times l} & 0_{l\times l} \\ I_l & \cdots & 0_{l\times l} & 0_{l\times l} \\ \vdots & \ddots & \vdots & \vdots \\ 0_{l\times l} & \cdots & I_l & 0_{l\times l} \end{bmatrix} \in \mathbb{R}^{lp\times lp}, \tag{3.1}$$

and define $E_1 \triangleq \begin{bmatrix} I_l \\ 0_{l(p-1)\times l} \end{bmatrix} \in \mathbb{R}^{lp\times l}$, where the dimension p is given by context.

Now, let $n_c \geq m$ and consider the $2ln_c$-order nonminimal state-space realization of (2.5)

$$\phi(k+1) = \mathcal{A}\phi(k) + \mathcal{B}u(k) + \mathcal{D}_1 W(k), \tag{3.2}$$
$$y(k) = \mathcal{C}\phi(k) + \mathcal{D}_2 W(k), \tag{3.3}$$

where

$$\mathcal{A} \triangleq \begin{bmatrix} \mathcal{N}_{n_c} & 0_{ln_c\times ln_c} \\ 0_{ln_c\times ln_c} & \mathcal{N}_{n_c} \end{bmatrix} + E_1\mathcal{C}, \tag{3.4}$$

$$\mathcal{B} \triangleq \begin{bmatrix} 0_{ln_c\times l} \\ E_1 \end{bmatrix}, \quad \mathcal{D}_1 \triangleq E_1\mathcal{D}_2, \tag{3.5}$$

$$\mathcal{C} \triangleq [\, -\alpha_1 \quad -\alpha_2 \quad \cdots \quad -\alpha_m \quad 0_{l\times l(n_c-m)} \quad 0_{l\times l(d-1)} \quad \beta_d \quad \cdots \quad \beta_m \quad 0_{l\times l(n_c-m)} \,], \tag{3.6}$$

$$\mathcal{D}_2 \triangleq [\, \gamma_0 \quad \cdots \quad \gamma_m \,], \tag{3.7}$$

and

$$\phi(k) \triangleq \begin{bmatrix} y(k-1) \\ \vdots \\ y(k-n_c) \\ u(k-1) \\ \vdots \\ u(k-n_c) \end{bmatrix}, \quad W(k) \triangleq \begin{bmatrix} w(k) \\ \vdots \\ w(k-m) \end{bmatrix}. \tag{3.8}$$

The triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is stabilizable and detectable. However, $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is neither controllable nor observable. In particular, $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ has n controllable and observable eigenvalues, while the remaining $2ln_c - n$ eigenvalues are located at 0, and each of these eigenvalues is either uncontrollable or unobservable. More precisely, $(\mathcal{A}, \mathcal{B})$ has $ln_c - n$ uncontrollable eigenvalues at 0, while $(\mathcal{A}, \mathcal{C})$ has ln_c unobservable eigenvalues at 0. Note that in this basis, the state $\phi(k)$ contains only past values of the performance y and the control u.

Now, we consider the time-series controller

$$u(k) = \sum_{i=1}^{n_c} M_i u(k-i) + \sum_{i=1}^{n_c} N_i y(k-i), \tag{3.9}$$

where, for all $i = 1, \ldots, n_c$, $M_i \in \mathbb{R}^{l\times l}$ and $N_i \in \mathbb{R}^{l\times l}$. The control can be written as

$$u(k) = \theta\phi(k), \tag{3.10}$$

where

$$\theta \triangleq [\, N_1 \quad \cdots \quad N_{n_c} \quad M_1 \quad \cdots \quad M_{n_c} \,]. \tag{3.11}$$

The control (3.10), which is dynamic output feedback in terms of y, can be computed by recording and using n_c past values of the performance y and the control u. However, (3.10) is a full-state-feedback control law for the nonminimal state-space system (3.2)-(3.7). The closed-loop system consisting of (3.2)-(3.7) with the linear time-invariant feedback (3.10) is

$$\phi(k+1) = \tilde{\mathcal{A}}\phi(k) + \mathcal{D}_1 W(k), \tag{3.12}$$
$$y(k) = \mathcal{C}\phi(k) + \mathcal{D}_2 W(k), \tag{3.13}$$

where

$$\tilde{\mathcal{A}} \triangleq \mathcal{A} + \mathcal{B}\theta. \tag{3.14}$$

4. Ideal Fixed-Gain Controller

In this section, we prove existence and derive properties of an ideal fixed-gain controller of the form (3.9) for the open-loop system (2.1) and (2.2). This controller is used in subsequent sections to construct an error system for analyzing the closed-loop adaptive system. We stress that the ideal controller is not intended for implementation. An ideal fixed-gain controller consists of two distinct parts: first, a precompensator that cancels the transmission zeros of the open-loop system, and second, a deadbeat internal model controller that operates in feedback on the observable states of the precompensator cascaded with the open-loop system.

First, we demonstrate how to construct an ideal fixed-gain controller. Consider the precompensator

$$u_1(k) = -\sum_{i=1}^{m-d} \beta_d^{-1}\beta_{i+d}u_1(k-i) + u_2(k), \tag{4.1}$$

which has a minimal state-space realization of the form

$$\hat{x}_1(k+1) = \hat{A}_1\hat{x}_1(k) + \hat{B}_1u_2(k), \tag{4.2}$$

$$u_1(k) = \hat{C}_1\hat{x}_1(k) + u_2(k), \tag{4.3}$$

where $\hat{x}_1 \in \mathbb{R}^{\hat{n}_1}$ and $\hat{n}_1$ is the McMillan degree of $\hat{G}_1(z) \triangleq \beta(z)^{-1}z^{m-d}\beta_d$, which is the transfer function from u_2 to u_1. Note that $\hat{n}_1 \leq l(m-d)$. The poles of the precompensator $\hat{G}_1(z)$ are exactly the transmission zeros of the open-loop transfer function $G_{yu}(z)$. Furthermore, assumption (A2) implies that the transmission zeros of $G_{yu}(z)$, and thus the poles of $\hat{G}_1(z)$, are asymptotically stable. Therefore, the cascade $G_{yu}(z)\hat{G}_1(z) = \alpha(z)^{-1}\beta(z)\beta(z)^{-1}z^{m-d}\beta_d = \alpha(z)^{-1}z^{m-d}\beta_d$ has asymptotically stable pole-zero cancellation. Let $n_{\rm o}$ be the McMillan degree of $G_{yu}(z)\hat{G}_1(z)$, and note that $n_{\rm o} \leq lm$.

