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Retrospective Cost Adaptive Control for Nonminimum-Phase Discrete-Time Systems, Part 1: The Ideal Controller and Error System

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Abstract— **We present a direct adaptive controller for discrete-time (and thus sampled-data) systems that are possibly nonminimum phase. The adaptive control algorithm requires limited model information, specifically, knowledge of the first nonzero Markov parameter and the nonminimum-phase zeros (if any) of the transfer function from the control to the performance. This adaptive control algorithm is effective for stabilization as well as for command following and disturbance rejection, where the command and disturbance spectra are unknown. The novel aspect of this controller is the use of a retrospective performance, which is minimized using either an instantaneous or cumulative retrospective cost function.**

I. Introduction

One of the major challenges in direct adaptive control is the existence of nonminimum-phase zeros – many direct adaptive control methodologies rely on the assumption that the plant is minimum phase [1]–[5]. Another challenge in adaptive control is command following and disturbance rejection, where the command and disturbance spectra are unknown and the disturbances are unmeasured. In addition, for disturbance rejection problems, some adaptive control methods require that the range of the disturbance input matrix is contained in the range of the control input matrix, meaning that the disturbance can be rejected directly by the input without using the system dynamics [5], [6].

In this paper and its companion paper [7], we present a discrete-time adaptive control algorithm that addresses several of these common challenges. In particular, we present an adaptive controller that is effective for plants that are nonminimum phase, provided that we have estimates of the nonminimum-phase zeros. Furthermore, this adaptive controller does not require that the disturbance input matrix be matched to the control input matrix. Finally, this adaptive controller is effective for command following and disturbance rejection where the spectrum of the commands and disturbances is unknown and the disturbance is unmeasured.

The discrete-time adaptive control literature includes [2], [4], [8]–[12]. In [4], [9], discrete-time adaptive control laws are presented for stabilization and command following of minimum-phase systems. An extension is given in [11], which addresses the combined stabilization, command following, and disturbance rejection problem. The results of [4], [9], [11] are restricted to minimum-phase systems.

Another class of discrete-time adaptive controllers use a retrospective cost [13]–[15]. Retrospective cost adaptive control uses a retrospective performance measure, in which the performance measurement is modified based on the difference between the actual past control inputs and the recomputed past control inputs, assuming that the current controller had been used in the past. These retrospective cost adaptive controllers have been demonstrated to be effective for nonminimum-phase systems; however, a proof of the closed-loop stability properties has been an open problem. A similar controller construction is used in [16] to control continuous-time minimum-phase systems, which have real nonminimum-phase zeros that arise from sampling.

This paper is the first part of a pair of papers, which together present the retrospective cost adaptive controller and stability analysis. This paper is intended to be read in conjunction with [7]. This first paper focuses on the existence and properties of an ideal control law as well as the construction of a closed-loop error system (i.e., a system constructed by taking the difference between the closed-loop system with the ideal controller in feedback and the closed-loop system with an adaptive controller in feedback). The results in this paper are essential to the development of the adaptive law and the closed-loop stability analysis presented in [7].

II. Problem Formulation

Consider the discrete-time system

$$x(k+1) = Ax(k) + Bu(k) + D_1 w(k), \tag{1}$$

$$y(k) = Cx(k) + D_2 w(k), \tag{2}$$

where $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}$, $u(k) \in \mathbb{R}$, $w(k) \in \mathbb{R}^{l_w}$, and $k \geq 0$. Our goal is to develop an adaptive output feedback controller, which generates a control signal u that drives the performance variable y to zero in the presence of the exogenous signal w. Note that w can represent either a command signal to be followed, an external disturbance to be rejected, or both.

Next, define the transfer function $G_{yu}(\mathbf{z}) \overset{\triangle}{=} C(\mathbf{z}I - A)^{-1}B = \sum_{i=d}^{\infty} \mathbf{z}^{-i} H_i$, where the relative degree $d \geq 1$ is the smallest positive integer i such that the ith Markov parameter $H_i \overset{\triangle}{=} CA^{i-1}B$ is nonzero. We make the following assumptions.

(A1) The triple (A, B, C) is controllable and observable.

(A2) If $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$, and rank $\begin{bmatrix} A - \lambda I & B \\ C & 0 \end{bmatrix} < n+1$, then λ is known.

(A3) d is known.

(A4) The first nonzero Markov parameter H_d is known.

(A5) There exists an integer $\bar{n}$ such that $n \leq \bar{n}$ and $\bar{n}$ is known.

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(A6) $y(k)$ is measured and available for feedback.

(A7) The exogenous signal $w(k)$ is generated by

$$x_w(k+1) = A_w x_w(k), \quad w(k) = C_w x_w(k), \quad (3)$$

where $x_w \in \mathbb{R}^{n_w}$ and A_w has distinct eigenvalues, all of which are on the unit circle, and none of which coincide with a zero of (A, B, C).

(A8) There exists an integer $\bar{n}_w$ such that $n_w \leq \bar{n}_w$ and $\bar{n}_w$ is known.

(A9) A, B, C, D_1, D_2, A_w, C_w, n, and n_w are not known.

Next, let the transfer function $G_{yu}(\mathbf{z})$ be given by $G_{yu}(\mathbf{z}) = H_d \frac{\beta(\mathbf{z})}{\alpha(\mathbf{z})}$, where $\alpha(\mathbf{z}) = \mathbf{z}^n + \alpha_1 \mathbf{z}^{n-1} + \cdots + \alpha_n$ and $\beta(\mathbf{z}) = \mathbf{z}^{n-d} + \beta_{d+1}\mathbf{z}^{n-d-1} + \cdots + \beta_n$ are coprime monic polynomials, and $\alpha_1, \ldots, \alpha_n, \beta_{d+1}, \ldots, \beta_n \in \mathbb{R}$.

