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Stabilization of a Specified Equilibrium in the Inverted Equilibrium Manifold of the 3D Pendulum

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***Abstract*— This paper treats the asymptotic stabilization of a specified equilibrium in the inverted equilibrium manifold of the 3D pendulum. This attitude stabilization problem is solved by use of Lyapunov methods applied to closed loop dynamics that evolve on the tangent bundle $T\mathrm{SO}(3)$. A smooth controller is proposed that achieves almost global asymptotic stabilization of the specified equilibrium; the controller provides freedom to influence the local dynamics of the closed loop near the specified equilibrium as well as some freedom to shape the manifold of solutions that do not converge to the specified equilibrium.**

I. Introduction

Pendulum models have provided a rich source of examples that have motivated and illustrated many recent developments in nonlinear dynamics and in nonlinear control [1]. An overview of pendulum control problems was given in [2], which provides motivation for the importance of such control problems. The 3D pendulum is a rigid body, supported at a fixed pivot, that has three rotational degrees of freedom; it is acted on by a uniform gravity force and by a control moment. Dynamics and control problems for the 3D pendulum were first introduced in [3]. Stabilization results for the 3D pendulum have been presented in [4].

The 3D pendulum has two disjoint equilibrium manifolds, namely the hanging and the inverted equilibrium manifold. In [4], we studied stabilization of these equilibrium manifolds. In this paper, we consider the problem of the stabilization of a specified equilibrium in the inverted equilibrium manifold. These control problems exemplify attitude stabilization problems on $\mathrm{SO}(3)$. The results are derived by using Lyapunov functions that are suited to the 3D pendulum problem. Also, the attitude is expressed in terms of a rotation matrix, in particular avoiding the use of Euler angles and other non-global attitude representations.

Due to a topological obstruction, it is not possible to globally asymptotically stabilize an inverted equilibrium using a continuous feedback controller. Thus, we propose a continuous controller that *almost* globally asymptotically stabilizes the inverted equilibrium.

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II. Mathematical Models of the 3D Pendulum and Equilibrium Structure

The attitude of the 3D pendulum is represented by a rotation matrix R, viewed as an element of the special orthogonal group $\mathrm{SO}(3)$. The angular velocity of the 3D pendulum with respect to the inertial frame, resolved in the body-fixed frame, is denoted by ω in $\mathbb{R}^3$. We note that even though global representations are used, the feedback controllers introduced in this paper could be expressed in terms of feedback using any other attitude representation, such as Euler angle or quaternions.

The constant inertia matrix, resolved in the body-fixed frame, is denoted by J. The vector from the pivot to the center of mass of the 3D pendulum, resolved in the body-fixed frame, is denoted by ρ. The symbol g denotes the constant acceleration due to gravity.

Standard techniques yield the equations of motion for the 3D pendulum. The dynamics are given by the Euler-Poincaré equation which includes the moment due to gravity and a control moment $u \in \mathbb{R}^3$ which represents the control torque applied to the 3D pendulum, resolved in the body-fixed frame

$$J\dot{\omega} = J\omega \times \omega + mg\rho \times R^{\mathrm{T}} e_3 + u, \tag{1}$$

where $e_3 = [0\ 0\ 1]^{\mathrm{T}}$ denotes unit vector in the direction of gravity in the inertial frame. The rotational kinematics equations are

$$\dot{R} = R\widehat{\omega}, \tag{2}$$

where $R \in \mathrm{SO}(3)$, $\omega \in \mathbb{R}^3$ and

$$\widehat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{3}$$

Note that $a \times b = \widehat{a}b$.

The equations of motion (1) and (2) for the 3D pendulum model has dynamics that evolve on the tangent bundle $T\mathrm{SO}(3)$ [5]. Note that since $e_3 = [0\ 0\ 1]^{\mathrm{T}}$ denotes the unit vector in the direction of gravity in the specified inertial frame, $R^{\mathrm{T}} e_3$ in (1) denotes the dimensionless unit vector in the direction of gravity resolved in the body-fixed frame.

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To further understand the dynamics of the 3D pendulum, we study the equilibria of (1) and (2). Equating the RHS of (1) and (2) to zero with $u = 0$ yields

$$J\omega_{\rm e} \times \omega_{\rm e} + mg\rho \times R_{\rm e}^{\rm T} e_3 = 0, \quad (4)$$

$$R_{\rm e}\widehat{\omega}_{\rm e} = 0. \quad (5)$$

Now $R_{\rm e}\widehat{\omega}_{\rm e} = 0$ if and only if $\omega_{\rm e} = 0$. Substituting $\omega_{\rm e} = 0$ in (4), we obtain $\rho \times R_{\rm e}^{\rm T} e_3 = 0$. Hence,

$$R_{\rm e}^{\rm T} e_3 = \pm \frac{\rho}{\|\rho\|}. \quad (6)$$

Hence an attitude $R_{\rm e}$ is an equilibrium attitude if and only if the direction of gravity resolved in the body-fixed frame, $R_{\rm e}^{\rm T} e_3$, is collinear with the vector ρ. If $R_{\rm e}^{\rm T} e_3$ is in the same direction as the vector ρ, then $(0, R_{\rm e})$ is a hanging equilibrium of the 3D pendulum; if $R_{\rm e}^{\rm T} e_3$ is in the opposite direction as the vector ρ, then $(0, R_{\rm e})$ is an inverted equilibrium of the 3D pendulum.