Define the pseudo-input $e(k) \triangleq u(k) - u_1(k)$, and cascade the precompensator (4.2), (4.3) with the open-loop system (2.1), (2.2) to obtain

$$\begin{bmatrix} x(k+1) \\ \hat{x}_1(k+1) \end{bmatrix} = \begin{bmatrix} A & B\hat{C}_1 \\ 0 & \hat{A}_1 \end{bmatrix}\begin{bmatrix} x(k) \\ \hat{x}_1(k) \end{bmatrix} + \begin{bmatrix} B \\ \hat{B}_1 \end{bmatrix}u_2(k) + \begin{bmatrix} B \\ 0 \end{bmatrix}e(k) + \begin{bmatrix} D_1 \\ 0 \end{bmatrix}w(k), \tag{4.4}$$

$$y(k) = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x(k) \\ \hat{x}_1(k) \end{bmatrix} + D_2w(k). \tag{4.5}$$

Since the poles of $\hat{G}_1(z)$ cancel the transmission zeros of $G_{yu}(z)$, it follows that

$$\left(\begin{bmatrix} A & B\hat{C}_1 \\ 0 & \hat{A}_1 \end{bmatrix}, \begin{bmatrix} B \\ \hat{B}_1 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}\right) \tag{4.6}$$

is not minimal. However, since (A, B) and $(\hat{A}_1, \hat{B}_1)$ are controllable, it follows that (4.6) is controllable. Thus,

$$\left(\begin{bmatrix} A & B\hat{C}_1 \\ 0 & \hat{A}_1 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}\right) \tag{4.7}$$

is not observable. In fact, it follows from the pole-zero cancellations between $\hat{G}_1(z)$ and $G_{yu}(z)$ that the unobservable modes of (4.7) are exactly the poles of $\hat{G}_1(z)$, all of which are asymptotically stable.

Next, let $\hat{n}_2 \geq n_{\rm o} + 2ln_w$, let $\hat{x}_2 \in \mathbb{R}^{\hat{n}_2}$, and let

$$\hat{x}_2(k+1) = \hat{A}_2\hat{x}_2(k) + \hat{B}_2y(k), \tag{4.8}$$

$$u_2(k) = \hat{C}_2\hat{x}_2(k), \tag{4.9}$$

be an internal model controller for the observable states of (4.4) and (4.5) that guarantees perfect command following and disturbance rejection in finite time, that is, (4.8) and (4.9) is a deadbeat internal model controller. Thus, an ideal fixed-gain controller consists of the precompensator (4.2), (4.3) and a deadbeat internal model controller (4.8), (4.9). Define the transfer function matrix of the deadbeat internal model controller (4.8) and (4.9) by $\hat{G}_2(z) \triangleq \hat{C}_2(zI - \hat{A}_2)^{-1}\hat{B}_2$. The following result guarantees the existence of an ideal fixed-gain controller and provides several useful properties associated with this controller.

Theorem 4.1. *Consider the closed-loop system (3.12), (3.13) and recall the definitions for $\tilde{\mathcal{A}}$, $\mathcal{B}$, and $\mathcal{C}$ given by (3.4), (3.6), and (3.14). Furthermore, let $n_{\rm c} \geq n_{\rm o} + 2ln_w + m - d$. Then there exists a linear output-feedback controller (3.9) such that the following statements hold.*

(i) For all initial conditions $\phi(0)$ and $x_w(0)$ and all integers $k \geq n_{\rm o} + n_{\rm c} + d - m$, $y(k) = 0$.
(ii) $\tilde{\mathcal{A}}$ is asymptotically stable.
(iii) For $i = 1, 2, 3, \ldots$,

$$\mathcal{C}\tilde{\mathcal{A}}^{i-1}\mathcal{B} = \begin{cases} \beta_d, & i = d, \\ 0, & i \neq d. \end{cases} \tag{4.10}$$

Proof. We show that a times-series representation of the fixed-gain controller (4.2), (4.3), (4.8), and (4.9) exists and satisfies *(i)-(iii)*.

First, consider the cascade (4.4),(4.5), and recall that (4.6) is controllable but not observable. Furthermore, the unobservable modes of (4.7) are precisely the poles of $\hat{G}_1(z)$, all of which are asymptotically stable. Therefore, it follows from the Kalman decomposition that there exists a nonsingular matrix $T \in \mathbb{R}^{(n+\hat{n}_1)\times(n+\hat{n}_1)}$ such that $\begin{bmatrix} A_{\rm o} & 0 \\ A_{21} & A_{\bar{\rm o}} \end{bmatrix} = T\begin{bmatrix} A & B\hat{C}_1 \\ 0 & \hat{A}_1 \end{bmatrix}T^{-1}$, and $\begin{bmatrix} C_{\rm o} & 0 \end{bmatrix} = \begin{bmatrix} C & 0 \end{bmatrix}T^{-1}$, where $(A_{\rm o}, C_{\rm o})$ is observable and $A_{\bar{\rm o}}$ is asymptotically stable.