Next, consider the factorization of $\beta(\mathbf{z})$ given by $\beta(\mathbf{z}) = \beta_{\rm u}(\mathbf{z})\beta_{\rm s}(\mathbf{z})$, where $\beta_{\rm u}(\mathbf{z})$ and $\beta_{\rm s}(\mathbf{z})$ are monic polynomials and the roots of $\beta(\mathbf{z})$ that lie on or outside the unit circle are the roots of $\beta_{\rm u}(\mathbf{z})$. More precisely, if $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$, and $\beta(\lambda) = 0$, then $\beta_{\rm u}(\lambda) = 0$ and $\beta_{\rm s}(\lambda) \neq 0$. Furthermore, we can write $\beta_{\rm u}(\mathbf{z}) = \mathbf{z}^{n_{\rm u}} + \beta_{{\rm u},1}\mathbf{z}^{n_{\rm u}-1} + \cdots + \beta_{{\rm u},n_{\rm u}-1}\mathbf{z} + \beta_{{\rm u},n_{\rm u}}$ and $\beta_{\rm s}(\mathbf{z}) = \mathbf{z}^{n-n_{\rm u}-d} + \beta_{{\rm s},1}\mathbf{z}^{n-n_{\rm u}-d-1} + \cdots + \beta_{{\rm s},n-n_{\rm u}-d-1}\mathbf{z} + \beta_{{\rm s},n-n_{\rm u}-d}$, where $\beta_{{\rm u},1}, \ldots, \beta_{{\rm u},n_{\rm u}}, \beta_{{\rm s},1}, \ldots, \beta_{{\rm s},n-n_{\rm u}-d} \in \mathbb{R}$, and $n_{\rm u} \leq n-d$ is the degree of $\beta_{\rm u}(\mathbf{z})$.

Next, we define the transfer function matrix $G_{yw}(\mathbf{z}) \triangleq C(\mathbf{z}I - A)^{-1}D_1 + D_2$, and let G_{yw} have a matrix-fraction description of the form $G_{yw} = \frac{1}{\alpha(\mathbf{z})}\gamma(\mathbf{z})$, which is not necessarily coprime, where $\gamma(\mathbf{z}) = \mathbf{z}^n\gamma_0 + \mathbf{z}^{n-1}\gamma_1 + \cdots + \gamma_n$, and $\gamma_0, \ldots, \gamma_n \in \mathbb{R}^{1\times l_w}$. Therefore, for $k \geq n$, the state-space system (1), (2) has the time-series representation

$$y(k) = \sum_{i=1}^{n} -\alpha_i y(k-i) + \sum_{i=d}^{n} \beta_i u(k-i) + \sum_{i=0}^{n} \gamma_i w(k-i). \quad (4)$$

Since, by assumption (A1), (A, B, C) is minimal, it follows that the nonminimum-phase zeros of (A, B, C) are exactly the roots of $\beta_{\rm u}(\mathbf{z})$. Therefore, assumption (A2) is equivalent to the assumption that the polynomial $\beta_{\rm u}(\mathbf{z})$ is known and the degree $n_{\rm u}$ of $\beta_{\rm u}(\mathbf{z})$ is known.

Assumption (A9) implies that the spectrum of the command and disturbance signals is unknown. Furthermore, we stress that $y(k)$ is the only signal available for feedback, that is, a direct measurement of $w(k)$ is not assumed to be available.

III. Nonminimal-State-Space Realization

We use a nonminimal-state-space realization of the time-series model (4) whose state consists entirely of measured information, specifically, y and u. To construct the nonminimal-state-space realization of the time-series system (4), we introduce the following notation. For a positive integer p, define

$$E_p \triangleq \begin{bmatrix} 1 \\ 0_{(p-1)\times 1} \end{bmatrix}, \quad \mathcal{N}_p \triangleq \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{p\times p},$$

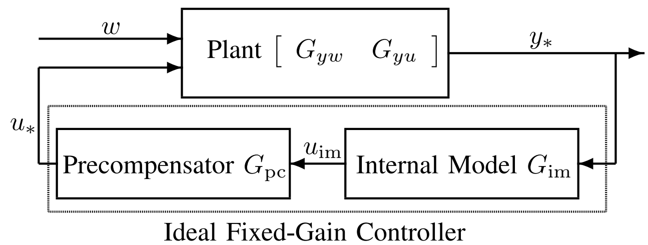

Fig. 1. Closed-loop system with the ideal fixed-gain controller.

Now, let $n_{\rm c} \geq n$, and, for all $k \geq n_{\rm c}$, consider the $2n_{\rm c}^{\rm th}$-order nonminimal-state-space realization of (4) given by

$$\phi(k+1) = \mathcal{A}\phi(k) + \mathcal{B}u(k) + \mathcal{D}_1 W(k), \quad (5)$$
$$y(k) = \mathcal{C}\phi(k) + \mathcal{D}_2 W(k), \quad (6)$$

where

$$\mathcal{A} \triangleq \mathcal{A}_{\rm nil} + E_{2n_{\rm c}}\mathcal{C}, \qquad \mathcal{B} \triangleq \begin{bmatrix} 0_{n_{\rm c}\times 1} \\ E_{n_{\rm c}} \end{bmatrix}, \quad (7)$$

$$\mathcal{C} \triangleq \begin{bmatrix} -\alpha_1 & \cdots & -\alpha_n & 0_{1\times(n_{\rm c}-n)} \\ 0_{1\times(d-1)} & \beta_d & \cdots & \beta_n & 0_{1\times(n_{\rm c}-n)} \end{bmatrix}, \quad (8)$$

$$\mathcal{D}_1 \triangleq E_{2n_{\rm c}}\mathcal{D}_2, \qquad \mathcal{D}_2 \triangleq \begin{bmatrix} \gamma_0 & \cdots & \gamma_n \end{bmatrix}, \quad (9)$$

$$\mathcal{A}_{\rm nil} \triangleq \begin{bmatrix} \mathcal{N}_{n_{\rm c}} & 0_{n_{\rm c}\times n_{\rm c}} \\ 0_{n_{\rm c}\times n_{\rm c}} & \mathcal{N}_{n_{\rm c}} \end{bmatrix}, \quad (10)$$

$$\phi(k) \triangleq \begin{bmatrix} y(k-1) & \cdots & y(k-n_{\rm c}) \\ u(k-1) & \cdots & u(k-n_{\rm c}) \end{bmatrix}^{\rm T}, \quad (11)$$

$$W(k) \triangleq \begin{bmatrix} w^{\rm T}(k) & \cdots & w^{\rm T}(k-n) \end{bmatrix}^{\rm T}. \quad (12)$$

The triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is stabilizable and detectable but is neither controllable nor observable. In particular, $(\mathcal{A}, \mathcal{B})$ has $n_{\rm c} - n$ uncontrollable eigenvalues at 0, while $(\mathcal{A}, \mathcal{C})$ has $2n_{\rm c} - n$ unobservable eigenvalues at 0. Note that in this basis, the state $\phi(k)$ contains only past values of y and u.

