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Unbiased Minimum-variance Filtering for Input Reconstruction

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Abstract— **In this paper, we introduce the concept of input and state observability, that is, conditions under which both the unknown input and state can be estimated from the output measurements. We discuss sufficient and necessary conditions for a discrete-time system to be input and state observable. Next, we derive an unbiased minimum-variance filter to estimate the unknown input and the state, when the state space matrices are known. We show that the Kalman filter and other filters in the literature are special cases of the filter derived in this paper. Finally, we present an illustrative example.**

I. Introduction

Systems with unknown inputs have received considerable attention in the past [3–8, 10–23, 25–28]. An active area of research is state reconstruction with known model equations and unknown inputs. Among the popular approaches are full-order observers [4, 6, 15, 16, 28], reduced-order observers [7, 8, 19, 21], geometric approach [3], generalized inverse approach [19, 21], trial-and-error approach [26], and the singular value decomposition [8]. A widely used approach is to model the unknown inputs as outputs of a known dynamic system and incorporate the input dynamics with the plant dynamics [1, 13]. However, this approach increases the dimension of the observer and is limited to specific types of inputs.

The unknown inputs in a dynamical system may represent unknown external drivers, input uncertainty, state uncertainty, or instrument faults. Thus unknown-input reconstruction has several important applications in uncertainty estimation and fault detection. Input reconstruction also has applications in filtering and coding theory. In some early work, input reconstruction is achieved through system inversion [23, 25]. More recently, methods for input reconstruction using optimal filters are developed in [4, 10, 14, 27].

An unbiased minimum-variance filter for discrete-time stochastic systems with arbitrary unknown inputs is derived in [16]. Finally, [9] presents an alternative derivation of the filter in [16] and uses the filter to reconstruct the unknown inputs. A related filter is presented in [24].

A related problem is the concept of input and state observability, which is the ability to reconstruct the inputs and states using only output measurements. Necessary and sufficient conditions for input and state observability for continuous-time systems in terms of the invariant zeros of the system are presented in [4, 8, 12, 14, 19]. Input and state observability for discrete-time systems are considered in [14].

In this paper, we adopt a rigorous approach to examine conditions under which both the input and state can be estimated from the output measurements. We discuss necessary and sufficient conditions for a discrete-time system to be input and state observable and derive simple tests for input and state observability. Since no assumptions on the input are made, the unknown input can be either an unmodeled exogenous signal or an unknown function of the states.

Once the necessary and sufficient conditions for input and state observability are presented, we develop optimal filtering techniques that take advantage of input and state observability. The Kalman filter and the filters derived in [16, 24] are shown to be special cases of the unbiased minimum-variance filter derived in this paper.

Finally, we present an example to illustrate the methods.

II. Input and State Observability

A. No Feedthrough Case

Consider the time-varying system

$$x_{k+1} = A_k x_k + H_k e_k, \qquad \text{(II.1)}$$
$$y_k = C_k x_k, \qquad \text{(II.2)}$$

where $x_k \in \mathbb{R}^n$, $e_k \in \mathbb{R}^p$, $y_k \in \mathbb{R}^l$, $A_k \in \mathbb{R}^{n\times n}$, $H_k \in \mathbb{R}^{n\times p}$, and $C_k \in \mathbb{R}^{l\times n}$. Without loss of generality, we assume $l \le n$, $\max_k[\text{rank}(C_k)] = l > 0$, and $\max_k[\text{rank}(H_k)] = p > 0$. No assumption on the inputs e_k are made. Hence, the signal e_k can either be an exogenous input or a nonlinear function of the states. For the sake of simplicity, we assume that the initial time $k_0 = 0$.

Throughout this paper, r denotes a nonnegative integer. Furthermore, for convenience, every vector or matrix with zero rows or zero columns is an empty matrix. Define the data vectors $\mathcal{Y}_r \in \mathbb{R}^{(r+1)l}$ and $\mathcal{E}_r \in \mathbb{R}^{(r+1)p}$ as

$$\mathcal{Y}_r \triangleq \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_r \end{bmatrix}, \quad \mathcal{E}_r \triangleq \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_r \end{bmatrix}. \qquad \text{(II.3)}$$

Definition II.1. Let $r \ge 1$. Then the *input and state*

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unobservable subspace $\mathfrak{U}_r$ of (II.1), (II.2) is the subspace

$$\mathfrak{U}_r \triangleq \left\{ \begin{bmatrix} x_0 \\ \mathcal{E}_{r-1} \end{bmatrix} \in \mathbb{R}^{n+rp} \colon \mathcal{Y}_r = 0 \right\}. \quad \text{(II.4)}$$