Now, defining $\begin{bmatrix} x_{\rm o}(k) \\ x_{\bar{\rm o}}(k) \end{bmatrix} \triangleq T\begin{bmatrix} x(k) \\ \hat{x}_1(k) \end{bmatrix}$ and applying this change of basis to the cascade (4.4) and (4.5) yields

$$\begin{bmatrix} x_{\rm o}(k+1) \\ x_{\bar{\rm o}}(k+1) \end{bmatrix} = \begin{bmatrix} A_{\rm o} & 0 \\ A_{21} & A_{\bar{\rm o}} \end{bmatrix}\begin{bmatrix} x_{\rm o}(k) \\ x_{\bar{\rm o}}(k) \end{bmatrix} + \begin{bmatrix} B_{\rm o} \\ B_{\bar{\rm o}} \end{bmatrix}u_2(k) + \begin{bmatrix} B_{e,{\rm o}} \\ B_{e,\bar{\rm o}} \end{bmatrix}e(k) + \begin{bmatrix} D_{1,{\rm o}} \\ D_{1,\bar{\rm o}} \end{bmatrix}w(k), \tag{4.11}$$

$$y(k) = \begin{bmatrix} C_{\rm o} & 0 \end{bmatrix}\begin{bmatrix} x_{\rm o}(k) \\ x_{\bar{\rm o}}(k) \end{bmatrix} + D_2w(k), \tag{4.12}$$

where $x_{\rm o} \in \mathbb{R}^{n_{\rm o}}$ and $\begin{bmatrix} B_{\rm o} \\ B_{\bar{\rm o}} \end{bmatrix} = T\begin{bmatrix} B \\ \hat{B}_1 \end{bmatrix}$, $\begin{bmatrix} B_{e,{\rm o}} \\ B_{e,\bar{\rm o}} \end{bmatrix} = T\begin{bmatrix} B \\ 0 \end{bmatrix}$, and $\begin{bmatrix} D_{1,{\rm o}} \\ D_{1,\bar{\rm o}} \end{bmatrix} = T\begin{bmatrix} D_1 \\ 0 \end{bmatrix}$. Note that $(A_{\rm o}, B_{\rm o}, C_{\rm o})$ is a minimal realization of the transfer function matrix $G_{\rm o}(z) \triangleq C_{\rm o}[zI - A_{\rm o}]^{-1}B_{\rm o} = G_{yu}(z)\hat{G}_1(z) = \alpha(z)^{-1}z^{m-d}\beta_d$.

Next, we consider a deadbeat internal model controller of the form (4.8), (4.9) designed for the observable subsystem of (4.11), (4.12) given by

$$x_{\rm o}(k+1) = A_{\rm o}x_{\rm o}(k) + B_{\rm o}u_2(k) + B_{e,{\rm o}}e(k) + D_{1,{\rm o}}w(k), \tag{4.13}$$

$$y(k) = C_{\rm o}x_{\rm o}(k) + D_2w(k). \tag{4.14}$$

The invariant zeros of $(A_{\rm o}, B_{\rm o}, C_{\rm o})$ are located at the origin and thus do not coincide with the eigenvalues of A_w by assumption (A9). Since, in addition, $(A_{\rm o}, B_{\rm o}, C_{\rm o})$ is minimal and the dimension of y equals the dimension of u, it follows from [4, Corollary 4.1] that, for all $\hat{n}_2 \geq n_{\rm o} + 2ln_w$, there exists a linear discrete-time controller (4.8) and (4.9) such that the closed-loop dynamics matrix of (4.8), (4.9), (4.13), and (4.14), which represents the feedback interconnection of $G_{\rm o}$ and $\hat{G}_2$ and given by

$$\left[\begin{array}{cc} A_{\rm o} & B_{\rm o}\hat{C}_2 \\ \hat{B}_2 C_{\rm o} & \hat{A}_2 \end{array}\right], \tag{4.15}$$

is nilpotent, and such that, with $e(k) \equiv 0$, for all initial conditions $(x_{\rm o}(0), x_{\bar{\rm o}}(0), \hat{x}_2(0), x_w(0))$ and all integers $k \geq n_{\rm o}+\hat{n}_2$, $y(k) = 0$. The closed-loop system (4.8), (4.9), (4.11), and (4.12) is

$$\left[\begin{array}{c} x_{\rm o}(k+1) \\ \hat{x}_2(k+1) \\ x_{\bar{\rm o}}(k+1) \end{array}\right] = \left[\begin{array}{ccc} A_{\rm o} & B_{\rm o}\hat{C}_2 & 0 \\ \hat{B}_2 C_{\rm o} & \hat{A}_2 & 0 \\ A_{21} & B_{\bar{\rm o}}\hat{C}_2 & A_{\bar{\rm o}} \end{array}\right] \left[\begin{array}{c} x_{\rm o}(k) \\ \hat{x}_2(k) \\ x_{\bar{\rm o}}(k) \end{array}\right] + \left[\begin{array}{c} B_{e,{\rm o}} \\ 0 \\ B_{e,\bar{\rm o}} \end{array}\right] e(k) + \left[\begin{array}{c} D_{1,{\rm o}} \\ \hat{B}_2 D_2 \\ D_{1,\bar{\rm o}} \end{array}\right] w(k), \tag{4.16}$$

$$y(k) = \left[\begin{array}{ccc} C_{\rm o} & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_{\rm o}(k) \\ \hat{x}_2(k) \\ x_{\bar{\rm o}}(k) \end{array}\right] + D_2 w(k). \tag{4.17}$$

Since (4.15) is nilpotent and $A_{\bar{\rm o}}$ is asymptotically stable, it follows that

$$\left[\begin{array}{ccc} A_{\rm o} & B_{\rm o}\hat{C}_2 & 0 \\ \hat{B}_2 C_{\rm o} & \hat{A}_2 & 0 \\ A_{12} & B_{\bar{\rm o}}\hat{C}_2 & A_{\bar{\rm o}} \end{array}\right] \tag{4.18}$$

is asymptotically stable.