IV. Ideal Fixed-Gain Controller

In this section, we prove the existence of an ideal fixed-gain controller for the open-loop system (1), (2). This controller, whose structure is illustrated in Figure 1, is used in the next section to construct an error system for analyzing the closed-loop adaptive system. An ideal fixed-gain controller consists of two distinct parts, specifically, a precompensator, which cancels the stable zeros of the open-loop transfer function $G_{yu}(\mathbf{z})$, and an internal model controller, which operates in feedback on the observable states of the precompensator cascaded with the open-loop system. For all $k \geq n_{\rm c}$, consider the strictly proper ideal fixed-gain controller

$$u_*(k) = \sum_{i=1}^{n_{\rm c}} M_{i,*}u_*(k-i) + \sum_{i=1}^{n_{\rm c}} N_{i,*}y_*(k-i), \quad (13)$$

where, for all $i = 1, \ldots, n_{\rm c}$, $M_{i,*} \in \mathbb{R}$ and $N_{i,*} \in \mathbb{R}$, and the ideal performance $y_*(k)$ is the output of the system (1),

(2) with $u(k) = u_*(k)$ for all $k \geq n_c$. For all $k \geq n_c$, the ideal control $u_*(k)$ can be written as

$$u_*(k) = \phi_*^{\mathrm{T}}(k)\theta_*, \tag{14}$$

where $\theta_*(k) \stackrel{\triangle}{=} \left[\begin{array}{cccccc} N_{1,*} & \cdots & N_{n_c,*} & M_{1,*} & \cdots & M_{n_c,*} \end{array}\right]^{\mathrm{T}}$, and

$$\phi_*(k) \stackrel{\triangle}{=} \left[\begin{array}{ccc} y_*(k-1) & \cdots & y_*(k-n_c) \\ u_*(k-1) & \cdots & u_*(k-n_c) \end{array}\right]^{\mathrm{T}}.$$

Therefore, for all $k \geq n_c$, the ideal closed-loop system consisting of (5)-(10) and the ideal fixed-gain controller (14) is given by

$$\phi_*(k+1) = \mathcal{A}_*\phi_*(k) + \mathcal{D}_1 W(k), \tag{15}$$

$$y_*(k) = \mathcal{C}\phi_*(k) + \mathcal{D}_2 W(k), \tag{16}$$

where $\mathcal{A}_* \stackrel{\triangle}{=} \mathcal{A} + \mathcal{B}\theta_*^{\mathrm{T}} = \mathcal{A}_{\mathrm{nil}} + \left[\begin{array}{c} E_{n_c}\mathcal{C} \\ E_{n_c}\theta_*^{\mathrm{T}} \end{array}\right]$.

Let $\mathbf{q}$ and $\mathbf{q}^{-1}$ denote the forward-shift and backward-shift operators, respectively.

Theorem IV.1. *Let*

$$n_c \geq 2n + 2n_w - n_u - d, \tag{17}$$

and let $d_*(\mathbf{z})$ be an asymptotically stable monic polynomial of degree $n_* \stackrel{\triangle}{=} n_c + n_u + d$. Then there exists an ideal fixed-gain controller (13) of order n_c such that the following statements hold for the ideal closed-loop system consisting of (15), (16):

(i) *$\mathcal{A}_*$ is asymptotically stable.*

(ii) *For all initial conditions $x_w(0)$ and $\phi_*(n_c)$ of the closed-loop system (15) and (16) with the exogenous input $w(k)$, which is given by (3),* $\lim_{k\to\infty} y_*(k) = 0$.

(iii) *For all initial conditions $x_w(0)$ and $\phi_*(n_c)$ of the closed-loop system (15) and (16) with the exogenous input $w(k)$, which is given by (3), and, for all $k \geq k_0 \stackrel{\triangle}{=} n_* + n_c = 2n_c + n_u + d$,*

$$D_*(\mathbf{q}^{-1})y_*(k) = 0, \tag{18}$$

where $D_*(\mathbf{q}^{-1}) \stackrel{\triangle}{=} \mathbf{q}^{-n_*}d_*(\mathbf{q})$.

(iv) *The transfer function $G_*(\mathbf{z}) \stackrel{\triangle}{=} \mathcal{C}(\mathbf{z}I - \mathcal{A}_*)^{-1}\mathcal{B}$ is given by*

$$G_*(\mathbf{z}) = \frac{H_d\beta_u(\mathbf{z})\mathbf{z}^{n_c}}{d_*(\mathbf{z})}. \tag{19}$$