We define $\Gamma_r \in \mathbb{R}^{(r+1)l\times n}$, $M_r \in \mathbb{R}^{(r+1)l\times rp}$, and $\Psi_r \in \mathbb{R}^{(r+1)l\times(n+rp)}$ by

$$\Gamma_r \triangleq \begin{bmatrix} C_0 \\ C_1A_0 \\ C_2A_1A_0 \\ \vdots \\ C_rA_{r-1}\cdots A_0 \end{bmatrix}, \quad M_r \triangleq$$
$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ C_1H_0 & 0 & \cdots & 0 \\ C_2A_1H_0 & C_2H_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_rA_{r-1}\cdots A_1H_0 & C_rA_{r-1}\cdots A_2H_1 & \cdots & C_rH_{r-1} \end{bmatrix}, \quad \text{(II.5)}$$

and

$$\Psi_r \triangleq \begin{bmatrix} \Gamma_r & M_r \end{bmatrix}. \quad \text{(II.6)}$$

Note that M_0 is an empty matrix and thus $\Psi_0 = \Gamma_0 = C_0$. Next, from (II.1), (II.2), we can write

$$\mathcal{Y}_r = \Gamma_r x_0 + M_r\mathcal{E}_{r-1} = \Psi_r \begin{bmatrix} x_0 \\ \mathcal{E}_{r-1} \end{bmatrix}, \quad \text{(II.7)}$$

so that

$$\mathfrak{U}_r = \mathcal{N}(\Psi_r), \quad \text{(II.8)}$$

where $\mathcal{N}$ denotes null space. Next, define the positive integer

$$r_0 \triangleq \begin{cases} \max\{\lceil \frac{n-l}{l-p} \rceil, 1\}, & p < l, \\ 1, & p = l, \end{cases} \quad \text{(II.9)}$$

where $\lceil a \rceil$ denotes the smallest integer greater than or equal to a. Note that r_0 is not defined in the case $p > l$.

Proposition II.1. Assume that $n \geq 2$ and $p \leq l$. Then $r_0 \leq n-1$.

Proof. Suppose $p = l$. Then $n-1 \geq 1 = r_0$. Next, suppose $p < l$. If $\lceil \frac{n-l}{l-p} \rceil \leq 1$ then $n-1 \geq 1 = r_0$. If $\lceil \frac{n-l}{l-p} \rceil > 1$, then, since $n-1 > n-l$ and $l-p \geq 1$, it follows that $r_0 = \lceil \frac{n-l}{l-p} \rceil \leq \lceil n-l \rceil \leq \lceil n-1 \rceil = n-1$. □

Proposition II.2. Let $r \geq 1$. If $\mathfrak{U}_r = \{0\}$, then the following statements hold:

1) $p \leq l$.
2) If $p = l$, then $p = l = n$.
3) $r \geq r_0$.
4) $\text{rank}(\Gamma_{n-1}) = n$.
5) $\text{rank}(C_rH_{r-1}) = p$.

Proposition II.3. Assume that either $p < l$ or $p = l = n$. Then $n + rp \leq (r+1)l$ for all $r \geq r_0$.

Proof. Suppose $p = l = n$. Then $n + rp = (r+1)l$ for all $r > 0$. Next, suppose $p < l$, let $r \geq r_0$ and assume $(r+1)l < n + rp$ so that $rl - rp < n - l$. Hence $r < \frac{n-l}{l-p}$, and thus $\lceil \frac{n-l}{l-p} \rceil \leq r_0 < \frac{n-l}{l-p}$, which is a contradiction. Thus $n + rp \leq (r+1)l$. □

Proposition II.3 implies that if $p < l$ or $p = l = n$, then for all $r \geq r_0$ the number of columns of Ψ_r is less than or equal to the number of rows of Ψ_r.

Definition II.2. The system (II.1), (II.2) is *input and state observable* if $\mathfrak{U}_r = \{0\}$ for all $r \geq r_0$.

Definition II.2 implies that if (II.1), (II.2) is input and state observable, then, for all $r \geq r_0$, the initial condition x_0 and input sequence $\{e_i\}_{i=0}^{r-1}$ are uniquely determined from the measured output sequence $\{y_i\}_{i=0}^{r}$.

Theorem II.1. The following statements are equivalent:

1) (II.1), (II.2) is input and state observable.
2) For all $r \geq r_0$, $\mathcal{Y}_r = 0$ if and only if $\begin{bmatrix} x_0 \\ \mathcal{E}_{r-1} \end{bmatrix} = 0$.
3) For all $r \geq r_0$, $\text{rank}(\Psi_r) = n + rp$.
4) $\text{rank}(\Psi_{n-1}) = n + (n-1)p$ and for all $r \geq r_0$, $\text{rank}(C_rH_{r-1}) = p$.