To show *(i)*, we write the transfer function matrix of (4.8), (4.9) as $\hat{G}_2(z) = \hat{M}(z)^{-1}\hat{N}(z)$, where $\hat{M}(z) = z^{\hat{n}_2}I_l + z^{\hat{n}_2-1}\hat{M}_1 + z^{\hat{n}_2-2}\hat{M}_2 + \cdots + z\hat{M}_{\hat{n}_2-1} + \hat{M}_{\hat{n}_2}$, and $\hat{N}(z) = z^{\hat{n}_2-1}\hat{N}_1 + z^{\hat{n}_2-2}\hat{N}_2 + \cdots + z\hat{N}_{\hat{n}_2-1} + \hat{N}_{\hat{n}_2}$, where, for $i = 1, \ldots, \hat{n}_2$, $\hat{M}_i \in \mathbb{R}^{l\times l}$ and $\hat{N}_i \in \mathbb{R}^{l\times l}$. Therefore, (4.8), (4.9) has the time series representation

$$u_2(k) = -\sum_{i=1}^{\hat{n}_2} \hat{M}_i u_2(k-i) + \sum_{i=1}^{\hat{n}_2} \hat{N}_i y(k-i). \tag{4.19}$$

The ideal fixed-gain controller, which consists of the precompensator (4.1) and the deadbeat internal model controller (4.19), is given by (3.9) with $u(k) = u_1(k)$ and $n_{\rm c} \triangleq \hat{n}_2 + m - d$, where, for $i = 1, 2, \ldots, n_{\rm c}$,

$$M_i \triangleq -\beta_d^{-1}\beta_{d+i} - \sum_{j=1}^{i} \hat{M}_j \beta_d^{-1}\beta_{d+i-j}, \tag{4.20}$$

$$N_i \triangleq \hat{N}_i, \tag{4.21}$$

where, for all $i > m$, $\beta_i = 0$, and, for all $i > \hat{n}_2$, $\hat{M}_i = \hat{N}_i = 0$.

Next, consider the $2ln_{\rm c}$-order nonminimal state-space realization of the controller (3.9), (4.20), and (4.21) given by

$$\phi_{\rm c}(k+1) = \mathcal{A}_{\rm c}\phi_{\rm c}(k) + \mathcal{B}_{\rm c}y(k), \tag{4.22}$$

$$u_1(k) = \mathcal{C}_{\rm c}\phi_{\rm c}(k), \tag{4.23}$$

where

$$\mathcal{A}_{\rm c} \triangleq \left[\begin{array}{cc} \mathcal{N}_{n_{\rm c}} & 0_{ln_{\rm c}\times ln_{\rm c}} \\ 0_{ln_{\rm c}\times ln_{\rm c}} & \mathcal{N}_{n_{\rm c}} \end{array}\right] + \left[\begin{array}{c} 0_{ln_{\rm c}\times ln_{\rm c}} \\ E_1 \end{array}\right]\theta, \tag{4.24}$$

$$\mathcal{B}_{\rm c} \triangleq E_1, \quad \mathcal{C}_{\rm c} \triangleq \theta, \tag{4.25}$$

and

$$\phi_{\rm c}(k) \triangleq \left[\begin{array}{c} y(k-1) \\ \vdots \\ y(k-n_{\rm c}) \\ u_1(k-1) \\ \vdots \\ u_1(k-n_{\rm c}) \end{array}\right]. \tag{4.26}$$

Note that $\mathcal{A}_{\rm c} = \mathcal{A} + \mathcal{B}\mathcal{C}_{\rm c} - \mathcal{B}_{\rm c}\mathcal{C}$.

Therefore, the closed-loop system (3.2)-(3.7) and (4.22)-(4.25) is

$$\left[\begin{array}{c} \phi(k+1) \\ \phi_{\rm c}(k+1) \end{array}\right] = \left[\begin{array}{cc} \mathcal{A} & \mathcal{B}\mathcal{C}_{\rm c} \\ \mathcal{B}_{\rm c}\mathcal{C} & \mathcal{A}_{\rm c} \end{array}\right] \left[\begin{array}{c} \phi(k) \\ \phi_{\rm c}(k) \end{array}\right] + \left[\begin{array}{c} \mathcal{B} \\ 0 \end{array}\right] e(k) + \left[\begin{array}{c} \mathcal{D}_1 \\ \mathcal{B}_{\rm c}\mathcal{D}_2 \end{array}\right] W(k), \tag{4.27}$$

$$y(k) = \left[\begin{array}{cc} \mathcal{C} & 0 \end{array}\right] \left[\begin{array}{c} \phi(k) \\ \phi_{\rm c}(k) \end{array}\right] + \mathcal{D}_2 W(k). \tag{4.28}$$

The closed-loop system (4.27) and (4.28) is a nonminimal representation of the closed-loop system (4.16) and (4.17). Furthermore, every unobservable or uncontrollable mode of (4.27) and (4.28) is located at zero. Thus, the spectrum of

$$\left[\begin{array}{cc} \mathcal{A} & \mathcal{B}\mathcal{C}_{\rm c} \\ \mathcal{B}_{\rm c}\mathcal{C} & \mathcal{A}_{\rm c} \end{array}\right] \tag{4.29}$$

consists of the eigenvalues of (4.18) as well as $4ln_{\rm c} - n - \hat{n}_1 - \hat{n}_2$ eigenvalues located at 0. Therefore, since (4.18) is asymptotically stable, it follows that (4.29) is asymptotically stable. Furthermore, since (4.27) and (4.28) is a nonminimal representation of (4.16) and (4.17), it follows that, with $e(k) \equiv 0$, for all initial conditions $(\phi(0), x_w(0))$ and all integers $k \geq n_{\rm o} + \hat{n}_2 = n_{\rm o} + n_{\rm c} + d - m$, $y(k) = 0$. Thus, we have verified *(i)*.