Proof. We construct a time-series representation of the ideal fixed-gain controller depicted in Figure 1 and show that it satisfies *(i)*-*(iv)*. To construct this controller, define $n_{\mathrm{pc}} \stackrel{\triangle}{=} n - n_u - d \geq 0$, and consider the strictly proper precompensator, with input u_{im}, given by

$$u_*(k) = -\sum_{i=1}^{n_{\mathrm{pc}}} \beta_{\mathrm{s},i}u_*(k-i) + u_{\mathrm{im}}(k - n_{\mathrm{pc}}), \tag{20}$$

which has a minimal state-space realization of the form

$$x_{\mathrm{pc}}(k+1) = A_{\mathrm{pc}}x_{\mathrm{pc}}(k) + B_{\mathrm{pc}}u_{\mathrm{im}}(k), \tag{21}$$

$$u_*(k) = C_{\mathrm{pc}}x_{\mathrm{pc}}(k), \tag{22}$$

where $x_{\mathrm{pc}} \in \mathbb{R}^{n_{\mathrm{pc}}}$. If $n_{\mathrm{pc}} = 0$, then the precompensator is absent. Next, cascading the precompensator (21), (22) with the open-loop system (1), (2) yields

$$\begin{aligned} \left[\begin{array}{c} x_*(k+1) \\ x_{\mathrm{pc}}(k+1) \end{array}\right] &= \left[\begin{array}{cc} A & BC_{\mathrm{pc}} \\ 0 & A_{\mathrm{pc}} \end{array}\right]\left[\begin{array}{c} x_*(k) \\ x_{\mathrm{pc}}(k) \end{array}\right] \\ &\quad + \left[\begin{array}{c} 0 \\ B_{\mathrm{pc}} \end{array}\right]u_{\mathrm{im}}(k) + \left[\begin{array}{c} D_1 \\ 0 \end{array}\right]w(k), \\ y_*(k) &= \left[\begin{array}{cc} C & 0 \end{array}\right]\left[\begin{array}{c} x_*(k) \\ x_{\mathrm{pc}}(k) \end{array}\right] + D_2 w(k), \end{aligned} \tag{23}$$

where $x_*(k)$ is the state of (1), (2) with $u(k) = u_*(k)$. It follows from (20) that the transfer function from u_{im} to u_* is $G_{\mathrm{pc}}(\mathbf{z}) \stackrel{\triangle}{=} C_{\mathrm{pc}}(\mathbf{z}I - A_{\mathrm{pc}})^{-1}B_{\mathrm{pc}} = \frac{1}{\beta_{\mathrm{s}}(\mathbf{z})}$, which implies that the poles of $G_{\mathrm{pc}}(\mathbf{z})$ are exactly the asymptotically stable zeros of $G_{yu}(\mathbf{z})$. Therefore, the cascade (23) has the transfer function representation $G_{yu}(\mathbf{z})G_{\mathrm{pc}}(\mathbf{z}) = H_d\frac{\beta_u(\mathbf{z})\beta_{\mathrm{s}}(\mathbf{z})}{\alpha(\mathbf{z})}\frac{1}{\beta_{\mathrm{s}}(\mathbf{z})} = H_d\frac{\beta_u(\mathbf{z})}{\alpha(\mathbf{z})}$, which entails asymptotically stable pole-zero cancellation. Since the poles of $G_{\mathrm{pc}}(\mathbf{z})$ cancel the minimum-phase zeros of $G_{yu}(\mathbf{z})$, it follows that

$$\left(\left[\begin{array}{cc} A & BC_{\mathrm{pc}} \\ 0 & A_{\mathrm{pc}} \end{array}\right], \left[\begin{array}{c} 0 \\ B_{\mathrm{pc}} \end{array}\right], \left[\begin{array}{cc} C & 0 \end{array}\right]\right) \tag{24}$$

is not minimal. However, we show that (24) is controllable. Since (A, B) is controllable and $(A_{\mathrm{pc}}, B_{\mathrm{pc}}, C_{\mathrm{pc}})$ has no zeros, it follows that, for all $\lambda \in \mathbb{C}$,

$$\begin{aligned} &\operatorname{rank}\left[\begin{array}{ccc} \lambda I_n - A & BC_{\mathrm{pc}} & 0 \\ 0 & \lambda I_{n_{\mathrm{pc}}} - A_{\mathrm{pc}} & B_{\mathrm{pc}} \end{array}\right] \\ &= \operatorname{rank}\left(\left[\begin{array}{ccc} \lambda I_n - A & B & 0 \\ 0 & 0 & I_{n_{\mathrm{pc}}} \end{array}\right]\right. \\ &\quad \times \left.\left[\begin{array}{ccc} I_n & 0 & 0 \\ 0 & 0 & C_{\mathrm{pc}} \\ 0 & B_{\mathrm{pc}} & \lambda I_{n_{\mathrm{pc}}} - A_{\mathrm{pc}} \end{array}\right]\right) = n + n_{\mathrm{pc}}. \end{aligned}$$

Thus, the cascade (23) is controllable but not observable. In fact, it follows from the pole-zero cancellations between $G_{\mathrm{pc}}(\mathbf{z})$ and $G_{yu}(\mathbf{z})$ that the unobservable modes of (24) are exactly the poles of $G_{\mathrm{pc}}(\mathbf{z})$, all of which are asymptotically stable. Therefore, it follows from the Kalman decomposition that there exists a nonsingular matrix $T \in \mathbb{R}^{(n+n_{\mathrm{pc}})\times(n+n_{\mathrm{pc}})}$ such that

$$T\left[\begin{array}{cc} A & BC_{\mathrm{pc}} \\ 0 & A_{\mathrm{pc}} \end{array}\right]T^{-1} = \left[\begin{array}{cc} A_{\mathrm{o}} & 0 \\ A_{21} & A_{\bar{\mathrm{o}}} \end{array}\right],$$

$$\left[\begin{array}{cc} C & 0 \end{array}\right]T^{-1} = \left[\begin{array}{cc} C_{\mathrm{o}} & 0 \end{array}\right],$$

where $A_{\mathrm{o}} \in \mathbb{R}^{n\times n}$, $(A_{\mathrm{o}}, C_{\mathrm{o}})$ is observable, and $A_{\bar{\mathrm{o}}} \in \mathbb{R}^{n_{\mathrm{pc}}\times n_{\mathrm{pc}}}$ is asymptotically stable.