Proof. From Definition II.1 and Definition II.2 it follows that 1) ⇒ 2). Using (II.7), 2) ⇒ 3). To prove 3) ⇒ 4), since for all $r \geq r_0$, $\text{rank}(\Psi_r) = n + rp$, it follows that $\text{rank}(C_rH_{r-1}) = p$. Hence, for all $\hat{r} \geq r_0$ $\text{rank}(\Psi_{\hat{r}}) = \text{rank}(\Psi_{\hat{r}-1}) + p$. Hence, since $n-1 \geq r_0$, we have $\text{rank}(\Psi_{n-1}) = n + (n-1)p$. Finally to show 4) ⇒ 1), we consider two cases. First, suppose $n = 1$. In this case C_k and H_k are nonzero scalars, and hence it follows that $\text{rank}(\Psi_r) = n + rp$ for all $r \geq r_0$ and hence $\mathfrak{U}_r = \{0\}$ for all $r \geq r_0$. Next, suppose $n \geq 2$. In this case, 4) implies that $\text{rank}(\Psi_r) = \text{rank}(\Psi_{r-1}) + p$ for all $r \geq r_0$. Next, since $n-1 \geq r_0$, it follows that, for all $r \geq r_0$, $\text{rank}(\Psi_r) = \text{rank}(\Psi_{n-1}) + (r-n+1)p$. Thus $\text{rank}(\Psi_r) = n + rp$ for all $r \geq r_0$ and hence $\mathfrak{U}_r = \{0\}$ for all $r \geq r_0$. □

Theorem II.1 shows that (II.1), (II.2) is input and state observable if and only if Ψ_r has full column rank for all $r \geq r_0$. In this case the unique solution of (II.7) is

$$\begin{bmatrix} x_0 \\ \mathcal{E}_{r-1} \end{bmatrix} = \Psi_r^\dagger \mathcal{Y}_r, \quad \text{(II.10)}$$

where † represents the Moore-Penrose generalized inverse $\Psi_r^\dagger = (\Psi_r^{\mathrm{T}}\Psi_r)^{-1}\Psi_r^{\mathrm{T}}$.

B. Feedthrough Case

Next, we consider the system

$$x_{k+1} = A_kx_k + H_ke_k, \quad \text{(II.11)}$$
$$y_k = C_kx_k + G_ke_k, \quad \text{(II.12)}$$

where $G_k \in \mathbb{R}^{l\times p}$, while A_k, H_k, C_k, x_k, e_k, and y_k are defined as in (II.1), (II.2). Without loss of generality, we assume $l \leq n$, $\max_k[\text{rank}(C_k)] = l > 0$, and $\max_k \left(\text{rank} \begin{bmatrix} H_k \\ G_k \end{bmatrix}\right) = p > 0$. Due to the term $G_k e_k$, the output y_k is directly affected by e_k as well as by the past values of e_k. Therefore, we have

$$\mathcal{Y}_r = \bar{\Psi}_r \begin{bmatrix} x_0 \\ \mathcal{E}_r \end{bmatrix}, \tag{II.13}$$

where $\bar{\Psi}_r \triangleq \begin{bmatrix} \Gamma_r & \bar{M}_r \end{bmatrix} \in \mathbb{R}^{(r+1)l \times [n+(r+1)p]}$, and

$$\bar{M}_r = \begin{bmatrix} G_0 & 0 & \cdots & 0 & 0 \\ C_1 H_0 & G_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ C_{r-1}A_{r-2}\cdots A_1 H_0 & C_{r-1}A_{r-2}\cdots A_2 H_1 & \cdots & G_{r-1} & 0 \\ C_r A_{r-1}\cdots A_1 H_0 & C_r A_{r-1}\cdots A_2 H_1 & \cdots & C_r H_{r-1} & G_r \end{bmatrix}. \tag{II.14}$$

Furthermore, we have the following definition.

Definition II.3. Let $r \geq 0$. Then the *input and state unobservable subspace* $\bar{\mathfrak{U}}_r$ of (II.11), (II.12) is the subspace

$$\bar{\mathfrak{U}}_r \triangleq \left\{ \begin{bmatrix} x_0 \\ \mathcal{E}_r \end{bmatrix} \in \mathbb{R}^{n+(r+1)p} \colon \mathcal{Y}_r = 0 \right\}. \tag{II.15}$$

The input and state unobservable subspace is given by $\bar{\mathfrak{U}}_r = \mathcal{N}(\bar{\Psi}_r)$. Next, if $p < l$ then define the integer

$$\bar{r}_0 \triangleq \left\lceil \tfrac{n}{l-p} \right\rceil - 1. \tag{II.16}$$

Since $n > l - p$ it follows that $\bar{r}_0 \geq 1$.