To show *(ii)*, consider the change of basis

$$\left[\begin{array}{cc} \tilde{\mathcal{A}} & \mathcal{B}\mathcal{C}_{\rm c} \\ 0 & \mathcal{A}_{\rm nil} \end{array}\right] = \left[\begin{array}{cc} I & 0 \\ -I & I \end{array}\right] \left[\begin{array}{cc} \mathcal{A} & \mathcal{B}\mathcal{C}_{\rm c} \\ \mathcal{B}_{\rm c}\mathcal{C} & \mathcal{A}_{\rm c} \end{array}\right] \left[\begin{array}{cc} I & 0 \\ I & I \end{array}\right], \tag{4.30}$$

$$\left[\begin{array}{c} \mathcal{B} \\ -\mathcal{B} \end{array}\right] = \left[\begin{array}{cc} I & 0 \\ -I & I \end{array}\right] \left[\begin{array}{c} \mathcal{B} \\ 0 \end{array}\right], \tag{4.31}$$

$$\left[\begin{array}{cc} \mathcal{C} & 0 \end{array}\right] = \left[\begin{array}{cc} \mathcal{C} & 0 \end{array}\right] \left[\begin{array}{cc} I & 0 \\ I & I \end{array}\right], \tag{4.32}$$

where $\mathcal{A}_{\rm nil} \triangleq \left[\begin{array}{cc} \mathcal{N}_{n_{\rm c}} & 0_{ln_{\rm c}\times ln_{\rm c}} \\ 0_{ln_{\rm c}\times ln_{\rm c}} & \mathcal{N}_{n_{\rm c}} \end{array}\right]$ is nilpotent. Since (4.29) is asymptotically stable and $\mathcal{A}_{\rm nil}$ is nilpotent, it follows from (4.30) that $\tilde{\mathcal{A}}$ is asymptotically stable, verifying *(ii)*.

To show *(iii)*, we compute the closed-loop Markov parameters from the pseudo-input $e(k)$ to the performance $y(k)$ using a state-space realization of the closed-loop system and a transfer function matrix representation of the closed-loop system. First, consider the nonminimal state-space realization (4.27) and (4.28). For $i = 1, 2, \ldots$, define the Markov parameters

$$\begin{aligned} H_i &\triangleq \left[\begin{array}{cc} \mathcal{C} & 0 \end{array}\right] \left[\begin{array}{cc} \mathcal{A} & \mathcal{B}\mathcal{C}_{\rm c} \\ \mathcal{B}_{\rm c}\mathcal{C} & \mathcal{A}_{\rm c} \end{array}\right]^{i-1} \left[\begin{array}{c} \mathcal{B} \\ 0 \end{array}\right] \\ &= \left[\begin{array}{cc} \mathcal{C} & 0 \end{array}\right] \left[\begin{array}{cc} \tilde{\mathcal{A}} & \mathcal{B}\mathcal{C}_{\rm c} \\ 0 & \mathcal{A}_{\rm nil} \end{array}\right]^{i-1} \left[\begin{array}{c} \mathcal{B} \\ -\mathcal{B} \end{array}\right] \\ &= \mathcal{C}\tilde{\mathcal{A}}^{i-1}\mathcal{B} + \sum_{j=1}^{i-1} -\mathcal{C}\tilde{\mathcal{A}}^{j-1}\mathcal{B}M_{i-j}, \end{aligned} \tag{4.33}$$

where $M_i = 0$ for all $i > n_c$.

Next, consider the transfer function matrix representation of the open-loop system

$$\begin{aligned} y &= G_{yu}(z)u + G_{yw}(z)w \\ &= G_{yu}(z)u_1 + G_{yu}(z)e + G_{yw}w \\ &= G_{yu}(z)\hat{G}_1(z)\hat{G}_2(z)y + G_{yu}(z)e + G_{yw}w, \end{aligned} \quad (4.34)$$

which implies that the closed-loop system is $y = \tilde{G}_{ye}e + \tilde{G}_{yw}$, where $\tilde{G}_{ye} \triangleq [I_l - G_{yu}(z)\hat{G}_1(z)\hat{G}_2(z)]^{-1}G_{yu} = [I_l - \alpha(z)^{-1}z^{m-d}\beta_d\hat{M}(z)^{-1}\hat{N}(z)]^{-1}\alpha(z)^{-1}\beta(z) = [\alpha(z)-z^{m-d}\beta_d\hat{M}(z)^{-1}\hat{N}(z)]^{-1}\beta(z) = \tilde{D}(z)^{-1}\hat{M}(z)\beta_d^{-1}\beta(z)$, $\tilde{G}_{yw} \triangleq [I_l - G_{yu}(z)\hat{G}_1(z)\hat{G}_2(z)]^{-1}G_{yw} = \tilde{D}(z)^{-1}\hat{M}(z)\beta_d^{-1}\gamma(z)$, and $\tilde{D}(z) \triangleq \hat{M}(z)\beta_d^{-1}\alpha(z) - z^{m-d}\hat{N}(z)$. Notice that $\tilde{D}(z)$ can be written as

$$\tilde{D}(z) = z^{m+\hat{n}_2}\beta_d^{-1} + z^{m+\hat{n}_2-1}\tilde{D}_1 + \cdots + \tilde{D}_{m+\hat{n}_2}, \quad (4.35)$$

where, for $i = 1, 2, \ldots, m + \hat{n}_2$, $\tilde{D}_i \in \mathbb{R}^{l\times l}$. Since the closed-loop dynamics (4.15) are nilpotent, it follows that the poles of $\tilde{G}_{ye}$ and $\tilde{G}_{yw}$ are located at zero; in particular, det $\tilde{D}(z) = z^{l(m+\hat{n}_2)}$det β_d^{-1}. In fact, it follows from (4.35) that the coefficients of the deadbeat controller $\hat{M}(z)^{-1}\hat{N}(z)$ can be chosen so that $\tilde{D}_1 = \cdots = \tilde{D}_{m+\hat{n}_2} = 0$, and thus

$$\tilde{G}_{ye}(z) = \left[z^{m+\hat{n}_2}\beta_d^{-1}\right]^{-1}\tilde{N}(z) = z^{-m-\hat{n}_2}\beta_d\tilde{N}(z), \quad (4.36)$$

where $\tilde{N}(z) \triangleq \hat{M}(z)\beta_d^{-1}\beta(z) = z^{m+\hat{n}_2}\tilde{N}_0 + z^{m+\hat{n}_2-1}\tilde{N}_1 + \cdots + z\tilde{N}_{m+\hat{n}_2-1} + N_{m+\hat{n}_2}$, and

$$\tilde{N}_i = \begin{cases} 0, & 0 \le i < d, \\ I_l, & i = d, \\ \beta_d^{-1}\beta_i + \sum_{j=1}^{i-d}\hat{M}_j\beta_d^{-1}\beta_{i-j}, & d < i \le m + \hat{n}_2. \end{cases}$$