According to (6), there is a smooth manifold of hanging equilibria and a smooth manifold of inverted equilibria, and these two equilibrium manifolds are clearly disjoint. The former is the hanging equilibrium manifold; the latter is the inverted equilibrium manifold.

III. Asymptotic Stabilization of a Specified Inverted Equilibrium

Let $(0, R_{\rm d})$ denote a specified equilibrium in the inverted equilibrium manifold of the 3D pendulum given by (1) and (2). In this section we present controllers that stabilize this specified equilibrium $(0, R_{\rm d})$.

Let $\Phi : [0, \infty) \to [0, \infty)$ be a C^2 function such that

$$\Phi(0) = 0 \;\text{ and }\; \Phi'(x) > 0 \text{ for all } x \in [0, \infty). \quad (7)$$

Let $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 function satisfying

$$\begin{cases} \Psi'(0) \text{ is positive definite}, \\ \mathscr{P}(x) \le x^{\rm T}\Psi(x) \le \alpha(\|x\|) \text{ for all } x \in \mathbb{R}^3, \end{cases} \quad (8)$$

where $\mathscr{P} : \mathbb{R}^3 \to \mathbb{R}$ is a positive definite function and $\alpha(\cdot)$ is a class-$\mathcal{K}$ function [6]. Given $\mathbf{a} = [a_1 \; a_2 \; a_3]^{\rm T} \in \mathbb{R}^3$, denote

$$\Omega_{\mathbf{a}}(R) \triangleq a_1\Big[(R_{\rm d}^{\rm T} e_1) \times (R^{\rm T} e_1)\Big] + a_2\Big[(R_{\rm d}^{\rm T} e_2) \times (R^{\rm T} e_2)\Big] + a_3\Big[(R_{\rm d}^{\rm T} e_3) \times (R^{\rm T} e_3)\Big]. \quad (9)$$

Further, let $A \in \mathbb{R}^{3\times 3}$, be a diagonal matrix defined as

$$A \triangleq \text{diag}(\mathbf{a}). \quad (10)$$

We study feedback controllers of the form

$$u = -\Psi(\omega) + \kappa\Big((R_{\rm d}^{\rm T} e_3) \times (R^{\rm T} e_3)\Big) + \Phi'\Big(\text{trace}(A - AR_{\rm d}R^{\rm T})\Big)\Omega_{\mathbf{a}}(R), \quad (11)$$

where $\kappa \ge mg\|\rho\|$.

The controller (11) requires measurements of the angular velocity and attitude, in the form of the rotation matrix R, of the 3D pendulum. The angular velocity dependent term $\Psi(\omega)$ in (11) provides damping, while the attitude dependent term in (11) can be viewed as a modification or shaping of the gravity potential.

Unlike feedback-linearization based approaches, the control law (11) requires no knowledge of the moment of inertia or of the location of the center of mass of the 3D pendulum relative to the pivot. However, the constant κ is an upper bound on the gravity moment about the pivot. Hence a bound on $mg\|\rho\|$ must be known.

We subsequently show that $(\omega, R) = (0, R_{\rm d})$ is an equilibrium of the closed-loop consisting of (1), (2) and (11) and it is almost globally asymptotically stable with locally exponential convergence.

A. Equilibrium Structure of the Closed-Loop

In this section, we study the equilibria in $T\mathrm{SO}(3)$ of the closed-loop system consisting of (1), (2) and (11). Define

$$\bar{\mathbf{a}} \triangleq [a_1 \; a_2 \; \bar{a}_3]^{\rm T}, \quad (12)$$

where

$$\bar{a}_3 \triangleq a_3 + \frac{\kappa - mg\|\rho\|}{\Phi'\Big(\text{trace}(A - AR_{\rm d}R^{\rm T})\Big)} \ge a_3. \quad (13)$$

Since $(0, R_{\rm d})$ lies in the inverted equilibrium manifold, it follows from (6) that $R_{\rm d}^{\rm T} e_3 = -\frac{\rho}{\|\rho\|}$. Substituting (11) in (1) and (2), and simplifying, we express the closed-loop system as

$$\begin{cases} J\dot{\omega} = J\omega \times \omega - \Psi(\omega) + \Phi'\Big(\text{trace}(A - AR_{\rm d}R^{\rm T})\Big)\Omega_{\bar{\mathbf{a}}}(R), \\ \dot{R} = R\widehat{\omega}. \end{cases} \quad (14)$$

Lemma 1: Consider the closed-loop system (14) of a 3D pendulum given by (1) and (2), with controller (11), where the functions Φ and Ψ satisfy (7) and (8), $\kappa \ge mg\|\rho\|$ and A defined in (10) satisfies $0 < 2a_1 < a_1 + a_2 < a_3$. Then, the closed-loop (14) has four equilibrium solutions given by

$$\mathscr{E} = \Big\{(\omega, R) \in T\mathrm{SO}(3) : \omega = 0, R = MR_{\rm d}, \; M \in \mathcal{M}_c\Big\}, \quad (15)$$

where

$$\mathcal{M}_c \triangleq \Big\{\text{diag}(1,1,1), \; \text{diag}(-1,1,-1), \; \text{diag}(1,-1,-1), \text{diag}(-1,-1,1)\Big\}. \quad (16)$$