Now, defining $\left[\begin{array}{c} x_{\mathrm{o}}(k) \\ x_{\bar{\mathrm{o}}}(k) \end{array}\right] \stackrel{\triangle}{=} T\left[\begin{array}{c} x(k) \\ x_{\mathrm{pc}}(k) \end{array}\right]$, where $x_{\mathrm{o}}(k) \in \mathbb{R}^n$ and $x_{\bar{\mathrm{o}}}(k) \in \mathbb{R}^{n_{\mathrm{pc}}}$, and applying this change of basis to the cascade (23) yields

$$\begin{aligned} \left[\begin{array}{c} x_{\mathrm{o}}(k+1) \\ x_{\bar{\mathrm{o}}}(k+1) \end{array}\right] &= \left[\begin{array}{cc} A_{\mathrm{o}} & 0 \\ A_{21} & A_{\bar{\mathrm{o}}} \end{array}\right]\left[\begin{array}{c} x_{\mathrm{o}}(k) \\ x_{\bar{\mathrm{o}}}(k) \end{array}\right] \\ &\quad + \left[\begin{array}{c} B_{\mathrm{o}} \\ B_{\bar{\mathrm{o}}} \end{array}\right]u_{\mathrm{im}}(k) + \left[\begin{array}{c} D_{1,\mathrm{o}} \\ D_{1,\bar{\mathrm{o}}} \end{array}\right]w(k), \end{aligned} \tag{25}$$

and

$$y_*(k) = \begin{bmatrix} C_{\rm o} & 0 \end{bmatrix} \begin{bmatrix} x_{\rm o}(k) \\ x_{\bar{\rm o}}(k) \end{bmatrix} + D_2 w(k), \quad (26)$$

where $\begin{bmatrix} B_{\rm o} \\ B_{\bar{\rm o}} \end{bmatrix} = T \begin{bmatrix} 0 \\ B_{\rm pc} \end{bmatrix}$ and $\begin{bmatrix} D_{1,{\rm o}} \\ D_{1,\bar{\rm o}} \end{bmatrix} = T \begin{bmatrix} D_1 \\ 0 \end{bmatrix}$.

Next, define $n_{\rm im} \stackrel{\triangle}{=} n_{\rm c} - n_{\rm pc} = n_{\rm c} - n + n_{\rm u} + d$, and note that it follows from (17) that $n_{\rm im} \ge n + 2n_w$. Let $x_{\rm im} \in \mathbb{R}^{n_{\rm im}}$ and consider the controller

$$x_{\rm im}(k+1) = A_{\rm im} x_{\rm im}(k) + B_{\rm im} y_*(k), \quad (27)$$
$$u_{\rm im}(k) = C_{\rm im} x_{\rm im}(k), \quad (28)$$

which is an internal model controller for the observable states of (25), (26) given by

$$x_{\rm o}(k+1) = A_{\rm o} x_{\rm o}(k) + B_{\rm o} u_{\rm im}(k) + D_{1,{\rm o}} w(k), \quad (29)$$
$$y_*(k) = C_{\rm o} x_{\rm o}(k) + D_2 w(k). \quad (30)$$

Therefore, the closed-loop system (27)-(30) is given by

$$\tilde{x}_{\rm o}(k+1) = \tilde{A}_{\rm o} \tilde{x}_{\rm o}(k) + \tilde{D}_{\rm o} w(k), \quad (31)$$
$$y_*(k) = \tilde{C}_{\rm o} \tilde{x}_{\rm o}(k) + D_2 w(k), \quad (32)$$

where

$$\tilde{A}_{\rm o} \stackrel{\triangle}{=} \begin{bmatrix} A_{\rm o} & B_{\rm o} C_{\rm im} \\ B_{\rm im} C_{\rm o} & A_{\rm im} \end{bmatrix}, \quad \tilde{D}_{\rm o} \stackrel{\triangle}{=} \begin{bmatrix} D_{1,{\rm o}} \\ B_{\rm im} D_2 \end{bmatrix},$$
$$\tilde{C}_{\rm o} \stackrel{\triangle}{=} \begin{bmatrix} C_{\rm o} & 0 \end{bmatrix}, \quad \tilde{x}_{\rm o}(k) \stackrel{\triangle}{=} \begin{bmatrix} x_{\rm o}(k) \\ x_{\rm im}(k) \end{bmatrix}.$$

Note that the dimension of the closed-loop system (31), (32) is $n + n_{\rm im} = n + n_{\rm c} - n_{\rm pc} = n_{\rm c} + n_{\rm u} + d = n_*$.

Since, by assumption (A7), the zeros of $(A_{\rm o}, B_{\rm o}, C_{\rm o})$, which are the roots of $\beta_{\rm u}(\mathbf{z})$, do not coincide with the eigenvalues of A_w; $(A_{\rm o}, B_{\rm o}, C_{\rm o})$ is minimal; and the dimension of y equals the dimension of u, it follows from [17, Theorem 4.1] that there exists an internal model controller (27), (28) such that $\tilde{A}_{\rm o}$ is asymptotically stable, $\det(\mathbf{z} I_{n_*} - \tilde{A}_{\rm o}) = d_*(\mathbf{z})$, and, for all initial conditions $(x_{\rm o}(0), x_{\bar{\rm o}}(0), x_{\rm im}(0), x_w(0))$, $\lim_{k\to\infty} y_*(k) = 0$.

Next, the closed-loop system (25)-(28), which includes the unobservable states $x_{\bar{\rm o}}$, is given by

$$\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{D} w(k), \quad (33)$$
$$y_*(k) = \tilde{C}\tilde{x}(k) + D_2 w(k), \quad (34)$$

where

$$\tilde{A} \stackrel{\triangle}{=} \begin{bmatrix} \tilde{A}_{\rm o} & 0 \\ \tilde{A}_{21} & A_{\bar{\rm o}} \end{bmatrix}, \quad \tilde{D} \stackrel{\triangle}{=} \begin{bmatrix} \tilde{D}_{\rm o} \\ D_{1,\bar{\rm o}} \end{bmatrix},$$
$$\tilde{C} \stackrel{\triangle}{=} \begin{bmatrix} \tilde{C}_{\rm o} & 0 \end{bmatrix}, \quad \tilde{x}(k) \stackrel{\triangle}{=} \begin{bmatrix} \tilde{x}_{\rm o}(k) \\ x_{\bar{\rm o}}(k) \end{bmatrix}, \quad (35)$$

and $\tilde{A}_{21} \stackrel{\triangle}{=} \begin{bmatrix} A_{21} & B_{\bar{\rm o}} C_{\rm im} \end{bmatrix}$. Since $\tilde{A}_{\rm o}$ and $A_{\bar{\rm o}}$ are asymptotically stable, it follows that $\tilde{A}$ is asymptotically stable.