Proposition II.4. Let $r \geq 0$. If $\bar{\mathfrak{U}}_r = \{0\}$, then the following statements hold:

1) $p < l$.
2) $n > 1$.
3) $r \geq \bar{r}_0$.
4) $\text{rank}(\Gamma_{n-1}) = n$.
5) $\text{rank}(G_r) = p$.

Definition II.4. The system (II.11), (II.12) is *input and state observable* if $\bar{\mathfrak{U}}_r = \{0\}$ for all $r \geq \bar{r}_0$.

Theorem II.2. *The following statements are equivalent:*

1) *(II.11), (II.12) is input and state observable.*
2) *For all $r \geq \bar{r}_0$, $\mathcal{Y}_r = 0$ if and only if $\begin{bmatrix} x_0 \\ \mathcal{E}_r \end{bmatrix} = 0$.*
3) *For all $r \geq \bar{r}_0$ rank$(\bar{\Psi}_r) = n + (r+1)p$.*
4) *rank$(\bar{\Psi}_{n-1}) = n(p+1)$ and for all $k \geq r_0$, rank$(G_k) = p$.*

Finally, if (II.11), (II.12) is input and state observable, then Theorem II.2 implies that $\bar{\Psi}_r$ is full column rank for all $r \geq \bar{r}_0$. In this case the unique solution of (II.13) is

$$\begin{bmatrix} x_0 \\ \mathcal{E}_r \end{bmatrix} = \bar{\Psi}_r^{\dagger} \mathcal{Y}_r. \tag{II.17}$$

III. Unbiased Minimum-variance Filter for Input and State Observable Systems

Consider the time-varying system

$$x_{k+1} = A_k x_k + B_k u_k + H_k e_k + w_k, \tag{III.1}$$
$$y_k = C_k x_k + D_k u_k + G_k e_k + v_k. \tag{III.2}$$

where x_k, y_k, e_k, A_k, C_k, H_k and G_k are defined as in section 2, while $u_k \in \mathbb{R}^m$, $B_k \in \mathbb{R}^{n\times m}$ and $D_k \in \mathbb{R}^{l\times m}$. We assume that A_k, B_k, C_k, D_k, H_k, and G_k are known, while e_k is unknown. $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^l$ are unknown Gaussian white noise sequences with known covariances Q_k and R_k respectively. We say that (III.1), (III.2) is input and state observable if it is input and state observable with $B_k \equiv 0$ and $D_k \equiv 0$. We assume that (III.1), (III.2) is input and state observable.

We consider a filter of the form

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + L_{k+1}(y_{k+1} - C_{k+1}\hat{x}_{k+1|k} - D_{k+1}u_{k+1}), \tag{III.3}$$
$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + B_k u_k. \tag{III.4}$$

The state estimation error is

$$\varepsilon_k \triangleq x_{k+1} - \hat{x}_{k+1|k+1}, \tag{III.5}$$

and the error covariance matrix is defined as

$$P_{k+1|k+1} \triangleq \mathbb{E}\left[\varepsilon_{k+1}\varepsilon_{k+1}^{\mathrm{T}}\right], \tag{III.6}$$

where $\mathbb{E}$ is the expected value. The filter is unbiased if and only if

$$\mathbb{E}[x_{k+1} - \hat{x}_{k+1|k+1}] = 0. \tag{III.7}$$

Furthermore, (III.7) can be written as

$$\mathbb{E}[A_k\varepsilon_k + H_k e_k + w_k - L_{k+1}(C_{k+1}A_k\varepsilon_k + C_{k+1}H_k e_k + C_{k+1}w_k + v_{k+1} + G_{k+1}e_{k+1})] = 0. \tag{III.8}$$

Since e_k is arbitrary, (III.8) implies

$$L_{k+1}G_{k+1} = 0, \tag{III.9}$$

and

$$(I - L_{k+1}C_{k+1})H_k = 0. \tag{III.10}$$

Lemma III.1. *If (III.9) and (III.10) are satisfied, the error covariance matrix $P_{k+1|k+1}$ is given by*

$$P_{k+1|k+1} = L_{k+1}\tilde{R}_{k+1}L_{k+1}^{\mathrm{T}} - F_{k+1}L_{k+1}^{\mathrm{T}} - L_{k+1}F_{k+1}^{\mathrm{T}} + P_{k+1|k}, \tag{III.11}$$

where

$$P_{k+1|k} \triangleq A_k P_{k|k} A_k^{\mathrm{T}} + Q_k, \tag{III.12}$$
$$\tilde{R}_{k+1} \triangleq C_{k+1}P_{k+1|k}C_{k+1}^{\mathrm{T}} + R_{k+1}, \tag{III.13}$$
$$F_{k+1} \triangleq P_{k+1|k}C_{k+1}^{\mathrm{T}}. \tag{III.14}$$

