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State Estimation for Large-Scale Systems Based on Reduced-Order Error-Covariance Propagation

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***Abstract*— We compare several reduced-order Kalman filters for discrete-time LTI systems based on reduced-order error-covariance propagation. These filters use combinations of balanced model truncation and complementary steady-state covariance compensation. After describing each method, we compare their performance through numerical studies using a compartmental model example. These methods are aimed at large-scale data-assimilation problems where reducing computational complexity is critical.**

I. INTRODUCTION

The development of reduced-order state estimators has been of interest for several decades; representative work includes [1–15]. Most of these techniques involve data injection with an estimator whose order is less than the order of the plant. The estimator dynamics are typically obtained from the full-order dynamics by a truncation or projection process, while the estimator gain is obtained from a steady-state or updated error-covariance matrix based on the full-order dynamics.

For large-scale systems, however, reduced-order filters based on a full-order error covariance may not be feasible. In particular, the effort needed to compute the steady-state error covariance or to update the time-dependent error covariance is significant, namely, $\mathrm{O}(n^3)$ for a system of order n. Relevant applications include systems modeled by discretized partial differential equations such as weather forecasting [16–21], where state estimation is generally referred to as data assimilation [22].

To overcome the $\mathrm{O}(n^3)$-computational burden of full-order-error-covariance-based estimation, we are interested in reduced-order filters based on a reduced-order error covariance. One such technique is developed in [16], where balancing is used to obtain a reduced-order model that provides the basis for the error-covariance update. By using the reverse transformation to convert the reduced-order error covariance to a full-order error covariance in the original basis, data injection is performed on the full-order model so that estimates of all states are obtained in the original, physically meaningful basis. Although performance bounds are not available for this technique, the approach is consistent with the use of balancing in model reduction [23] while reducing the computational burden of the error-covariance update.

In the present paper we compare the performance of the algorithm developed in [16] with several alternative algorithms. These alternative algorithms use balancing or truncation in various combinations to achieve a reduced-order-error-covariance for data injection with either the full-order model or a reduced-order model. Some of these algorithms use an initial balancing transformation, while others use an initial model truncation along with a steady-state error covariance. Algorithms that avoid the need for a balancing step are desirable when the system order is sufficiently high that balancing and transformation are prohibitive. For example, in weather applications, a state dimension greater than 10^6 is commonplace [16–21].

As in [16], our study is primarily numerical, although we provide analytical performance bounds for the complementary steady-state error-covariance filters. Our goals in the present paper are thus to 1) clarify the nature of the reduced-order-error-covariance estimation problem, 2) present a collection of reduced-order-error-covariance estimators that are potentially useful in practice, and 3) numerically compare the performance of these filters on representative examples. This study is a precursor to the development of estimators for large-scale systems with nonlinear dynamics; preliminary results are described in [24].

In the classical Kalman filter, the full-order error covariance is propagated to obtain the estimator gain by which measurements are injected into the full state to obtain optimal state estimates under uncertain disturbances and measurement noises. However, for large scale systems, propagation of the full-order error covariance is computationally infeasible. Hence, we consider reduced-order error-covariance filters. In the following subsections, we describe these filters. To fix notation, we begin with a brief review of the full-order Kalman filter.

II. FULL-ORDER KALMAN FILTER (FOKF)

Consider the discrete-time LTI system

$$x_{k+1} = Ax_k + Gw_k, \tag{2.1}$$

$$y_k = Cx_k + v_k, \tag{2.2}$$

where $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^d$, $y_k, v_k \in \mathbb{R}^p$ and A, G, C are known real matrices of appropriate size. The plant disturbance Gw_k has the covariance $Q_k \triangleq \mathcal{E}[Gw_k w_k^{\mathrm{T}} G^{\mathrm{T}}]$, while the sensor noise v_k has the covariance $R_k \triangleq \mathcal{E}[v_k v_k^{\mathrm{T}}]$. The objective is to obtain an estimate x_k^{f} of the state x_k using measurements y_k to minimize $\mathrm{tr} P_k^{\mathrm{f}}$, where $e_k^{\mathrm{f}} \in \mathbb{R}^n$ is

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defined by

$$e_k^{\mathrm{f}} \triangleq x_k - x_k^{\mathrm{f}} \tag{2.3}$$

and the state-error covariance $P_k^{\mathrm{f}} \in \mathbb{R}^{n\times n}$ is defined by

$$P_k^{\mathrm{f}} \triangleq \mathcal{E}[e_k^{\mathrm{f}} e_k^{\mathrm{fT}}]. \tag{2.4}$$

The full-order Kalman filter is expressed in two steps, namely, the *forecast step*, which uses the model, and the *data assimilation step*, where the measurement is used to update the states. These steps can be summarized as follows:

Forecast Step

$$x_{k+1}^{\mathrm{f}} = A x_k^{\mathrm{da}}, \tag{2.5}$$

$$P_{k+1}^{\mathrm{f}} = A P_k^{\mathrm{da}} A^{\mathrm{T}} + Q_k^{\mathrm{f}}. \tag{2.6}$$