Therefore, it follows from (4.20) that

$$\tilde{N}_i = \begin{cases} 0, & 0 \le i < d, \\ I_l, & i = d, \\ -M_{i-d}, & d < i \le m + \hat{n}_2. \end{cases}$$

It follows from (4.36) that the Markov parameters from the pseudo-input e to the performance y are $H_i = \beta_d\tilde{N}_i$ for $i = 1, 2, \ldots, m + \hat{n}_2$ and $H_i = 0$ for $i > m + \hat{n}_2$, which implies

$$H_i = \begin{cases} 0, & 0 \le i < d, \\ \beta_d, & i = d, \\ -\beta_d M_{i-d}, & d < i \le m + \hat{n}_2, \\ 0, & i > m + \hat{n}_2. \end{cases} \quad (4.37)$$

Then property *(iii)* follows from comparing the expressions for H_i given by (4.33) and (4.37).

□

5. Error System

We now construct an error system using the ideal fixed-gain controller and a controller whose gains are updated by an adaptive law. Let

$$n_c \ge 2\bar{n} + 2l\bar{n}_w - d \ge n_o + m + 2ln_w - d, \quad (5.1)$$

and let $u^*(k) = \sum_{i=1}^{n_c} M_i^* u^*(k-i) + \sum_{i=1}^{n_c} N_i^* y^*(k-i)$ be the ideal fixed-gain controller given by Theorem 4.1, where y^* is the output of the ideal system with the dynamics (3.2)-(3.7) and the control $u^*(k)$. Note that the lower bound on n_c given by (5.1) is known by assumptions (A3), (A6), and (A10). The ideal controller can be expressed as

$$u^*(k) = \theta^*\phi^*(k), \quad (5.2)$$

where $\theta^* \triangleq \begin{bmatrix} N_1^* & \cdots & N_{n_c}^* & M_1^* & \cdots & M_{n_c}^* \end{bmatrix}$ and

$$\phi^*(k) \triangleq \begin{bmatrix} y^*(k-1) \\ \vdots \\ y^*(k-n_c) \\ u^*(k-1) \\ \vdots \\ u^*(k-n_c) \end{bmatrix}. \quad (5.3)$$

The closed-loop system consisting of (3.2)-(3.7) with the ideal feedback (5.2) is

$$\phi^*(k+1) = \tilde{\mathcal{A}}\phi^*(k) + \mathcal{D}_1 W(k), \quad (5.4)$$

$$y^*(k) = \mathcal{C}\phi^*(k) + \mathcal{D}_2 W(k), \quad (5.5)$$

where $\tilde{\mathcal{A}} \triangleq \mathcal{A} + B\theta^*$ is asymptotically stable.

Next, consider the controller

$$u(k) = \sum_{i=1}^{n_c} M_i(k)u(k-i) + \sum_{i=1}^{n_c} N_i(k)y(k-i), \quad (5.6)$$

where, for all $i = 1, \ldots, n_c$, $M_i : \mathbb{N} \to \mathbb{R}^{l\times l}$ and $N_i : \mathbb{N} \to \mathbb{R}^{l\times l}$ are given by the adaptive law presented in the following section. The control can be expressed as

$$u(k) = \theta(k)\phi(k), \quad (5.7)$$

where

$$\theta(k) \triangleq \begin{bmatrix} N_1(k) & \cdots & N_{n_c}(k) & M_1(k) & \cdots & M_{n_c}(k) \end{bmatrix}. \quad (5.8)$$

Therefore, the closed-loop system consisting of (3.2)-(3.7) with the time-varying feedback (5.7) is given by

$$\phi(k+1) = \tilde{\mathcal{A}}\phi(k) + \mathcal{B}\tilde{\theta}(k)\phi(k) + \mathcal{D}_1 W(k), \quad (5.9)$$

$$y(k) = \mathcal{C}\phi(k) + \mathcal{D}_2 W(k), \quad (5.10)$$

where $\tilde{\theta}(k) \triangleq \theta(k) - \theta^*$.

Now, we construct an error system from the ideal closed-loop system (5.4),(5.5) and the closed-loop system (5.9),(5.10). Define the error state $\tilde{\phi}(k) \triangleq \phi(k) - \phi^*(k)$, and subtract (5.4),(5.5) from (5.9),(5.10) to obtain

$$\tilde{\phi}(k+1) = \tilde{\mathcal{A}}\tilde{\phi}(k) + \mathcal{B}\tilde{\theta}(k)\phi(k), \quad (5.11)$$

$$\tilde{y}(k) = \mathcal{C}\tilde{\phi}(k), \quad (5.12)$$

where $\tilde{y}(k) \triangleq y(k) - y^*(k)$.

The following lemma shows that $y(k)$ is linear in the estimation error $\tilde{\theta}$. This lemma is essential for developing the adaptive law and analyzing the stability of the error system.