Proof: To obtain the equilibria of the closed-loop system, equate the RHS of (14) to zero, which yields

$$J\omega \times \omega - \Psi(\omega) + \Phi'\Big(\text{trace}(A - AR_{\rm d}R^{\rm T})\Big)\Omega_{\bar{\mathbf{a}}}(R) = 0, \quad (17)$$

$$\omega = 0. \quad (18)$$

Substituting $\omega = 0$ into (17) and noting that $\Phi'(x) > 0$ for all $x \in [0, \infty)$ yields

$$\Omega_{\bar{\mathbf{a}}}(R) = a_1 \widehat{R_\mathrm{d}^\mathrm{T} e_1} R^\mathrm{T} e_1 + a_2 \widehat{R_\mathrm{d}^\mathrm{T} e_2} R^\mathrm{T} e_2 + \bar{a}_3 \widehat{R_\mathrm{d}^\mathrm{T} e_3} R^\mathrm{T} e_3 = 0. \tag{19}$$

It can be shown that $\Omega_{\bar{\mathbf{a}}}(R) = 0$ implies $\Omega_{\mathbf{a}}(R) = 0$.

Now define $f : \mathrm{SO}(3) \to \mathbb{R}$ by $f(R) = \mathrm{trace}(A R_\mathrm{d} R^\mathrm{T})$ where $A \triangleq \mathrm{diag}(\mathbf{a})$. Then, it can be shown that the tangent map of f [5] denoted as $Tf : T_R\mathrm{SO}(3) \to \mathbb{R}$ is given by $Tf(R\widehat{\eta}) = \eta^\mathrm{T} \Omega_{\mathbf{a}}(R)$, where $\eta \in \mathbb{R}^3$ and $R\widehat{\eta} \in T_R\mathrm{SO}(3)$ represents the lifted action of $R \in \mathrm{SO}(3)$ on $\widehat{\eta} \in \mathfrak{so}(3)$. It follows that the set of all points in $\mathrm{SO}(3)$ that satisfy $\Omega_{\mathbf{a}}(R) = 0$ are the critical points of the function f.

Since A is a diagonal matrix with distinct positive eigenvalues, Theorem 1.1 in [7] implies that f is a perfect Morse function [7]. Hence, f has exactly four critical points on $\mathrm{SO}(3)$ that are the four solutions of (19). Therefore, the equilibrium solutions lie in $\mathscr{E}$ as given in (15). ■

Remark 1: Note that the desired equilibrium $(0, R_\mathrm{d}) \in \mathscr{E}$. Each of the other three equilibrium solutions in $\mathscr{E}$ corresponds to an attitude configuration formed by the desired attitude R_d, followed by a rotation about one of the three body fixed axes by 180 degrees.

B. Local Analysis of the Closed-Loop

Consider a perturbation of the initial conditions about an equilibrium $(0, R_\mathrm{e}) \in \mathscr{E}$ given in (15) in terms of a perturbation parameter $\varepsilon \in \mathbb{R}$. We express the perturbation in the rotation matrix using exponential coordinates [5], [8], [9]. Let the perturbation in the initial condition for attitude be given as $R(0, \varepsilon) = R_\mathrm{e} e^{\varepsilon \widehat{\delta\Theta}}$, where $R_\mathrm{e} R_\mathrm{d}^\mathrm{T} \in \mathcal{M}_c$ and $\delta\Theta \in \mathbb{R}^3$ is a constant vector. The perturbation in the initial condition for angular velocity is given as $\omega(0, \varepsilon) = \varepsilon \delta\omega$, where $\delta\omega \in \mathbb{R}^3$ is a constant vector. Note that if $\varepsilon = 0$ then, $(\omega(0,0), R(0,0)) = (0, R_\mathrm{e})$ and hence

$$(\omega(t,0), R(t,0)) \equiv (0, R_\mathrm{e}) \tag{20}$$

for all time $t \in \mathbb{R}$. This simply represents the unperturbed equilibrium solution.

Next, consider the solution to the perturbed equations of motion for the closed-loop 3D pendulum given by (14). Then differentiating both sides of the perturbed closed-loop with respect to ε and substituting $\varepsilon = 0$ yields

$$J\dot{\omega}_\varepsilon(t,0) = -\Psi'(0)\omega_\varepsilon(t,0) + \Phi'(\mathrm{trace}(A - AR_\mathrm{d}R_\mathrm{e}^\mathrm{T}))\Omega_{\bar{\mathbf{a}}}(R_\varepsilon(t,0)), \tag{21}$$

$$\dot{R}_\varepsilon(t,0) = R_\mathrm{e}\widehat{\omega}_\varepsilon(t,0). \tag{22}$$

where

$$\omega_\varepsilon(t,0) \triangleq \left.\frac{\partial\, \omega(t,\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} \text{ and } R_\varepsilon(t,0) \triangleq \left.\frac{\partial R(t,\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