Proof.

$$\begin{aligned}
&P_{k+1|k+1}\\
&\quad= \mathbb{E}\left[\varepsilon_{k+1}\varepsilon_{k+1}^{\mathrm{T}}\right]\\
&\quad= \mathbb{E}\Big[\big(A_k x_k + H_k e_k + w_k - A_k\hat{x}_{k|k}\\
&\qquad -L_{k+1}(C_{k+1}A_k x_k + C_{k+1}H_k e_k + C_{k+1}w_k\\
&\qquad +G_k e_{k+1} + v_{k+1} - C_{k+1}A_k\hat{x}_{k|k})\big)\\
&\qquad \times\big(A_k x_k + H_k e_k + w_k - A_k\hat{x}_{k|k}\\
&\qquad -L_{k+1}(C_{k+1}A_k x_k + C_{k+1}H_k e_k + C_{k+1}w_k\\
&\qquad +G_k e_{k+1} + v_{k+1} - C_{k+1}A_k\hat{x}_{k|k})\big)^{\mathrm{T}}\Big]. \quad \text{(III.15)}
\end{aligned}$$

Since (III.9) and (III.10) are satisfied, (III.15) becomes

$$\begin{aligned}
&P_{k+1|k+1}\\
&\quad= \mathbb{E}\,[([I - L_{k+1}C_{k+1}]A_k\varepsilon_k + L_{k+1}v_{k+1}\\
&\qquad -[I - L_{k+1}C_{k+1}]w_k)\\
&\qquad \times([I - L_{k+1}C_{k+1}]A_k\varepsilon_k + L_{k+1}v_{k+1}\\
&\qquad -[I - L_{k+1}C_{k+1}]w_k)^{\mathrm{T}}\Big]. \quad \text{(III.16)}
\end{aligned}$$

Noting that $\mathcal{E}[\varepsilon_k w_k^{\mathrm{T}}] = 0$, $\mathbb{E}[\varepsilon_k v_{k+1}^{\mathrm{T}}] = 0$ and $\mathbb{E}[w_k v_{k+1}^{\mathrm{T}}] = 0$, we have that

$$\begin{aligned}
P_{k+1|k+1} &= L_{k+1}R_{k+1}L_{k+1}^{\mathrm{T}}\\
&\quad +(I - L_{k+1}C_{k+1})[A_k P_{k|k}A_k^{\mathrm{T}} + Q_k]\\
&\quad \times(I - L_{k+1}C_{k+1})^{\mathrm{T}} \quad \text{(III.17)}
\end{aligned}$$

Furthermore, using (III.12) - (III.14) and (III.17), it follows that (III.11) holds. □

Next, we define the cost function J as the trace of the error covariance matrix

$$J(L_{k+1}) = \mathrm{tr}\mathbb{E}[\varepsilon_{k+1}\varepsilon_{k+1}^{\mathrm{T}}] = \mathrm{tr}P_{k+1|k+1}. \quad \text{(III.18)}$$

Theorem III.1. *The unbiased minimum-variance gain L_{k+1} in the filter (III.3) that minimizes the cost function (III.18) subject to the constraints (III.9) and (III.10) is given by*

$$\begin{aligned}
L_{k+1} &= \big(F_{k+1}\\
&\quad -\Omega_{k+1}(\Phi_{k+1}^{\mathrm{T}}\tilde{R}_{k+1}^{-1}\Phi_{k+1}^{\mathrm{T}})^{-1}\Phi_{k+1}^{\mathrm{T}}\big)\,\tilde{R}_{k+1}^{-1}, \quad \text{(III.19)}
\end{aligned}$$

where

$$\Phi_{k+1} \triangleq \begin{bmatrix} -G_{k+1} & V_{k+1} \end{bmatrix} \quad \text{(III.20)}$$

$$V_{k+1} \triangleq C_{k+1}H_k, \quad \text{(III.21)}$$

$$\Omega_{k+1} \triangleq \begin{bmatrix} 0_{n\times p} & H_k \end{bmatrix} - F_{k+1}\tilde{R}_{k+1}^{-1}\Phi_{k+1} \quad \text{(III.22)}$$