Lemma 5.1. *Consider the error system (5.11) and (5.12). For all integers $k \ge n_o + n_c + d - m$,*

$$\tilde{y}(k) = y(k) = \beta_d\tilde{\theta}(k-d)\phi(k-d). \quad (5.13)$$

Proof. Substituting (5.11) into (5.12) yields

$$\tilde{y}(k) \triangleq \sum_{i=1}^{\infty} \mathcal{C}\tilde{\mathcal{A}}^{i-1}\mathcal{B}\tilde{\theta}(k-i)\phi(k-i). \quad (5.14)$$

It now follows from *(iii)* of Theorem 4.1 and (5.14) that $\tilde{y}(k) = \beta_d\tilde{\theta}(k-d)\phi(k-d)$. Furthermore, it follows from *(i)* of Theorem 4.1 that, for all $k \ge n_o + n_c + d - m$, $y^*(k) = 0$, that is, $\tilde{y}(k) = y(k)$. □

6. Adaptive Controller and Stability Analysis

We now present the adaptive law for the controller (5.7), (5.8) and analyze the properties of the closed-loop error system. Consider the adaptive law

$$\theta(k+1) = \theta(k-d) - \frac{\bar{\beta}_d^{\mathrm{T}} y(k)\phi^{\mathrm{T}}(k-d)}{1+\eta(k)\phi^{\mathrm{T}}(k-d)\phi(k-d)}, \quad (6.1)$$

where $\eta : \mathbb{N} \to (0,\infty)$ satisfies the conditions

(C1) $\eta(k)$ is bounded.

(C2) For all $k \in \mathbb{N}$, $\eta(k) \geq \lambda_{\max}(\bar{\beta}_d\bar{\beta}_d^{\mathrm{T}})$.

Subtracting θ^* from (6.1) yields the estimator-error update equation

$$\tilde{\theta}(k+1) = \tilde{\theta}(k-d) - \frac{\bar{\beta}_d^{\mathrm{T}} y(k)\phi^{\mathrm{T}}(k-d)}{1+\eta(k)\phi^{\mathrm{T}}(k-d)\phi(k-d)}. \quad (6.2)$$

Now, we present our main result on adaptive stabilization, command following, and disturbance rejection.

Theorem 6.1. *Consider the open-loop system (2.1) and (2.2) satisfying assumptions (A1)-(A12) and the adaptive feedback controller (5.1), (5.7), (5.8), and (6.1) satisfying conditions (C1) and (C2), with the error dynamics (6.2). Then, for all initial conditions $x(0)$ and $\theta(0)$, $\theta(k)$ is bounded, $u(k)$ is bounded, and $\lim_{k\to\infty} y(k) = 0$.*

Proof. Consider the positive-definite, radially unbounded Lyapunov-like function $V(\tilde{\theta}(k),\ldots,\tilde{\theta}(k-d)) \triangleq \sum_{i=0}^{d} \operatorname{tr} \tilde{\theta}^{\mathrm{T}}(k-i)\tilde{\theta}(k-i)$, and the Lyapunov-like difference $\Delta V(k) \triangleq V(\tilde{\theta}(k+1),\ldots,\tilde{\theta}(k-d+1)) - V(\tilde{\theta}(k),\ldots,\tilde{\theta}(k-d))$. Evaluating $\Delta V(k)$ along the trajectories of the error equation (6.2) yields

$$\begin{aligned}
\Delta V(k) &= \operatorname{tr}\left[\tilde{\theta}^{\mathrm{T}}(k+1)\tilde{\theta}(k+1) - \tilde{\theta}^{\mathrm{T}}(k-d)\tilde{\theta}(k-d)\right] \\
&= -\frac{\phi^{\mathrm{T}}(k-d)\tilde{\theta}^{\mathrm{T}}(k-d)\bar{\beta}_d^{\mathrm{T}} y(k)}{1+\eta(k)\phi^{\mathrm{T}}(k-d)\phi(k-d)} \\
&\quad - \frac{y^{\mathrm{T}}(k)\bar{\beta}_d\tilde{\theta}(k-d)\phi(k-d)}{1+\eta(k)\phi^{\mathrm{T}}(k-d)\phi(k-d)} \\
&\quad + \frac{y^{\mathrm{T}}(k)\bar{\beta}_d\bar{\beta}_d^{\mathrm{T}} y(k)\phi^{\mathrm{T}}(k-d)\phi(k-d)}{(1+\eta(k)\phi^{\mathrm{T}}(k-d)\phi(k-d))^2} \\
&\leq -\frac{\phi^{\mathrm{T}}(k-d)\tilde{\theta}^{\mathrm{T}}(k-d)\bar{\beta}_d^{\mathrm{T}} y(k)}{1+\eta(k)\phi^{\mathrm{T}}(k-d)\phi(k-d)} \\
&\quad - \frac{y^{\mathrm{T}}(k)\bar{\beta}_d\tilde{\theta}(k-d)\phi(k-d)}{1+\eta(k)\phi^{\mathrm{T}}(k-d)\phi(k-d)} \\
&\quad + \frac{y^{\mathrm{T}}(k)y(k)}{1+\eta(k)\phi^{\mathrm{T}}(k-d)\phi(k-d)}. \quad (6.3)
\end{aligned}$$

Note that the last inequality follows from condition (C2). Next, Lemma 5.1 implies that, for all $k \geq k_{\mathrm{s}} \triangleq n_{\mathrm{o}} + n_{\mathrm{c}} + d - m$,

$$\begin{aligned}
\Delta V(k) \leq &-\frac{y^{\mathrm{T}}(k)\beta_d^{-\mathrm{T}}\bar{\beta}_d^{\mathrm{T}} y(k)}{1+\eta(k)\phi^{\mathrm{T}}(k-d)\phi(k-d)} \\
&- \frac{y^{\mathrm{T}}(k)\bar{\beta}_d\beta_d^{-1} y(k)}{1+\eta(k)\phi^{\mathrm{T}}(k-d)\phi(k-d)} \\
&+ \frac{y^{\mathrm{T}}(k)y(k)}{1+\eta(k)\phi^{\mathrm{T}}(k-d)\phi(k-d)}. \quad (6.4)
\end{aligned}$$

Since, by assumption (A5), $\bar{\beta}_d\beta_d^{-1} + \beta_d^{-T}\bar{\beta}_d^{\mathrm{T}} > I$, it follows that, for all $k \geq k_{\mathrm{s}}$,

$$\Delta V(k) \leq -W(y(k),\phi(k-d)), \quad (6.5)$$

where $W(y(k),\phi(k-d)) \triangleq \sigma\frac{||y(k)||^2}{1+\eta(k)\phi^{\mathrm{T}}(k-d)\phi(k-d)}$, $\sigma \triangleq \lambda_{\min}(\bar{\beta}_d\beta_d^{-1} + \beta_d^{-T}\bar{\beta}_d^{\mathrm{T}} - I) > 0$, and $||\cdot||$ denotes the Euclidean norm.