Data Assimilation Step

$$K_k = P_k^{\mathrm{f}} C^{\mathrm{T}} (C P_k^{\mathrm{f}} C^{\mathrm{T}} + R_k)^{-1}, \tag{2.7}$$

$$P_k^{\mathrm{da}} = (I - K_k C) P_k^{\mathrm{f}}, \tag{2.8}$$

$$x_k^{\mathrm{da}} = x_k^{\mathrm{f}} + K_k (y_k - C x_k^{\mathrm{f}}). \tag{2.9}$$

III. Kalman Filter with Error-Propagation in Balanced-Reduction Model (KFEBRM)

Applying the similarity transformation $x_k = T\hat{x}_k$, the system (2.1), (2.2) becomes

$$\hat{x}_{k+1} = \hat{A}\hat{x}_k + \hat{G} w_k, \tag{3.1}$$

$$y_k = \hat{C}\hat{x}_k + v_k, \tag{3.2}$$

where $\hat{A} \triangleq T^{-1}AT,\ \hat{G} \triangleq T^{-1}G,$ and $\hat{C} \triangleq CT$.

We choose the transformation T such that the controllability and observability gramians of the transformed system (3.1), (3.2) are diagonal and equal, that is, the system (3.1), (3.2) is a balanced realization of the system (2.1), (2.2). Then, we reduce the transformed system by retaining the dominant subspace as determined by the Hankel singular values $\sigma_1, \ldots, \sigma_n$, which describe the relative importance of each transformed state. The Hankel singular values $\sigma_1, \ldots, \sigma_n$ are the diagonal entries of the diagonal matrix Σ given by

$$\Sigma \triangleq \hat{W}_{\mathrm{c}} = \hat{W}_{\mathrm{o}}, \tag{3.3}$$

where $\hat{W}_{\mathrm{c}}$,$\hat{W}_{\mathrm{o}} \in \mathbb{R}^{n\times n}$ are the controllability and observability gramians of the transformed system, respectively. The reduced model of order n_{r} is given by

$$\hat{x}_{\mathrm{r},k+1} = \hat{A}_{\mathrm{r}}\hat{x}_{\mathrm{r},k} + \hat{G}_{\mathrm{r}} w_k, \tag{3.4}$$

$$y_{\mathrm{r},k} = \hat{C}_{\mathrm{r}}\hat{x}_{\mathrm{r},k} + v_k, \tag{3.5}$$

where $\hat{A}_{\mathrm{r}} \triangleq (T^{-1})_{\mathrm{r}} A T_{\mathrm{r}}$, $\hat{G}_{\mathrm{r}} \triangleq (T^{-1})_{\mathrm{r}} G$, $\hat{C}_{\mathrm{r}} \triangleq CT_{\mathrm{r}}$, $T_{\mathrm{r}} \triangleq T\begin{bmatrix} I_{n_{\mathrm{r}}} & 0_{n_{\mathrm{r}}\times(n-n_{\mathrm{r}})}\end{bmatrix}^{\mathrm{T}}$, and $(T^{-1})_{\mathrm{r}} \triangleq \begin{bmatrix} I_{n_{\mathrm{r}}} & 0_{n_{\mathrm{r}}\times(n-n_{\mathrm{r}})}\end{bmatrix} T^{-1}$.

The method used in [16] propagates the error covariance for a model of order $n_{\mathrm{r}} < n$ truncated according to the Hankel singular values. Furthermore, at each time step k, the full-order error covariance is approximated using the reduced-order model-error covariance by means of

$$\hat{P}_k^{\mathrm{f}} = T_{\mathrm{r}} P_{\mathrm{r},k}^{\mathrm{f}} T_{\mathrm{r}}^{\mathrm{T}}, \tag{3.6}$$

where $P_{\mathrm{r},k}^{\mathrm{f}}$ is the $n_{\mathrm{r}} \times n_{\mathrm{r}}$ reduced-order error-covariance matrix propagated for the reduced-order model (3.4), (3.5), and $\hat{P}_k^{\mathrm{f}}$ is the $n \times n$ approximate full-order error covariance matrix. The resulting forecast and data assimilation steps are given as follows:

Forecast Step

$$x_{k+1}^{\mathrm{f}} = A x_k^{\mathrm{da}}, \tag{3.7}$$

$$P_{\mathrm{r},k+1}^{\mathrm{f}} = \hat{A}_{\mathrm{r}} P_{\mathrm{r},k}^{\mathrm{da}} \hat{A}_{\mathrm{r}}^{\mathrm{T}} + \hat{Q}_{\mathrm{r},k}^{\mathrm{f}}. \tag{3.8}$$