Proof. The cost function J can be written as

$$\begin{aligned}
J(L_{k+1}) &= \mathrm{tr}P_{k+1|k+1}\\
&= \mathrm{tr}[L_{k+1}\tilde{R}_{k+1}L_{k+1}^{\mathrm{T}} - F_{k+1}L_{k+1}^{\mathrm{T}}\\
&\quad -L_{k+1}C_{k+1}P_{k+1|k} + P_{k+1|k}]. \quad \text{(III.23)}
\end{aligned}$$

Thus the optimization problem is to minimize the cost function (III.23) subject to the constraints (III.9) and (III.10). If $\Lambda_k \in \mathbb{R}^{n\times 2q}$ is the matrix of Lagrange multipliers, then the Lagrangian is

$$\begin{aligned}
&\mathcal{L}(L_{k+1}) \triangleq J(L_{k+1})\\
&\quad +2\mathrm{tr}(\begin{bmatrix} L_{k+1}G_{k+1} & (I - L_{k+1}C_{k+1})H_k \end{bmatrix}\Lambda_{k+1}^{\mathrm{T}}). \quad \text{(III.24)}
\end{aligned}$$

Differentiating with respect to L_{k+1} and setting it to zero, we get

$$\begin{aligned}
&2\tilde{R}_{k+1}L_{k+1}^{\mathrm{T}} - 2C_{k+1}P_{k+1|k}\\
&\quad +2\begin{bmatrix} G_{k+1} & -C_{k+1}H_k \end{bmatrix}\Lambda_{k+1}^{\mathrm{T}} = 0, \quad \text{(III.25)}
\end{aligned}$$

while differentiating with respect to Λ_{k+1} and setting it to zero yields the constraints (III.9) and (III.10). Further, assuming R_k to be positive definite, we write (III.25) as

$$L_{k+1}^{\mathrm{T}} = \tilde{R}_{k+1}^{-1}(C_{k+1}P_{k+1|k} - \Phi_{k+1}\Lambda_{k+1}^{\mathrm{T}}). \quad \text{(III.26)}$$

Using (III.26) in (III.9) and (III.10), we obtain the matrix of Lagrange multipliers

$$\Lambda_{k+1} = \Omega_{k+1}(\Phi_{k+1}^{\mathrm{T}}\tilde{R}_{k+1}^{-1}\Phi_{k+1}^{\mathrm{T}})^{-1}. \quad \text{(III.27)}$$

Substituting (III.27) into (III.26), yields the optimal solution for L_{k+1} as

$$\begin{aligned}
L_{k+1} &= \big(F_{k+1}\\
&\quad -\Omega_{k+1}(\Phi_{k+1}^{\mathrm{T}}\tilde{R}_{k+1}^{-1}\Phi_{k+1}^{\mathrm{T}})^{-1}\Phi_{k+1}^{\mathrm{T}}\big)\,\tilde{R}_{k+1}^{-1}.
\end{aligned}$$

□

It is straightforward to check that L_{k+1} given by (III.19) satisfies the constraints (III.9) and (III.10). We now show that the filters presented in [16, 24] and the Kalman filter are special cases of the filter (III.19).

Proposition III.1. Suppose $H_k \equiv 0$ in (III.1), then the unbiased minimum-variance filter gain L_{k+1} satisfying (III.9) is given by

$$\begin{aligned}
L_{k+1} &= \Big[F_{k+1} - F_{k+1}\tilde{R}_{k+1}^{-1}G_{k+1}\\
&\quad \times(G_{k+1}^{\mathrm{T}}\tilde{R}_{k+1}^{-1}G_{k+1})^{-1}G_{k+1}^{\mathrm{T}}\Big]\,\tilde{R}_{k+1}^{-1}. \quad \text{(III.28)}
\end{aligned}$$

Furthermore, the covariance update equation is given by

$$\begin{aligned}
P_{k+1|k+1} &= P_{k+1|k} - F_{k+1}\tilde{R}_{k+1}^{-1}[I - G_{k+1}\\
&\quad (G_{k+1}^{\mathrm{T}}\tilde{R}_{k+1}^{-1}G_{k+1})^{-1}G_{k+1}^{\mathrm{T}}\tilde{R}_{k+1}^{-1}]F_{k+1}^{\mathrm{T}} \quad \text{(III.29)}
\end{aligned}$$

Proof. Substituting $H_k = 0$ in equation (III.19) yields (III.28). Next, we note that when $H_k = 0$ and (III.9) is statisfied, $P_{k+1|k+1}$ is again given by (III.11). Furthermore, using the expression for L_{k+1} from (III.28) in (III.11), we get (III.29). □

Proposition III.1 concides with the output-correction filter developed in [24]. Next, the following result is given in [16].