To show that $\tilde{\theta}(k)$ is bounded, summing (6.5) from k_{s} to k, where $k \geq k_{\mathrm{s}}$, yields $0 \leq V(\tilde{\theta}(k),\ldots,\tilde{\theta}(k-d)) \leq -\sum_{j=k_{\mathrm{s}}}^{k} W(y(j),\phi(j-d)) + V(\tilde{\theta}(k_{\mathrm{s}}),\ldots,\tilde{\theta}(k_{\mathrm{s}}-d)) \leq V(\tilde{\theta}(k_{\mathrm{s}}),\ldots,\tilde{\theta}(k_{\mathrm{s}}-d))$. Thus, V is bounded. Since $V(\tilde{\theta}(k),\ldots,\tilde{\theta}(k-d))$ is positive definite and radially unbounded, it follows that $\tilde{\theta}(k)$ is bounded. Thus, $\theta(k) = \tilde{\theta}(k) + \theta^*$ is bounded.

To show that $u(k)$ is bounded and $\lim_{k\to\infty} y(k) = 0$, we use the Key Technical Lemma [5, p. 181]. First, we show that $\lim_{k\to\infty} W(y(k),\phi(k-d)) = 0$. Since V is positive definite, using (6.5) implies that the limit

$$\begin{aligned}
0 &\leq \lim_{k\to\infty}\sum_{j=k_{\mathrm{s}}}^{k} W(y(j),\phi(j-d)) \\
&\leq -\lim_{k\to\infty}\sum_{j=k_{\mathrm{s}}}^{k}\Delta V(j) \\
&= V(\tilde{\theta}(k_{\mathrm{s}}),\ldots,\tilde{\theta}(k_{\mathrm{s}}-d)) - \lim_{k\to\infty} V(\tilde{\theta}(k),\ldots,\tilde{\theta}(k-d)) \\
&\leq V(\tilde{\theta}(k_{\mathrm{s}}),\ldots,\tilde{\theta}(k_{\mathrm{s}}-d)) \quad (6.6)
\end{aligned}$$

exists. Thus, $\lim_{k\to\infty} W(y(k),\phi(k-d)) = 0$.

Next, we show that $\phi(k-d)$ is linearly bounded by the performance $y(k)$. The triangle inequality implies that

$$\begin{aligned}
||\phi(k-d)|| &\leq \sum_{j=1}^{n_{\mathrm{c}}}||y(k-d-j)|| + \sum_{j=1}^{n_{\mathrm{c}}}||u(k-d-j)|| \\
&\leq n_{\mathrm{c}}\max_{0\leq\tau\leq k}||y(\tau)|| + \sum_{j=1}^{n_{\mathrm{c}}}\sum_{i=1}^{l}\left|\left|e_i^{\mathrm{T}} u(k-d-j)\right|\right|,
\end{aligned} \quad (6.7)$$

where e_i is the ith column of I_l. Next, assumption (A2) implies that the invariant zeros of the system (2.1)-(2.3) from u to y are asymptotically stable. Thus, it follows from [5, Lemma B.3.3] that there exist $c_1 \geq 0$ and $c_2 > 0$ such that, for all $i = 1,\ldots,l$ and for all $0 \leq j \leq k$,

$$\left|\left|e_i^{\mathrm{T}} u(j-d)\right|\right| \leq c_1 + c_2\max_{0\leq\tau\leq k}||y(\tau)||. \quad (6.8)$$

Combining (6.7) and (6.8) implies that $||\phi(k-d)|| \leq n_{\mathrm{c}}\max_{0\leq\tau\leq k}||y(\tau)|| + l n_{\mathrm{c}}(c_1 + c_2\max_{0\leq\tau\leq k}||y(\tau)||) = c_3 + c_4\max_{0\leq\tau\leq k}||y(\tau)||$, where $c_3 \triangleq c_1 l n_{\mathrm{c}}$ and $c_4 \triangleq n_{\mathrm{c}} + c_2 l n_{\mathrm{c}}$.

Since $\lim_{k\to\infty} W(y(k),\phi(k-d)) = 0$, $\eta(k)$ is bounded, and the linear boundedness condition above is satisfied, it follows from the Key Technical Lemma [5, p. 181] that $\phi(k)$ is bounded and $\lim_{k\to\infty} y(k) = 0$. Furthermore, since $\phi(k)$ is bounded it follows that $u(k)$ is bounded. □

References

[1] T. Kailath, *Linear Systems*. Englewood Cliffs, New Jersey: Prentice Hall, 1980.

[2] J. M. Maciejowski, *Multivariable Feedback Design*. Great Britain: Addison-Wesley, 1989.

[3] D. S. Bernstein, *Matrix Mathematics*. Princeton University Press, 2005.

[4] J. B. Hoagg and D. S. Bernstein, "Deadbeat internal model control for command following and disturbance rejection in discrete-time systems," in *Proc. Amer. Contr. Conf.*, Minneapolis, MN, 2006, pp. 194–199, also, submitted to *Lin. Alg. Appl.*

[5] G. C. Goodwin and K. S. Sin, *Adaptive Filtering, Prediction, and Control*. Englewood Cliffs, New Jersey: Prentice Hall, 1984.