Data Assimilation Step

$$K_{\mathrm{r},k} \triangleq P_{\mathrm{r},k}^{\mathrm{f}} \hat{C}_{\mathrm{r}}^{\mathrm{T}} (\hat{C}_{\mathrm{r}} P_{\mathrm{r},k}^{\mathrm{f}} \hat{C}_{\mathrm{r}}^{\mathrm{T}} + R_k)^{-1}, \tag{3.9}$$

$$K_k = T_{\mathrm{r}} K_{\mathrm{r},k} = \hat{P}_k^{\mathrm{f}} C^{\mathrm{T}} (C\hat{P}_k^{\mathrm{f}} C^{\mathrm{T}} + R_k)^{-1}, \tag{3.10}$$

$$P_{\mathrm{r},k}^{\mathrm{da}} = (I - K_{\mathrm{r},k}\hat{C}_{\mathrm{r}}) P_{\mathrm{r},k}^{\mathrm{f}}, \tag{3.11}$$

$$x_k^{\mathrm{da}} = x_k^{\mathrm{f}} + K_k (y_k - C x_k^{\mathrm{f}}). \tag{3.12}$$

IV. Localized Kalman Filter (LKF)

We now assume that the system (2.1), (2.2) can be partitioned as

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} G_1 w_k \\ G_2 w_k \end{bmatrix}, \tag{4.1}$$

$$y_k = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + v_k, \tag{4.2}$$

where $x_{1,k} \in \mathbb{R}^{n_1}$ and $x_{2,k} \in \mathbb{R}^{n_2}$. Note that y_k depends only on $x_{1,k}$, which means physically that y_k is a local measurement. Truncating (4.1), (4.2) yields

$$x_{1,k+1} = A_{11} x_{1,k} + G_1 w_k, \tag{4.3}$$

$$y_k = C_1 x_{1,k} + v_k, \tag{4.4}$$

which is used for error-covariance propagation and data injection as follows:

Forecast Step

$$x_{k+1}^{\mathrm{f}} = A x_k^{\mathrm{da}}, \tag{4.5}$$

$$P_{1,k+1}^{\mathrm{f}} = A_{11} P_{1,k}^{\mathrm{da}} + Q_{1,k}^{\mathrm{f}}. \tag{4.6}$$

Data Assimilation Step

$$K_{1,k} = P_{1,k}^{\mathrm{f}} C_1^{\mathrm{T}} (C_1 P_{1,k}^{\mathrm{f}} C_1^{\mathrm{T}} + R_k)^{-1}, \tag{4.7}$$

$$P_{1,k}^{\mathrm{da}} = (I - K_{1,k} C_1) P_{1,k}^{\mathrm{f}}, \tag{4.8}$$

$$x_{1,k}^{\mathrm{da}} = x_{1,k}^{\mathrm{f}} + K_{1,k} (y_k - C_1 x_{1,k}^{\mathrm{f}}), \tag{4.9}$$

$$x_{2,k}^{\mathrm{da}} = x_{2,k}^{\mathrm{f}}. \tag{4.10}$$

In (4.6)-(4.8), $P_{1,k}$ is defined as the state-error covariance of the truncated system (4.3), (4.4), that is,

$$P_{1,k}^{\mathrm{f}} \triangleq \mathcal{E}[e_{1,k}^{\mathrm{f}} e_{1,k}^{\mathrm{fT}}], \tag{4.11}$$

where $e_{1,k}^{\mathrm{f}} \triangleq x_{1,k} - x_{1,k}^{\mathrm{f}}$.

V. Localized Kalman Filter with Balanced Reduction (LKFBR)

To apply LKF to the balanced system (3.1), (3.2), we first partition the transformed system (3.1), (3.2) such that

$$\begin{bmatrix} \hat{x}_{1,k+1} \\ \hat{x}_{2,k+1} \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_{1,k} \\ \hat{x}_{2,k} \end{bmatrix} + \begin{bmatrix} \hat{G}_1 w_k \\ \hat{G}_2 w_k \end{bmatrix}, \tag{5.1}$$

$$y_k = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} \begin{bmatrix} \hat{x}_{1,k} \\ \hat{x}_{2,k} \end{bmatrix} + v_k, \tag{5.2}$$

where the dimension of $\hat{x}_{1,k}$ is determined according to the Hankel singular values. Truncating (5.1), (5.2) yields

$$\hat{x}_{1,k+1} = \hat{A}_{11}x_{1,k} + \hat{G}_1 w_k, \tag{5.3}$$

$$y_{\mathrm{r},k} = \hat{C}_1 x_{1,k} + v_k, \tag{5.4}$$

which is used for error-covariance propagation and data injection using the LKF procedures (4.5) - (4.10). Finally, in order to compare the estimates to those of LKF without balanced model reduction given in (4.5), we transform the estimates back to the original coordinates using $x_k^{\mathrm{f}} = T\hat{x}_k^{\mathrm{f}}$.