Proposition III.2. Suppose $G_k \equiv 0$ in (III.2) and assume (III.1), (III.2) is input and state observable. Then the unbiased minimum-variance filter gain L_{k+1} satisfying (III.10) is

$$L_{k+1} = H_k\Pi_k + F_{k+1}\tilde{R}_{k+1}^{-1}(I - V_{k+1}\Pi_k), \quad \text{(III.30)}$$

where Π_k is defined as

$$\Pi_k \triangleq (V_{k+1}^{\mathrm{T}}\tilde{R}_{k+1}^{-1}V_{k+1})^{-1}V_{k+1}^{\mathrm{T}}\tilde{R}_{k+1}^{-1}. \quad \text{(III.31)}$$

Furthermore, the covariance update equation becomes

$$\begin{aligned} P_{k+1|k+1} &= P_{k+1|k} - F_{k+1}\tilde{R}_{k+1}^{-1}F_{k+1}^{\mathrm{T}} + \\ &\quad (H_k - F_{k+1}\tilde{R}_{k+1}^{-1}V_{k+1})(V_{k+1}^{\mathrm{T}}\tilde{R}_{k+1}^{-1}V_{k+1})^{-1} \\ &\quad \times(H_k - F_{k+1}\tilde{R}_{k+1}^{-1}V_{k+1})^{\mathrm{T}}. \end{aligned} \quad \text{(III.32)}$$

Proof. The proof follows directly from Proposition III.1 and (III.11). □

Proposition III.3. If $G_k = 0$ and $H_k = 0$ in (III.1), (III.2), then the unbiased minimum-variance filter gain L_{k+1} reduces to the Kalman filter gain

$$L_{k+1} = F_{k+1}\tilde{R}_{k+1}^{-1}, \quad \text{(III.33)}$$

and the covariance update equation reduces to the Kalman filter covariance update equation

$$P_{k+1|k+1} = P_{k+1|k} - F_{k+1}\tilde{R}_{k+1}^{-1}F_{k+1}^{\mathrm{T}}. \quad \text{(III.34)}$$

Proof. Setting $H_k = 0$ and $G_k = 0$ in (III.19) and (III.11), we get (III.33) and (III.34) respectively. □

So far, we have discussed unbiased estimation of the state x_k in the presence of arbitrary unknown inputs e_k. Next, we discuss how the unknown inputs e_k are estimated, using the unbiased estimates $\hat{x}_{k|k}$ of the states x_k.

Proposition III.4. Consider (III.1), (III.2), and suppose that $\hat{x}_{k|k}$ is an unbiased estimate of x_k in (III.1). Then

$$\hat{e}_k = G_k^{\dagger}(y_k - C_k\hat{x}_{k|k} - D_k u_k), \quad \text{(III.35)}$$

is an unbiased estimate of e_k.

Proof. Since $l > p$, we can define $\hat{e}_k$ as

$$\hat{e}_k = G_k^{\dagger}(y_k - C_k\hat{x}_{k|k} - D_k u_k), \quad \text{(III.36)}$$

Taking expected value on both sides of (III.36) yields

$$\mathbb{E}[\hat{e}_k] = \mathbb{E}[G_k^{\dagger}(G_k e_k + C_k\varepsilon_k + v_k)] \quad \text{(III.37)}$$

Since $\hat{x}_{k|k}$ is an unbiased estimate of x_k, the state estimation error ε_k satisfies

$$\mathbb{E}[\varepsilon_k] = 0. \quad \text{(III.38)}$$

Using (III.38) and noting that v_k is zero mean, (III.37) becomes

$$\begin{aligned} \mathbb{E}[\hat{e}_k] &= \mathbb{E}[G_k^{\dagger}G_k e_k] \\ &= \mathbb{E}[e_k]. \end{aligned}$$ □

Proposition III.5. Consider (III.1), (III.2) and let $G_k = 0$. Suppose that $\hat{x}_{k|k}$ is an unbiased estimate of the states x_k of (III.1). Then

$$\hat{e}_k = H_k^{\dagger}L_{k+1}(y_{k+1} - C_{k+1}\hat{x}_{k+1|k} - D_{k+1}u_{k+1}), \quad \text{(III.39)}$$

is an unbiased estimate of e_k.