Define variables for the linearization $\Delta\omega, \Delta\Theta \in \mathbb{R}^3$ as $\Delta\omega(t) \triangleq \omega_\varepsilon(t,0)$ and $\widehat{\Delta\Theta}(t) \triangleq R_\mathrm{e}^\mathrm{T} R_\varepsilon(t,0)$. Then from (22) we obtain $\widehat{\Delta\dot{\Theta}}(t) = R_\mathrm{e}^\mathrm{T} \dot{R}_\varepsilon(t,0) = \widehat{\omega}_\varepsilon(t,0) = \widehat{\Delta\omega}(t)$. Therefore,

$$\Delta\dot{\Theta} = \Delta\omega. \tag{23}$$

Combining (21) and (23), and simplifying, we obtain

$$J\Delta\ddot{\Theta} + \Psi'(0)\,\Delta\dot{\Theta} + \mathsf{K}\Delta\Theta = 0, \tag{24}$$

where

$$\mathsf{K} = \Phi'\Big(\mathrm{trace}(A - AR_\mathrm{d}R_\mathrm{e}^\mathrm{T})\Big)\cdot \left[-a_1 \widehat{R_\mathrm{d}^\mathrm{T} e_1}\, \widehat{R_\mathrm{e}^\mathrm{T} e_1} - a_2 \widehat{R_\mathrm{d}^\mathrm{T} e_2}\, \widehat{R_\mathrm{e}^\mathrm{T} e_2} - \bar{a}_3 \widehat{R_\mathrm{d}^\mathrm{T} e_3}\, \widehat{R_\mathrm{e}^\mathrm{T} e_3} \right], \tag{25}$$

and $\bar{a}_3$ is given in (13). Now, since $M = R_\mathrm{e} R_\mathrm{d}^\mathrm{T} \in \mathcal{M}_c$, the identity $\widehat{Re_i} = R\widehat{e}_i R^\mathrm{T}$, where $R \in \mathrm{SO}(3)$ [8], yields

$$\widehat{R_\mathrm{d}^\mathrm{T} e_i}\, \widehat{R_\mathrm{e}^\mathrm{T} e_i} = \widehat{R_\mathrm{e}^\mathrm{T} M e_i}\, \widehat{R_\mathrm{e}^\mathrm{T} e_i} = R_\mathrm{e}^\mathrm{T} \widehat{Me_i}\, \widehat{e}_i R_\mathrm{e},$$

for $i \in \{1,2,3\}$. Thus, using the above, the expression for K in (25) can be written as

$$\mathsf{K} = \Phi'\Big(\mathrm{trace}(A - AR_\mathrm{d}R_\mathrm{e}^\mathrm{T})\Big) R_\mathrm{e}^\mathrm{T}\, \mathcal{Q}\, R_\mathrm{e}, \tag{26}$$

where

$$\mathcal{Q} = -a_1 \widehat{Me_1}\, \widehat{e}_1 - a_2 \widehat{Me_2}\, \widehat{e}_2 - \bar{a}_3 \widehat{Me_3}\, \widehat{e}_3 \tag{27}$$

and $M = R_\mathrm{e} R_\mathrm{d}^\mathrm{T} \in \mathcal{M}_c$, as in (16).

Lemma 2: Consider the closed-loop model of a 3D pendulum given by (1) and (2), with controller (11), where the functions Φ and Ψ satisfy (7) and (8), $\kappa \geq mg\|\rho\|$ and A defined in (10) satisfies $0 < 2a_1 < a_1 + a_2 < a_3$. Then the closed-loop equilibrium $(0, R_\mathrm{d}) \in \mathscr{E}$ is asymptotically stable and the convergence is locally exponential.

Proof: Combining equations (1), (2) and (11), we obtain the closed-loop system given by (14). Next, we linearize the dynamics of (14) about the equilibrium $(0, R_\mathrm{d})$ yielding equation (24) where $R_\mathrm{e} = R_\mathrm{d}$.

Now $\Psi'(0)$ is positive definite and M is the identity matrix. Hence, from (27)

$$\mathcal{Q} = -a_1\, \widehat{e}_1^{\,2} - a_2\, \widehat{e}_2^{\,2} - \bar{a}_3\, \widehat{e}_3^{\,2}$$

is positive definite. Next, since $\Phi'(\cdot)$ is positive and K is a similarity transform of $\mathcal{Q}$, K in (26) is positive definite. Thus, since K and $\Psi'(0)$ are positive definite, linear theory guarantees that the linearized system given by (24) is asymptotically stable. Hence, the equilibrium $(0, R_\mathrm{d})$ of (14) is locally asymptotically stable with local exponential convergence. ■

Consider the equilibria $(0, R_\mathrm{e})$ of the closed-loop (14) such that $R_\mathrm{e} \neq R_\mathrm{d}$. From Lemma 1, we express the three equilibria $(0, R_\mathrm{e}) \in \mathscr{E}$ such that $R_\mathrm{e} \neq R_\mathrm{d}$ as $R_{\mathrm{e},i} = M_i R_\mathrm{d}$, $i \in \{1,2,3\}$, where

$$M_1 = \mathrm{diag}(1,-1,-1), \quad M_2 = \mathrm{diag}(-1,1,-1), \quad \text{and} \quad M_3 = \mathrm{diag}(-1,-1,1). \tag{28}$$

We next show that the above three equilibria $(0, R_{e\,i})$, $i \in \{1,2,3\}$ of the closed-loop (14) are unstable and present the local properties of the closed-loop trajectories.