VI. Localized Kalman Filter with Residual-Subspace Balanced Truncation (LKFRBT)

We can account for the $x_{2,k}$ subsystem in the LKF algorithm by reducing the $x_{2,k}$ subsystem and then augmenting the $x_{1,k}$ subsystem with the reduced $x_{2,k}$ subsystem. To do this, the dynamics of $x_{2,k} \in \mathbb{R}^{n_2}$ in (4.1) are expressed as

$$x_{2,k+1} = A_{22}x_{2,k} + [A_{21}\ G_2]\begin{bmatrix} x_{1,k} \\ w_k \end{bmatrix}, \tag{6.1}$$

$$z_k = A_{12}x_{2,k}, \tag{6.2}$$

to which we apply balanced realization and reduction. The resulting reduced-order model is

$$\hat{x}_{2,\mathrm{r},k+1} = (\hat{T}^{-1})_{\mathrm{r}} A_{22}\hat{T}_{\mathrm{r}}\hat{x}_{2,\mathrm{r},k} + (\hat{T}^{-1})_{\mathrm{r}}[A_{21}\ G_2]\begin{bmatrix} x_{1,k} \\ w_k \end{bmatrix}, \tag{6.3}$$

$$z_{\mathrm{r},k} = A_{12}\hat{T}_{\mathrm{r}}x_{2,\mathrm{r},k}, \tag{6.4}$$

where $\hat{T}$ is the balanced transformation for $x_{2,k}$ subsystem (6.1), (6.2), $\hat{x}_{2,\mathrm{r},k} \in \mathbb{R}^{n_{\mathrm{r}}}$, where $n_{\mathrm{r}} < n_2$ is the reduced approximation of $\hat{x}_{2,k} \triangleq \hat{T}^{-1}x_{2,k}$, $\hat{T}_{\mathrm{r}} \triangleq \hat{T}\left[I_{n_{\mathrm{r}}}\ \ 0_{n_{\mathrm{r}}\times(n_2-n_{\mathrm{r}})}\right]^{\mathrm{T}}$, and $(\hat{T}^{-1})_{\mathrm{r}} \triangleq \left[I_{n_{\mathrm{r}}}\ \ 0_{n_{\mathrm{r}}\times(n_2-n_{\mathrm{r}})}\right]\hat{T}^{-1}$.

By replacing the corresponding terms of (4.1), (4.2) with terms of (6.3), (6.4), we obtain

$$\begin{bmatrix} x_{1,k+1} \\ \hat{x}_{2,\mathrm{r},k+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12}\hat{T}_{\mathrm{r}} \\ (\hat{T}^{-1})_{\mathrm{r}}A_{21} & (\hat{T}^{-1})_{\mathrm{r}}A_{22}\hat{T}_{\mathrm{r}} \end{bmatrix}\begin{bmatrix} x_{1,k} \\ \hat{x}_{2,\mathrm{r},k} \end{bmatrix} + \begin{bmatrix} G_1 \\ (\hat{T}^{-1})_{\mathrm{r}}G_2 \end{bmatrix} w_k, \tag{6.5}$$

$$y_k = \begin{bmatrix} C_1 & 0 \end{bmatrix}\begin{bmatrix} x_{1,k} \\ \hat{x}_{2,\mathrm{r},k} \end{bmatrix} + v_k. \tag{6.6}$$

The error covariance is propagated by the reduced system (6.5), (6.6) whose dimension is n_1+n_{r}. The forecast and data assimilation steps are the same as those of KFEBRM (3.7)-(3.12), replacing the reduced-order system (3.4), (3.5) with the system (6.5), (6.6), where $T_{\mathrm{r}} \in \mathbb{R}^{n\times(n_1+n_{\mathrm{r}})}$ is defined by

$$T_{\mathrm{r}} \triangleq \begin{bmatrix} I_{n_1\times n_1} & 0 \\ 0 & \hat{T}_{\mathrm{r}} \end{bmatrix}. \tag{6.7}$$

VII. Complementary Static Open-Loop Steady-State (OLSS) and Closed-Loop Steady-State (CLSS) Error-Covariance-Based Gain

KFEBRM, LKFBR, and LKFRBT account for interactions with the truncated subsystem by means of balanced reduction. Rather than using balanced reduction, we now compensate the reduced-order error-covariance of LKF with a complementary open-loop or closed-loop steady-state error-covariance. We begin by proving that the performance of an estimator that uses a steady-state open-loop or a closed-loop error-covariance-based static gain is better than or equal to the open-loop or the closed-loop performance, respectively. The proofs provide a justification for the complementary steady-state error-covariance approach.

We start from the closed-loop case. Then, the open-loop case is derived as the special case when the estimator gain is zero. Now consider the closed-loop estimator

$$\hat{x}_{\mathrm{CL},k+1} = A\hat{x}_{\mathrm{CL},k} + K(y_k - C\hat{x}_{\mathrm{CL},k}), \tag{7.1}$$

where K is an estimator gain chosen so that $A - KC$ is asymptotically stable. Define the closed-loop state-estimation error

$$e_{\mathrm{CL},k} \triangleq x_k - \hat{x}_k, \tag{7.2}$$

and the closed-loop error covariance

$$P_{\mathrm{CL},k} \triangleq \mathcal{E}\left[e_{\mathrm{CL},k}e_{\mathrm{CL},k}^{\mathrm{T}}\right]. \tag{7.3}$$

Subtracting (7.1) from (2.1) yields

$$e_{\mathrm{CL},k+1} = (A - KC)e_{\mathrm{CL},k} + Gw_k - Kv_k \tag{7.4}$$

so that

$$P_{\mathrm{CL},k+1} = (A-KC)P_{\mathrm{CL},k}(A-KC)^{\mathrm{T}} + Q + KRK^{\mathrm{T}}. \tag{7.5}$$