Proof. Since $l \geq p$, we can define $\hat{e}_k$ as

$$\hat{e}_k = H_k^{\dagger}L_{k+1}(y_{k+1} - C_{k+1}\hat{x}_{k+1|k} - D_{k+1}u_{k+1}), \quad \text{(III.40)}$$

where † denotes the Moore-Penrose generalized inverse. Next, we use (III.3) and (III.40) to get

$$\begin{aligned} \hat{e}_k &= H_k^{\dagger}(\hat{x}_{k+1|k+1} - \hat{x}_{k+1|k}) \\ &= H_k^{\dagger}(x_{k+1} + \varepsilon_{k+1} - A_k\hat{x}_{k|k} - B_k u_k) \\ &= H_k^{\dagger}(x_{k+1} - A_k x_k - B_k u_k + \varepsilon_{k+1} - A_k\varepsilon_k) \\ &= H_k^{\dagger}(H_k e_k + w_k + \varepsilon_{k+1} - A_k\varepsilon_k). \end{aligned} \quad \text{(III.41)}$$

Further, taking expected value on both sides of (III.41), yields

$$\mathbb{E}[\hat{e}_k] = \mathbb{E}[H_k^{\dagger}(H_k e_k + w_k + \varepsilon_{k+1} - A_k\varepsilon_k)], \quad \text{(III.42)}$$

Finally, noting that $\mathbb{E}[\varepsilon_k] = 0$ and the fact that w_k is zero-mean, we get

$$\mathbb{E}[\hat{e}_k] = H_k^{\dagger}H_k\mathbb{E}[e_k] = \mathbb{E}[e_k].$$ □

IV. EXAMPLE

We consider a discrete-time model of the Van der Pol oscillator

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} x_{1,k} + T_s x_{2,k} \\ x_{2,k} + T_s[(1 - x_{1,k}^2)x_{2,k} - x_{1,k}] \end{bmatrix}, \quad \text{(IV.1)}$$

where T_s is the sampling interval. We assume that the linear part of the dynamics is known perfectly, that is, the initial model is the linear part of the equations while $T_s x_{1,k}^2 x_{2,k}$ is assumed to be unknown and thus acts as the unknown input e_k. Measurements of the state x_2 are available, thus the output matrix is $C_k = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Since the nonlinear term appears only in the equation of the second state we take $H_k = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Figure 1 shows a plot of the actual states, the states from an open-loop simulation of the known model, and the estimates of the states from the unbiased minimum-variance filter. It is seen from the plot that the state estimates from the unbiased minimum-variance filter based on the known model approximates the actual states closely. Once the estimates of the states are obtained we then obtain a least squares estimate $\hat{e}_k$ of the unknown signal e_k by using (III.39). Figure 2 shows the actual unknown signal e_k and the estimate $\hat{e}_k$ of the unknown signal.

V. Conclusions

In this paper, we introduced the concept of input and state observability, that is, conditions under which both the unknown input and state can be estimated from the output measurements. We discussed sufficient and necessary conditions for a discrete-time system to be input and state observable. Next, we derived an unbiased minimum-variance filter to estimate the unknown input and the state. Finally, we presented an illustrative example.

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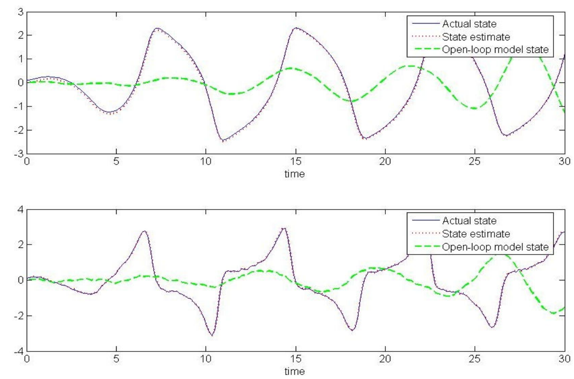

Fig. 1. Discrete-time Van-der Pol Oscillator Example. Comparison of the actual states of the system, the state estimates from the unbiased minimum-variance filter, and the states from an open-loop simulation of the known model equations.

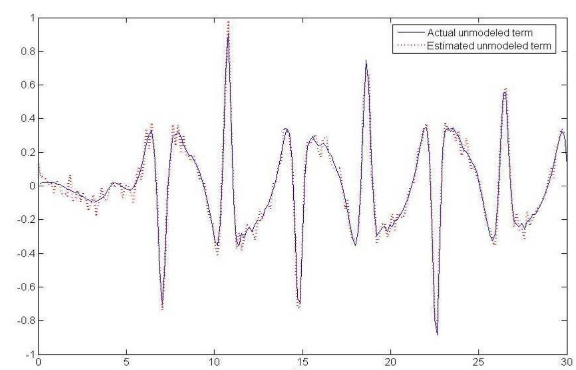

Fig. 2. Discrete-time Van-der Pol Oscillator Example. A plot showing the actual unknown input and the estimated unknown input.