To show *(i)* and *(ii)*, we show that, for all $k \ge n_{\rm c}$, the ideal fixed-gain controller, which is the cascade of the precompensator (21), (22) and the internal model controller (27), (28), can be expressed as the time-series controller (13). First, we denote the transfer function of the internal model controller (27), (28) by $G_{\rm im}(\mathbf{z}) \stackrel{\triangle}{=} C_{\rm im}(\mathbf{z}I - A_{\rm im})^{-1} B_{\rm im} = \frac{N_{\rm im}(\mathbf{z})}{M_{\rm im}(\mathbf{z})}$, where $M_{\rm im}(\mathbf{z}) = \mathbf{z}^{n_{\rm im}} + M_{{\rm im},1}\mathbf{z}^{n_{\rm im}-1} + \cdots + M_{{\rm im},n_{\rm im}}$ and $N_{\rm im}(\mathbf{z}) = N_{{\rm im},1}\mathbf{z}^{n_{\rm im}-1} + N_{{\rm im},2}\mathbf{z}^{n_{\rm im}-2} + \cdots + N_{{\rm im},n_{\rm im}}$, where, for $i = 1, \ldots, n_{\rm im}$, $M_{{\rm im},i} \in \mathbb{R}$ and $N_{{\rm im},i} \in \mathbb{R}$. Therefore, the ideal fixed-gain controller is given by the transfer function

$$G_{{\rm c},*}(\mathbf{z}) \stackrel{\triangle}{=} G_{\rm pc}(\mathbf{z}) G_{\rm im}(\mathbf{z}) = \frac{N_*(\mathbf{z})}{M_*(\mathbf{z})}, \quad (36)$$

where $M_*(\mathbf{z}) \stackrel{\triangle}{=} \beta_{\rm s}(\mathbf{z}) M_{\rm im}(\mathbf{z})$ and $N_*(\mathbf{z}) \stackrel{\triangle}{=} N_{\rm im}(\mathbf{z})$. Thus, $M_*(\mathbf{z})$ and $N_*(\mathbf{z})$ are given by

$$M_*(\mathbf{z}) = \mathbf{z}^{n_{\rm c}} - M_{1,*}\mathbf{z}^{n_{\rm c}-1} - \cdots - M_{n_{\rm c},*}, \quad (37)$$
$$N_*(\mathbf{z}) = N_{1,*}\mathbf{z}^{n_{\rm c}-1} + N_{2,*}\mathbf{z}^{n_{\rm c}-2} + \cdots + N_{n_{\rm c},*}, \quad (38)$$

where, for all $i = 1, 2, \ldots, n_{\rm c}$,

$$M_{i,*} \stackrel{\triangle}{=} -\beta_{{\rm s},i} - M_{{\rm im},i} - \sum_{j=1}^{i-1} \beta_{{\rm s},j} M_{{\rm im},i-j}, \quad (39)$$

$$N_{i,*} \stackrel{\triangle}{=} N_{{\rm im},i-n_{\rm c}+n_{\rm im}}, \quad (40)$$

where, for all $j > n_{\rm pc}$, $\beta_{{\rm s},j} = 0$, for all $j > n_{\rm im}$, $M_{{\rm im},j} = 0$, and, for all $j \le 0$, $N_{{\rm im},j} = 0$. Therefore, the ideal time-series controller is given by (13) with parameters (39) and (40).

To show *(i)*, note that, for all $k \ge n_{\rm c}$, the closed-loop system (15), (16) is a $2n_{\rm c}^{\rm th}$-order nonminimal-state-space realization of the closed-loop system (33), (34). Thus, the spectrum of $\mathcal{A}_*$ consists of the $n_{\rm c} + n$ eigenvalues of $\tilde{A}$ as well as $n_{\rm c} - n$ uncontrollable eigenvalues located at 0, which are exactly the uncontrollable eigenvalues of the open-loop dynamics, that is, the uncontrollable eigenvalues of $(\mathcal{A}, \mathcal{B})$. Therefore, since $\tilde{A}$ is asymptotically stable, it follows that $\mathcal{A}_*$ is asymptotically stable. Thus, we have verified *(i)*.

Next, since, for all $k \ge n_{\rm c}$, (15), (16) is a nonminimal representation of (33), (34), it follows that, for all initial conditions $\phi_*(n_{\rm c})$ and $x_w(0)$, $\lim_{k\to\infty} y_*(k) = 0$. Thus, we have verified *(ii)*.

To show *(iii)*, for all $k \ge n_{\rm c}$, the closed-loop system (33), (34) with the exogenous input $w(k)$, which is given by (3), can be written as

$$\begin{bmatrix} \tilde{x}(k+1) \\ x_w(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{D} C_w \\ 0 & A_w \end{bmatrix} \begin{bmatrix} \tilde{x}(k) \\ x_w(k) \end{bmatrix},$$
$$y_*(k) = \begin{bmatrix} \tilde{C} & D_2 C_w \end{bmatrix} \begin{bmatrix} \tilde{x}(k) \\ x_w(k) \end{bmatrix}.$$

Since $\lim_{k\to\infty} y_*(k) = 0$ and $\tilde{A}$ is asymptotically stable, it follows from [17, Lemma 2.1] that there exists $S \in \mathbb{R}^{(n+n_{\rm c})\times n_w}$ such that $\tilde{A}S - SA_w = \tilde{D}C_w$ and $\tilde{C}S = D_2 C_w$. Therefore, define $Q \stackrel{\triangle}{=} \begin{bmatrix} I_{n+n_{\rm c}} & S \\ 0 & I_{n_w} \end{bmatrix}$, and consider the change of basis

$$\tilde{x}_Q(k) \stackrel{\triangle}{=} Q \begin{bmatrix} \tilde{x}(k) \\ x_w(k) \end{bmatrix} = \begin{bmatrix} \tilde{x}(k) + S x_w(k) \\ x_w(k) \end{bmatrix}, \quad (41)$$

$$\tilde{A}_Q \stackrel{\triangle}{=} Q \begin{bmatrix} \tilde{A} & \tilde{D}C_w \\ 0 & A_w \end{bmatrix} Q^{-1} = \begin{bmatrix} \tilde{A} & 0 \\ 0 & A_w \end{bmatrix}, \quad (42)$$

$$\tilde{C}_Q \stackrel{\triangle}{=} \begin{bmatrix} \tilde{C} & D_2 C_w \end{bmatrix} Q^{-1} = \begin{bmatrix} \tilde{C} & 0 \end{bmatrix}, \quad (43)$$

which implies that, for all $k \geq n_c$,

$$\tilde{x}_Q(k+1) = \tilde{A}_Q \tilde{x}_Q(k), \quad y_*(k) = \tilde{C}_Q \tilde{x}_Q(k).$$