Since $A - KC$ is asymptotically stable, the steady-state closed-loop error covariance defined by

$$P_{\mathrm{CL}} \triangleq \lim_{k\to\infty} P_{\mathrm{CL},k} \tag{7.6}$$

exists and satisfies the Lyapunov equation

$$P_{\mathrm{CL}} = (A - KC)P_{\mathrm{CL}}(A - KC)^{\mathrm{T}} + Q + KRK^{\mathrm{T}}. \tag{7.7}$$

Next, using steady-state closed-loop error covariance we define the estimator gain

$$K_{\mathrm{CL}} \triangleq AP_{\mathrm{CL}}C^{\mathrm{T}}\hat{R}_{\mathrm{CL}}^{-1}, \tag{7.8}$$

where

$$\hat{R}_{\mathrm{CL}} \triangleq CP_{\mathrm{CL}}C^{\mathrm{T}} + R. \tag{7.9}$$

Consider an estimator based on the estimator gain in (7.8), that is,

$$\hat{x}_{k+1} = A\hat{x}_k + K_{\mathrm{CL}}(y_k - C\hat{x}_k). \tag{7.10}$$

Subtracting (7.10) from (2.1) yields the error dynamics

$$e_{k+1} = (A - K_{\mathrm{CL}}C)e_k + Gw_k - K_{\mathrm{CL}}v_k. \tag{7.11}$$

Then the error covariance defined in (7.3) is propagated using

$$P_{k+1} = (A-K_{\mathrm{CL}}C)P_k(A-K_{\mathrm{CL}}C)^{\mathrm{T}} + Q + K_{\mathrm{CL}}RK_{\mathrm{CL}}^{\mathrm{T}}. \tag{7.12}$$

The following result shows that the performance of the estimator based on the closed-loop error covariance P_{CL} is better than the performance of the estimator in (7.1).

Proposition VII.1. *Assume that $P_k \leqslant P_{\mathrm{CL}}$. Then, $P_{k+1} \leqslant P_{\mathrm{CL}}$.*

Proof. Subtracting (7.12) from (7.7) yields

$$P_{\rm CL} - P_{k+1} = (A-KC)P_{\rm CL}(A-KC)^{\rm T} \quad (7.13)$$
$$- (A-K_{\rm CL}C)P_k(A-K_{\rm CL}C)^{\rm T} + KRK^{\rm T} - K_{\rm CL}RK_{\rm CL}^{\rm T}.$$

Adding and subtracting $(A-K_{\rm CL}C)P_{\rm CL}(A-K_{\rm CL}C)^{\rm T}$ to the right hand side of (7.13) yields

$$\begin{aligned} P_{\rm CL}-P_{k+1} &= (A-K_{\rm CL}C)(P_{\rm CL}-P_k)(A-K_{\rm CL}C)^{\rm T}+KRK^{\rm T} \\ &-K_{\rm CL}RK_{\rm CL}^{\rm T}-KCP_{\rm CL}A^{\rm T}-AP_{\rm CL}C^{\rm T}K^{\rm T}+KCP_{\rm CL}C^{\rm T}K^{\rm T} \\ &+K_{\rm CL}CP_{\rm CL}A^{\rm T}+AP_{\rm CL}C^{\rm T}K_{\rm CL}^{\rm T}-K_{\rm CL}CP_{\rm CL}C^{\rm T}K_{\rm CL}^{\rm T}. \quad (7.14) \end{aligned}$$

Using (7.8) and (7.9) in (7.14) yields

$$\begin{aligned} P_{\rm CL}-P_{k+1} &= (A-K_{\rm CL}C)(P_{\rm CL}-P_k)(A-K_{\rm CL}C)^{\rm T} \quad (7.15) \\ &+K\hat{R}_{\rm CL}K^{\rm T}+K_{\rm CL}\hat{R}_{\rm CL}K_{\rm CL}^{\rm T}-K\hat{R}_{\rm CL}K_{\rm CL}^{\rm T}-K_{\rm CL}\hat{R}_{\rm CL}K^{\rm T}. \end{aligned}$$

Hence,

$$\begin{aligned} P_{\rm CL}-P_{k+1} =& (A-K_{\rm CL}C)(P_{\rm CL}-P_k)(A-K_{\rm CL}C)^{\rm T} \\ &+ (K_{\rm CL}-K)\hat{R}_{\rm CL}(K_{\rm CL}-K)^{\rm T}. \end{aligned} \quad (7.16)$$

Therefore, $P_{k+1} \leq P_{\rm CL}$. □

Hence, if $P_0 \leqslant P_{\rm CL}$, then for all $k \geqslant 0$, $P_k \leqslant P_{\rm CL}$. Next, for the open-loop case, let $K = 0$ and assume that A is asymptotically stable. We define $P_{\rm OL}$ as the corresponding open-loop steady-state error covariance. Then the Corollary VII.1 follows.

Corollary VII.1. Assume that $P_k \leqslant P_{\rm OL}$. Then $P_{k+1} \leqslant P_{\rm OL}$.

Based on Proposition VII.1 and Corollary VII.1, we combine LKF gain with the steady-state error-covariance-based gain to inject data into all of the states for potentially better performance than that of LKF alone.