Lemma 3: Consider the closed-loop model of a 3D pendulum given by (1) and (2), with controller (11), where the functions Φ and Ψ satisfy (7) and (8), $\kappa \geq mg\|\rho\|$ and A defined in (10) satisfies $0 < 2a_1 < a_1 + a_2 < a_3$. Consider an equilibrium $(0, R_{e\,i}) \in \mathscr{E}$, such that $R_{e\,i} \neq R_{\rm d}$, $i \in \{1,2,3\}$. Then, $(0, R_{e\,i})$ is unstable. Furthermore, there exists an invariant 3-dimensional submanifold $\mathcal{M}_1$, an invariant 4-dimensional submanifold $\mathcal{M}_2$, and an invariant 5-dimensional submanifold $\mathcal{M}_3$, respectively in $T\mathrm{SO}(3)$ whose complements are open and dense, such that **(a)** for all initial conditions $(\omega(0), R(0)) \in \mathcal{M}_i$, $i \in \{1,2,3\}$, the closed-loop solutions converge to the equilibrium $(0, R_{e\,i})$ and **(b)** for all initial conditions $(\omega(0), R(0)) \in T\mathrm{SO}(3)\backslash\mathcal{M}_i$, the closed-loop solutions do not converge to the equilibrium $(0, R_{e\,i})$, $i \in \{1,2,3\}$.

Proof: Combining equations (1), (2) and (11), we obtain the closed-loop system given by (14). Next, we linearize the dynamics of (14) about the equilibrium $(0, R_{e\,i})$, $i \in \{1,2,3\}$ yielding equation (24). Since, $R_{e\,i} \neq R_{\rm d}$, the three equilibria are given by $(0, R_{e\,i}) = (0, M_i R_{\rm d})$, where M_i, $i \in \{1,2,3\}$ is as given in (28).

Next, computing $\mathcal{Q}_i$ using (27) corresponding to the three attitude equilibria $(0, M_i R_{\rm d})$, $i \in \{1,2,3\}$ yields

$$
\begin{aligned}
\mathcal{Q}_1 &= \mathrm{diag}(-a_2 - \bar{a}_3, a_1 - \bar{a}_3, a_1 - a_2),\\
\mathcal{Q}_2 &= \mathrm{diag}(a_2 - \bar{a}_3, -a_1 - \bar{a}_3, -a_1 + a_2),\\
\mathcal{Q}_3 &= \mathrm{diag}(-a_2 + \bar{a}_3, -a_1 + \bar{a}_3, -a_1 - a_2).
\end{aligned}
$$

Since $0 < a_1 < a_2 < \bar{a}_3$, all eigenvalues of $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ lie in $\mathbb{R}\backslash\{0\}$ and each of $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ has a negative eigenvalue. Since $R_{e\,i} \in \mathrm{SO}(3)$, it follows from (26) that corresponding to $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$, all eigenvalues of the matrices $\mathsf{K}_1, \mathsf{K}_2$ and K_3 lie in $\mathbb{R}\backslash\{0\}$ and each of $\mathsf{K}_1, \mathsf{K}_2$ and K_3 has a negative eigenvalue. Hence (14) is unstable for each equilibrium $(0, R_{e\,i})$, $i \in \{1,2,3\}$ [9].

Next, since, $\Psi'(0)$ is positive definite and all eigenvalues of the matrices $\mathsf{K}_1, \mathsf{K}_2$ and K_3 lie in $\mathbb{R}\backslash\{0\}$, it follows that each equilibrium $(0, R_{e\,i}) \in \mathscr{E}$, $i \in \{1,2,3\}$ of (14) is *hyperbolic*. Theorem 3.2.1 in [10] guarantees that each equilibrium $(0, R_{e\,i}) \in \mathscr{E}$ of (14) has a nontrivial unstable manifold W_i^u. Let W_i^s denote its corresponding stable manifold. The tangent space to the stable manifold W_i^s at the equilibrium $(0, R_{e\,i})$ is the stable eigenspace of the linearized system (24), and hence is 3-dimensional, 4-dimensional and 5-dimensional, for $i \in \{1,2,3\}$, respectively. Since, the equilibria are hyperbolic, there are no center manifolds. Then, all trajectories near $(0, R_{e\,i})$ other than those in W_i^s diverge from that equilibrium. Since the dimension of the submanifold W_i^s is less than the dimension of the tangent bundle $T\mathrm{SO}(3)$, the complement of W_i^s is open and dense. Denoting $\mathcal{M}_i \triangleq W_i^s$, $i \in \{1,2,3\}$, the result follows. ∎

IV. Global Analysis of the Closed-Loop

In this section, we present global convergence properties of closed-loop trajectories.