Next, define $k_0 \triangleq n_* + n_c$, where n_* is the degree of the polynomial $d_*(\mathbf{z})$. Therefore, it follows from (41)-(43) that, for all $k \geq k_0$, $D_*(\mathbf{q}^{-1})y_*(k) = \tilde{C}_Q D_*(\mathbf{q}^{-1})\tilde{x}_Q(k) = \tilde{C}_Q d_*(\tilde{A}_Q)\tilde{x}_Q(k-n_*) = \tilde{C} d_*(\tilde{A})\left[\tilde{x}(k-n_*) + Sx_w(k-n_*)\right]$, and using (35) yields, for all $k \geq k_0$,

$$\begin{aligned} D_*(\mathbf{q}^{-1})y_*(k) &= \left[\begin{array}{cc} \tilde{C}_o & 0 \end{array} \right] \left[\begin{array}{cc} d_*(\tilde{A}_o) & 0 \\ \sharp & d_*(A_{\bar{o}}) \end{array} \right] \\ &\quad \times \left[\tilde{x}(k-n_*) + Sx_w(k-n_*)\right] \\ &= \left[\begin{array}{cc} \tilde{C}_o d_*(\tilde{A}_o) & 0 \end{array} \right] \\ &\quad \times \left[\tilde{x}(k-n_*) + Sx_w(k-n_*)\right], \end{aligned}$$

where $\sharp$ denotes an inconsequential entry. Since $\det(\mathbf{z}I_{n_*} - \tilde{A}_o) = d_*(\mathbf{z})$, the Cayley-Hamilton theorem implies that $d_*(\tilde{A}_o) = 0$. Therefore, for all $k \geq k_0$, $D_*(\mathbf{q}^{-1})y_*(k) = 0$.

To show *(iv)*, consider the transfer function representation of the open-loop system, which is given by

$$y = G_{yu}(\mathbf{z})u + G_{yw}(\mathbf{z})w, \tag{44}$$

and let $u(k) = u_c(k) + e(k)$, where $e(k)$ is an arbitrary signal, and $u_c(k)$ is the output of the ideal fixed-gain controller (13) with parameters (39) and (40) and the input $y(k)$, that is, $u_c = G_{c,*}(\mathbf{z})y$. It follows from (36) and (44) that the closed-loop system is given by $y = \tilde{G}_{ye}(\mathbf{z})e + \tilde{G}_{yw}(\mathbf{z})w$, where

$$\tilde{G}_{ye}(\mathbf{z}) \triangleq \frac{H_d \beta_u(\mathbf{z}) M_*(\mathbf{z})}{d_*(\mathbf{z})}, \tag{45}$$

$\tilde{G}_{yw}(\mathbf{z}) \triangleq \frac{M_{im}(\mathbf{z})\gamma(\mathbf{z})}{d_*(\mathbf{z})}$, and $d_*(\mathbf{z}) = \det(\mathbf{z}I_{n_*} - \tilde{A}_o) = \alpha(\mathbf{z})M_{im}(\mathbf{z}) - H_d\beta_u(\mathbf{z})N_{im}(\mathbf{z})$.

Next, consider the $2n_c^{\text{th}}$-order nonminimal-state-space realization of $u_c = G_{c,*}(\mathbf{z})y$ given by

$$\phi_c(k+1) = \mathcal{A}_c\phi_c(k) + \mathcal{B}_c y(k), \tag{46}$$

$$u_c(k) = \mathcal{C}_c\phi_c(k), \tag{47}$$

where

$$\mathcal{A}_c \triangleq \mathcal{A}_{nil} + \left[\begin{array}{c} 0_{n_c\times 2n_c} \\ E_{n_c}\theta_*^{\mathrm{T}} \end{array} \right], \ \mathcal{B}_c \triangleq \left[\begin{array}{c} E_{n_c} \\ 0_{n_c\times 1} \end{array} \right], \ \mathcal{C}_c \triangleq \theta_*^{\mathrm{T}},$$

$$\phi_c(k) \triangleq \left[\begin{array}{cccccc} y(k-1) & \cdots & y(k-n_c) & u_c(k-1) & \cdots & u_c(k-n_c) \end{array} \right]^{\mathrm{T}}.$$

Noting that $\mathcal{A}_c = \mathcal{A} + \mathcal{B}\mathcal{C}_c - \mathcal{B}_c\mathcal{C}$, the closed-loop system (5)-(10) and (46)-(47) is

$$\begin{aligned} \left[\begin{array}{c} \phi(k+1) \\ \phi_c(k+1) \end{array} \right] &= \left[\begin{array}{cc} \mathcal{A} & \mathcal{B}\mathcal{C}_c \\ \mathcal{B}_c\mathcal{C} & \mathcal{A}_c \end{array} \right] \left[\begin{array}{c} \phi(k) \\ \phi_c(k) \end{array} \right] \\ &\quad + \left[\begin{array}{c} \mathcal{B} \\ 0 \end{array} \right] e(k) + \left[\begin{array}{c} \mathcal{D}_1 \\ \mathcal{B}_c\mathcal{D}_2 \end{array} \right] W(k), \\ y(k) &= \left[\begin{array}{cc} \mathcal{C} & 0 \end{array} \right] \left[\begin{array}{c} \phi(k) \\ \phi_c(k) \end{array} \right] + \mathcal{D}_2 W(k), \end{aligned}$$

and the closed-loop transfer function from e to y is given by

$$\tilde{G}_{ye}(\mathbf{z}) = \left[\begin{array}{cc} \mathcal{C} & 0 \end{array} \right] \left(\mathbf{z}I_{4n_c} - \left[\begin{array}{cc} \mathcal{A} & \mathcal{B}\mathcal{C}_c \\ \mathcal{B}_c\mathcal{C} & \mathcal{A}_c \end{array} \right]\right)^{-1} \left[\begin{array}{c} \mathcal{B} \\ 0 \end{array} \right].$$