VIII. LKF with Complementary Open-Loop Steady-State Error Covariance (LKFCOLC)

At each time step, the local-system error-covariance $P_{1,k}^{\rm f}$ is propagated by (4.6), (4.7), and (4.8), whereas the open-loop steady-state error covariance is given by

$$P_{\rm OL} = \begin{bmatrix} P_{\rm OL,11} & P_{\rm OL,12} \\ P_{\rm OL,12} & P_{\rm OL,22} \end{bmatrix}, \quad (8.1)$$

where $P_{\rm OL}$ is the steady-state error covariance that satisfies

$$AP_{\rm OL}A^{\rm T} - P_{\rm OL} + Q = 0. \quad (8.2)$$

Note that $P_{\rm OL}$ is partitioned in 8.1 according to (4.1), (4.2).

Next, we inject data into the forecast state $x_{2,k}^{\rm f}$ of LKF using the open-loop steady-state covariance. That is, (4.10) is modified as

$$x_{2,k}^{\rm da} = x_{2,k}^{\rm f} + K_2(y_k - C_1 x_{1,k}^{\rm f}), \quad (8.3)$$

where

$$K_2 \triangleq P_{\rm OL,12}C_1^{\rm T}(C_1P_{\rm OL,11}C_1^{\rm T}+R_k)^{-1}. \quad (8.4)$$

Finally, the estimator gain K_k for full-state data injection composed of (4.9), (8.3) is given by

$$K_k \triangleq \begin{bmatrix} K_{1,k} \\ K_2 \end{bmatrix} = \begin{bmatrix} P_{1,k}^{\rm f}C_1^{\rm T}(C_1P_{1,k}^{\rm f}C_1^{\rm T}+R_k)^{-1} \\ P_{\rm OL,12}C_1^{\rm T}(C_1P_{\rm OL,11}C_1^{\rm T}+R_k)^{-1} \end{bmatrix}. \quad (8.5)$$

IX. LKF with Complementary Closed-Loop Steady-State Error Covariance (LKFCCLC)

The LKFCOLC technique may not have good performance when the complementary open-loop steady-state error covariance and optimal error covariance are significantly different. In this case, we use the complementary closed-loop steady-state covariance when LKF is applied. Hence, let $K_{\rm LKF} = [K_1^{\rm T}\ 0]^{\rm T}$, where K_1 is the steady-state Kalman gain of LKF given by (4.7), and let $P_{\rm CL}$ satisfy

$$\begin{aligned} (A-K_{\rm LKF}C)P_{\rm CL}(A-K_{\rm LKF}C)^{\rm T} - P_{\rm CL} \\ + K_{\rm LKF}RK_{\rm LKF}^{\rm T} + Q = 0. \end{aligned} \quad (9.1)$$

Now partition $P_{\rm CL}$ as

$$P_{\rm CL} = \begin{bmatrix} P_{\rm CL,11} & P_{\rm CL,12} \\ P_{\rm CL,12} & P_{\rm CL,22} \end{bmatrix}, \quad (9.2)$$

We obtain the estimator gain K_k by means of (8.5) replacing $P_{\rm OL,11}$, $P_{\rm OL,12}$ with $P_{\rm CL,11}$, $P_{\rm CL,12}$, respectively.

X. Simulation Example

We apply the methods introduced in sections II-IX to a compartmental model [25], which involves states whose values are nonnegative quantities. This compartmental model is based on the physics of the processes by which material or energy is exchanged among coupled subsystems. In addition, conservation laws account for the flow of such quantities among subsystems.

A schematic diagram of the compartmental model is shown at Fig 1. The total number of cells n is 25 for simulations with one state per cell. We assume that the states of the first five cells are measured. Hence, the size of the localized system n_1 is set to 5. All σ_{ii} 's are set to 0.1 and all σ_{ij} $(i \neq j)$ are set to 0.44.

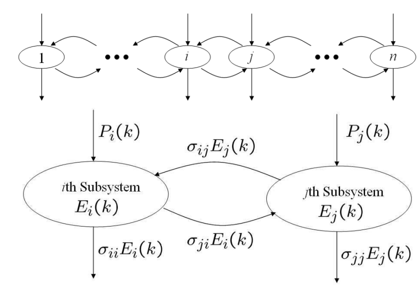

Fig. 1. Compartmental model involving interconnected subsystems.

We simulate two cases. Case 1 involves a single-input disturbance in which the input matrix G is the $n \times 1$ ones matrix. Hence, Figure 2(a) shows one dominant Hankel singular value. In Case 2, n mutually independent disturbances are spread out over all of the cells, and thus the disturbance input matrix G of (2.1) is the $n \times n$ identity matrix. In Case 2, as can be seen in Figure 2(b), the Hankel singular values decrease gradually and thus there is no definite model-truncation threshold.

Simulation results for KFEBRM, LKF, LKFBR, LKFRBT, LKFCOLC, and LKFCCLC are shown in figure 3. LKFCCLC shows the best performance in Case 2. In Case 1, KFEBRM and LKFBR show the best performance.