Theorem 1: Consider the closed-loop model of a 3D pendulum given by (1) and (2), with controller (11), where the functions Φ and Ψ satisfy (7) and (8), $\kappa \geq mg\|\rho\|$ and A defined in (10) satisfies $0 < 2a_1 < a_1 + a_2 < a_3$. Then, $(0, R_{\rm d})$ is an asymptotically stable equilibrium of the closed-loop (14) with local exponential convergence. Furthermore, there exists an invariant manifold $\mathsf{M} \subset T\mathrm{SO}(3)$, whose complement is open and dense such that for all initial conditions $(\omega(0), R(0)) \in T\mathrm{SO}(3)\backslash\mathsf{M}$, the solutions of the closed-loop system given by (14) satisfy $\lim\limits_{t\to\infty} \omega(t) = 0$ and $\lim\limits_{t\to\infty} R(t) = R_{\rm d}$. For all other initial conditions $(\omega(0), R(0)) \in \mathsf{M}$, the solutions of the closed-loop system given by (14) satisfy $\lim\limits_{t\to\infty}(\omega(t), R(t)) \in \mathscr{E}\backslash\{(0, R_{\rm d})\}$.

Proof: Consider the closed-loop system consisting of (1), (2) and (11) given by (14). Then, it immediately follows from Lemma 2 that $(0, R_{\rm d})$ is an asymptotically stable equilibrium of the closed-loop (14) with local exponential convergence.

Consider the following candidate Lyapunov function.

$$
\begin{aligned}
V(\omega, R) = \frac{1}{2}\omega^{\rm T} J\omega + (\kappa - mg\|\rho\|)(1 - e_3^{\rm T} R_{\rm d} R^{\rm T} e_3)\\
+ \Phi\Big(\mathrm{trace}(A - AR_{\rm d}R^{\rm T})\Big). \qquad (29)
\end{aligned}
$$

Note that $V(\omega, R) \geq 0$ for all $(\omega, R) \in T\mathrm{SO}(3)$ and $V(\omega, R) = 0$ if and only if $(\omega, R) = (0, R_{\rm d})$. Thus $V(\omega, R)$ is a positive definite function on $T\mathrm{SO}(3)$.

We show that the Lie derivative of the Lyapunov function along the closed-loop vector field of (14) is negative semidefinite. Denote the closed-loop vector field of (14) by Z. Then,

$$
\begin{aligned}
&\mathscr{L}_Z \Phi\Big(\mathrm{trace}(A - AR_{\rm d}R^{\rm T})\Big)\\
&\quad = -\Phi'\Big(\mathrm{trace}(A - AR_{\rm d}R^{\rm T})\Big)[\mathrm{trace}(AR_{\rm d}(R\widehat{\omega})^{\rm T})].
\end{aligned}
$$

Now, $\mathrm{trace}\Big(AR_{\rm d}(R\widehat{\omega})^{\rm T}\Big) = \mathrm{trace}\Big((R\widehat{\omega})R_{\rm d}^{\rm T}A\Big)^{\rm T} = \omega^{\rm T}\Big[a_1\widehat{R_{\rm d}^{\rm T}e_1}R^{\rm T}e_1 + a_2\widehat{R_{\rm d}^{\rm T}e_2}R^{\rm T}e_2 + a_3\widehat{R_{\rm d}^{\rm T}e_3}R^{\rm T}e_3\Big]$. Therefore, the derivative of the Lyapunov function along a solution of the closed-loop is

$$
\begin{aligned}
\dot{V}(\omega, R) &= \omega^{\rm T} J\dot{\omega} - (\kappa - mg\|\rho\|)e_3^{\rm T} R_{\rm d}\dot{R}^{\rm T} e_3\\
&\qquad + \mathscr{L}_Z \Phi\Big(\mathrm{trace}(A - AR_{\rm d}R^{\rm T})\Big),\\
&= \omega^{\rm T}\Big\{u - \kappa(R_{\rm d}^{\rm T}e_3 \times R^{\rm T}e_3)\\
&\qquad - \Phi'\Big(\mathrm{trace}(A - AR_{\rm d}R^{\rm T})\Big)\Omega_{\mathbf{a}}(R)\Big\}. \qquad (30)
\end{aligned}
$$

Substituting (11) into (30), we obtain $\dot{V}(\omega, R) = -\omega^{\rm T}\Psi(\omega) \leq -\mathscr{P}(\omega)$. Thus, the derivative of the Lyapunov function along a solution of the closed-loop system is negative semidefinite.

Recall that $\Phi(\cdot)$ is a strictly increasing monotone function and $\mathrm{SO}(3)$ is compact. Hence, for any $(\omega(0), R(0)) \in T\mathrm{SO}(3)$, the set $\mathscr{K} = \Big\{(\omega, R) \in T\mathrm{SO}(3) : V(\omega, R) \leq V(\omega(0), R(0))\Big\}$, is a compact, positively invariant set of the closed-loop.

By the invariant set theorem, it follows that all solutions that begin in $\mathscr{K}$ converge to the largest invariant set in $\dot{V}^{-1}(0)$ contained in $\mathscr{K}$. Now, since $\mathscr{P}$ is a positive definite function, $\dot{V}(\omega, R) \equiv 0$ implies $\omega \equiv 0$. Substituting this into the closed-loop system (14), it can be shown that

$$\dot{V}^{-1}(0) = \Big\{(\omega, R) \in T\mathrm{SO}(3) : \omega \equiv 0, \quad \Omega_{\bar{\mathrm{a}}}(R) \equiv 0\Big\},$$

where $\Omega_{\bar{\mathrm{a}}}(\cdot)$ is as given in (9). Thus, following the same arguments as in Lemma 1, it can be shown that the largest invariant set in $\dot{V}^{-1}(0)$ is given by (15). Note that each of the four points given in (15) correspond to an equilibrium of the closed-loop system in $T\mathrm{SO}(3)$. Hence, all solutions of the closed-loop system converge to one of the equilibrium solutions in $\mathscr{E} \cap \mathscr{K}$, where $\mathscr{E}$ is given in (15).

Next, consider an equilibrium $(0, R_{\mathrm{e},i}) \in \mathscr{E}$ such that $R_{\mathrm{e},i} \neq R_{\mathrm{d}}$, $i \in \{1, 2, 3\}$. Lemma 3 yields that the solutions of the closed-loop system except for solutions in the invariant submanifolds $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$, whose complement is open and dense, diverge from the equilibria $(0, R_{\mathrm{e},i})$, $i \in \{1, 2, 3\}$. Thus, solutions of the closed-loop system for initial conditions that do not lie in $\mathsf{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ must converge to the equilibrium $(0, R_{\mathrm{d}})$. Solutions of the closed-loop system (14) for initial conditions that lie in M converge to one of the equilibrium solutions in $\mathscr{E} \backslash \{(0, R_{\mathrm{d}})\}$. ■

Theorem 1 is the main result on asymptotic stabilization of a specified inverted equilibrium of the 3D pendulum. Under the indicated assumptions, almost global asymptotic stabilization is achieved. This is the best possible result for this stabilization problem, in the sense that global stabilization using smooth feedback is not achievable.

V. Simulation Results

In this section, we present examples that illustrate the effect of the functions Φ and Ψ on the closed-loop responses. We present simulation results for two distinct cases. These correspond to cases where the closed-loop is either *damped* with low frequency oscillations or *stiff* with fast initial transients and oscillatory behavior. This is analogous to the behavior observed in a linear closed-loop system with a linear PD-type control law. The parameter $\mathbf{a}$ in (12) effects the distribution of control moment along each of the three body fixed axis. We simulate the closed-loop dynamics of the 3D pendulum (1) and (2) with the controller (11). The parameters are chosen as $J = \mathrm{diag}(200, 300, 150)$ kg·m^2 and $mg\rho = 200\,[0\ \ 0\ \ 1]^{\mathrm{T}}$ N·m. The initial conditions for both damped and stiff closed-loop are chosen as $\omega(0) = [10\ 40\ 10]^{\mathrm{T}}$ deg/ sec, and

$$R(0) = \begin{bmatrix} 0.2065 & 0.8760 & -0.4359 \\ -0.9733 & 0.2294 & 0 \\ 0.1000 & 0.4243 & 0.9000 \end{bmatrix}.$$

The desired attitude in the inverted equilibrium manifold is chosen as $R_{\mathrm{d}} = \mathrm{diag}(-1, 1, -1)$. To present the simulation data, we plot the rotation angle $\Theta \triangleq \cos^{-1}\Big((\mathrm{trace}(R_{\mathrm{d}}^{\mathrm{T}}R) - 1)/2\Big)$ that is a scalar measure of the attitude error between R and R_{d}. Physically, Θ corresponds to the angle of rotation about an eigenaxis required to bring the spacecraft to the desired attitude R_{d}.

For the controller (11), choose $a_1 = 1$, $a_2 = 1.9$, and $a_3 = 3$. First consider the case for the damped closed-loop. The functions Φ and Ψ are chosen as $\Phi(x) = 10x$ and $\Psi(\omega) = P\omega$, where $P = \mathrm{diag}(10, 20, 30)$. The corresponding plots are shown in figures 1–3. Note that the closed-loop system converges in 150 seconds. Furthermore, since P has diagonal values that increase in magnitude, the damping along the body axes increases accordingly. As seen in Figure 1, ω_3 converges to zero before ω_2 and ω_1 converge to zero. Note that ω_1 takes the longest time to converge to zero.

Next, we study the closed-loop system for the stiff case. We choose $\Phi(x) = 20x$ and $\Psi(\omega) = P\omega$, where $P = \mathrm{diag}(5, 10, 15)$. Thus, the gain in Φ is doubled whereas the gain P is halved. The corresponding plots are shown in figures 4–6. Note that compared to the previous case, the closed-loop system oscillates more and the convergence time is nearly doubled to 300 seconds. As is clear from Figure 4, the previous comments on the distribution of dissipation hold true.

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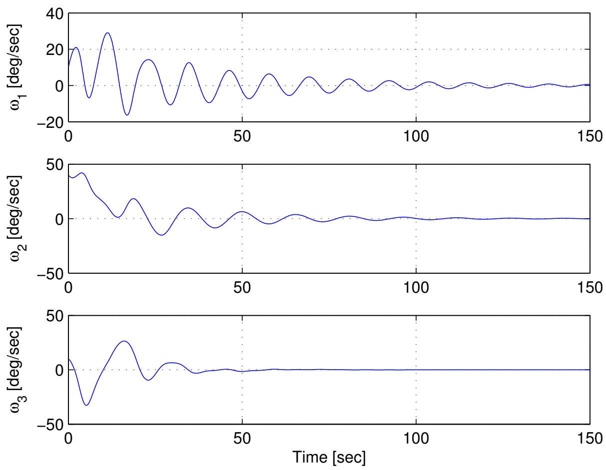

Fig. 1. Angular velocity of the 3D pendulum.