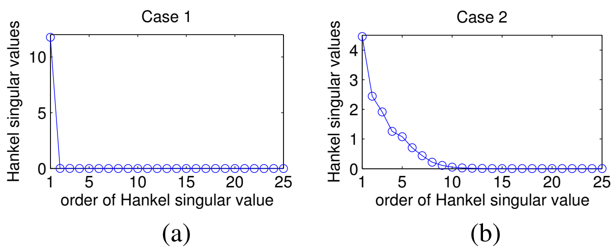

Fig. 2. Case 1 Hankel singular values (left) and Case 2 Hankel singular values (right)

Balanced-model-based methods perform well for Case 1 because of the rapidly decreasing Hankel singular values. However, the performance of LKFCOLC and LKFCCLC are comparable to that of the balanced-model-based methods. Moreover, LKFCCLC is more close to the optimal Kalman filter with higher model order than LKFCOLC. We summarize the properties and performance ranks of each method in Table I.

Estimators with an OLSS covariance-based static gain and CLSS covariance-based static gain consistently perform better than without the static gain as shown in figures 4. Moreover, figures 5 shows that LKF compensated by either OLSS or CLSS covariance show improved performance than LKF alone even when an erroneous Q is used to obtain the OLSS and CLSS covariances.

XI. Conclusions

We presented several Kalman filters for reduced-order error-covariance propagation and compared them through numerical studies. We conducted numerical studies for two extreme cases of Hankel singular values. In both cases, LKFCOLC and LKFCCLC show good performance. When there are a few dominant Hankel singular values, LKFCCLC can be applied efficiently without the need for a similarity transformation that may be prohibitive in large-scale systems.

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TABLE I: Comparisons of reduced-order error-covariance Kalman filters.

Properties of Methods / Method	number of states for data injection	model order for covariance propagation	requires balancing transform?	requires local measurements?	performance rank for Case 1	performance rank for Case 2
KFEBRM	25	5	yes	no	1	3
LKF	5	5	no	yes	5	5
LKFBR	5	5	yes	yes	1	3
LKFRBT	25	5	yes	yes	5	5
LKFCOLC	25	5	no	yes	3	2
LKFCCLC	25	5	no	yes	4	1