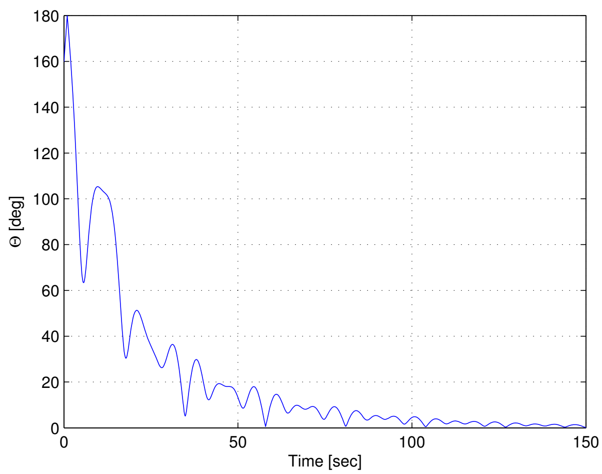

Fig. 2. Attitude Error of the 3D pendulum.

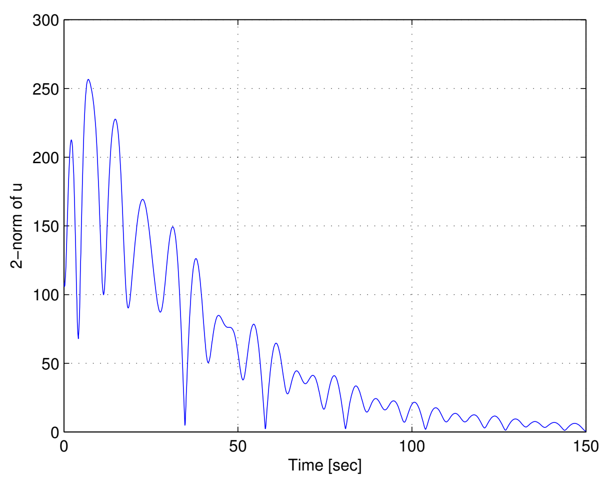

Fig. 3. Magnitude of the Control Torque.

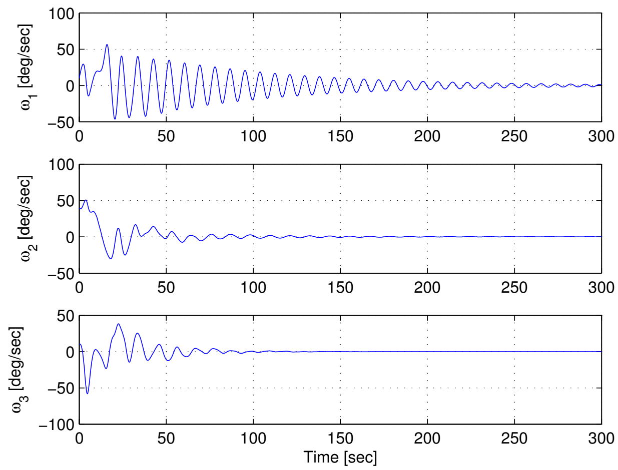

Fig. 4. Angular velocity of the 3D pendulum.

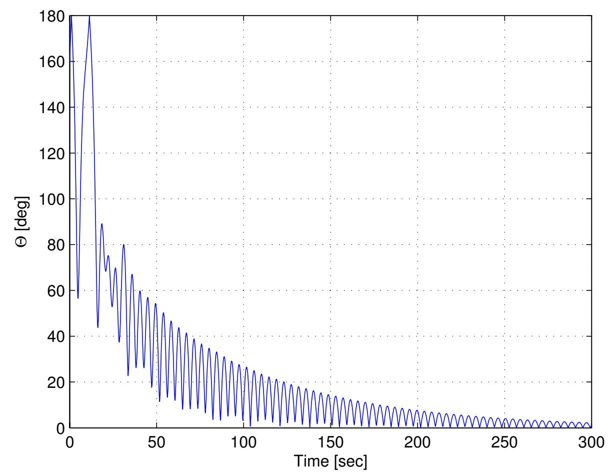

Fig. 5. Attitude Error of the 3D pendulum.

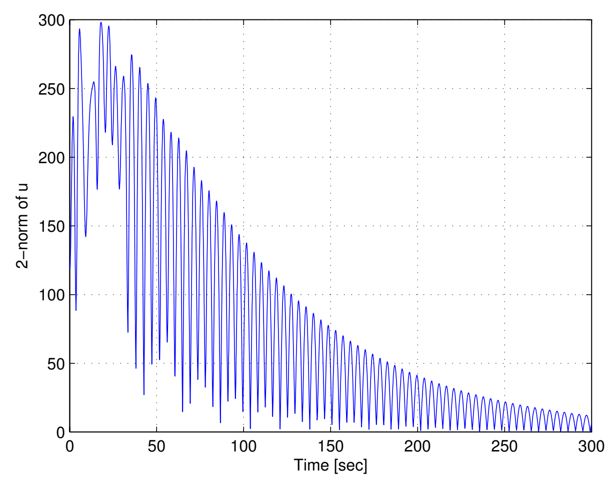

Fig. 6. Magnitude of the Control Torque.