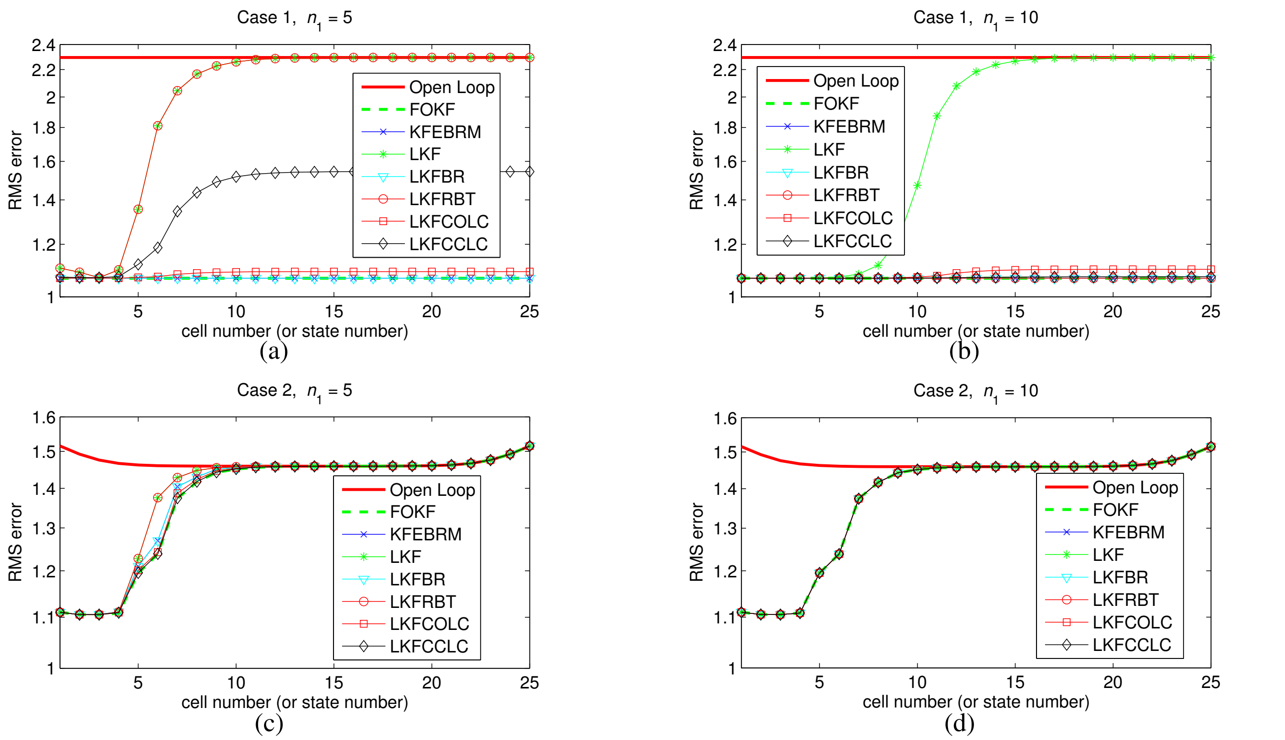

Fig. 3. RMS error of each cell (spatial distribution of errors) with respect to each method when the order of the reduced model is $n_1 = 5, 10$. (a) and (b) are for Case 1 while (c) and (d) are for Case 2. Note that LKF and LKFRBT are identical when $n_1 = 5$. When n_1 is fixed at 5, it can be seen from (a), (c) that KFEBRM and LKFBR show the best performance in Case 1 whereas LKFCCLC shows the best performance in Case 2. Balanced-model-based methods perform well for Case 1 because of the rapidly decreasing Hankel singular values. However, note that the performance of LKFCOLC and LKFCCLC are comparable to that of the balanced-model-based methods. Moreover, LKFCCLC is more close to the optimal Kalman filter with higher model order than LKFCOLC.

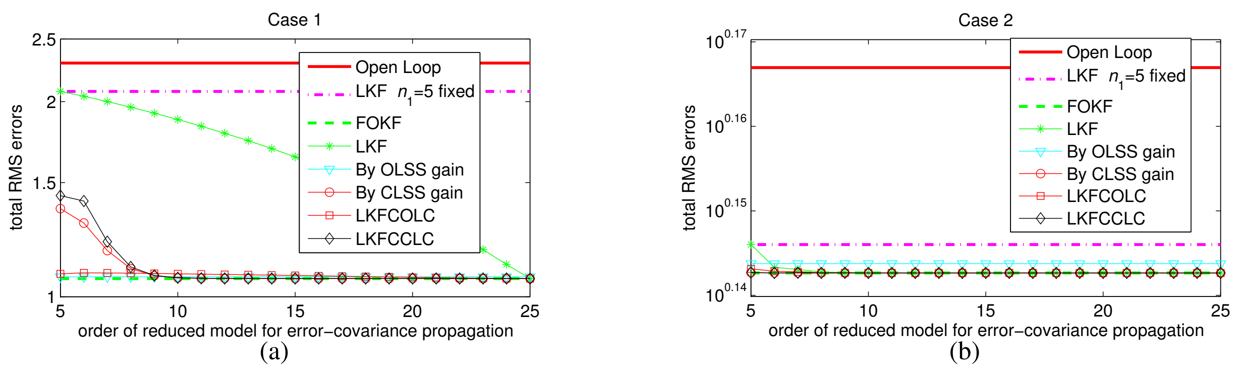

Fig. 4. Total RMS errors of estimators with open-loop steady-state (OLSS) covariance-based gain, closed-loop steady-state (CLSS) covariance-based gain, LKFCOLC and LKFCCLC when the order of the reduced model for error-covariance propagation is increased. LKF compensated by OLSS and CLSS covariance show significantly improved performance over LKF alone.

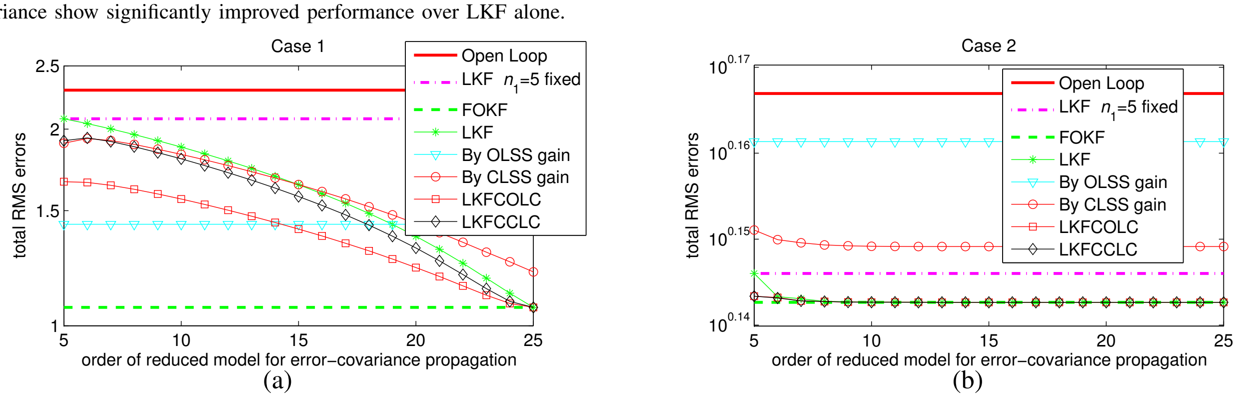

Fig. 5. Total RMS errors of estimators with open-loop steady-state (OLSS) covariance-based gain, closed-loop steady-state (CLSS) covariance-based gain, LKFCOLC and LKFCCLC when the order of the reduced model for error-covariance propagation is increased and $0.01Q$ is taken as an erroneous disturbance covariance to obtain OLSS and CLSS covariances. LKF compensated by OLSS and CLSS covariance show improved performance over LKF alone although the erroneous disturbance covariance is used in obtaining the OLSS and CLSS covariances.