Next, with the change of basis

$$\left[\begin{array}{cc} \mathcal{A}_* & \mathcal{B}\mathcal{C}_c \\ 0 & \mathcal{A}_{nil} \end{array} \right] = \left[\begin{array}{cc} I & 0 \\ -I & I \end{array} \right] \left[\begin{array}{cc} \mathcal{A} & \mathcal{B}\mathcal{C}_c \\ \mathcal{B}_c\mathcal{C} & \mathcal{A}_c \end{array} \right] \left[\begin{array}{cc} I & 0 \\ I & I \end{array} \right],$$

$$\left[\begin{array}{c} \mathcal{B} \\ -\mathcal{B} \end{array} \right] = \left[\begin{array}{cc} I & 0 \\ -I & I \end{array} \right] \left[\begin{array}{c} \mathcal{B} \\ 0 \end{array} \right],$$

$$\left[\begin{array}{cc} \mathcal{C} & 0 \end{array} \right] = \left[\begin{array}{cc} \mathcal{C} & 0 \end{array} \right] \left[\begin{array}{cc} I & 0 \\ I & I \end{array} \right],$$

it follows that

$$\begin{aligned} \tilde{G}_{ye}(\mathbf{z}) &= \left[\begin{array}{cc} \mathcal{C} & 0 \end{array} \right] \left[\begin{array}{cc} \mathbf{z}I - \mathcal{A}_* & -\mathcal{B}\mathcal{C}_c \\ 0 & \mathbf{z}I - \mathcal{A}_{nil} \end{array} \right]^{-1} \left[\begin{array}{c} \mathcal{B} \\ -\mathcal{B} \end{array} \right] \\ &= \mathcal{C}(\mathbf{z}I - \mathcal{A}_*)^{-1}\mathcal{B} \\ &\quad - \mathcal{C}(\mathbf{z}I - \mathcal{A}_*)^{-1}\mathcal{B}\mathcal{C}_c(\mathbf{z}I - \mathcal{A}_{nil})^{-1}\mathcal{B}, \\ &= G_*(\mathbf{z})\left[1 - \mathcal{C}_c(\mathbf{z}I - \mathcal{A}_{nil})^{-1}\mathcal{B}\right]. \end{aligned} \tag{48}$$

Since $\mathcal{C}_c = \theta_*^{\mathrm{T}}$, it follows from the structure of $\mathcal{A}_{nil}$ and $\mathcal{B}$ and (37) that $\frac{M_*(\mathbf{z})}{\mathbf{z}^{n_c}} = 1 - \mathcal{C}_c(\mathbf{z}I_{2n_c} - \mathcal{A}_{nil})^{-1}\mathcal{B}$, and thus

$$\tilde{G}_{ye}(\mathbf{z}) = G_*(\mathbf{z})\frac{M_*(\mathbf{z})}{\mathbf{z}^{n_c}}. \tag{49}$$

Finally, comparing (45) and (49) yields (19), verifying *(iv)*. □

V. Error System

We now construct an error system using the ideal fixed-gain controller (which is not implemented) and a controller whose gains are updated by an adaptive law, which is presented in [7]. Since n and n_w are unknown, the lower bound for the controller order n_c given by (17) is unknown. However, since, by assumptions (A5) and (A8), $n \leq \bar{n}$ and $n_w \leq \bar{n}_w$, it follows that the modified lower bound

$$n_c \geq 2\bar{n} + 2\bar{n}_w - n_u - d, \tag{50}$$

implies (17). Furthermore, assumptions (A2), (A3), (A5), and (A8) imply that the lower bound on n_c given by (50) is known. Therefore, we let n_c satisfy the known lower bound (50).

Next, for all $k \geq n_c$, consider the time-varying controller

$$u(k) = \sum_{i=1}^{n_c} M_i(k)u(k-i) + \sum_{i=1}^{n_c} N_i(k)y(k-i), \tag{51}$$

where, for all $i = 1, \ldots, n_c$, $M_i : \mathbb{N} \to \mathbb{R}$ and $N_i : \mathbb{N} \to \mathbb{R}$ are given by the adaptive law presented in the following section. For all $k \geq n_c$, the controller (51) can be expressed as

$$u(k) = \phi^{\mathrm{T}}(k)\theta(k), \tag{52}$$

where $\theta(k) \triangleq \left[\begin{array}{cccccc} N_1(k) & \cdots & N_{n_c}(k) & M_1(k) & \cdots & M_{n_c}(k) \end{array} \right]^{\mathrm{T}}$, and, for all $k \geq n_c$, $\phi(k)$ is defined by (11). Furthermore, for all nonnegative integers $k < n_c$, let $u(k)$ be given by (52), where, for all nonnegative integers $k < n_c$, $\phi(k) \in \mathbb{R}^{2n_c}$. Inserting (52) into (5) yields, for all $k \geq n_c$

$$\phi(k+1) = \mathcal{A}\phi(k) + \mathcal{B}\phi^{\mathrm{T}}(k)\theta(k) + \mathcal{D}_1 W(k). \tag{53}$$

Next, define $n_* \triangleq n_c + n_u + d$, and let $d_*(\mathbf{q})$ be an asymptotically stable monic polynomial of degree n_*. Furthermore, let $\theta_* \in \mathbb{R}^{2n_c}$ be the ideal fixed-gain controller given by Theorem IV.1, and, for all $k \geq n_c$, let $\phi_*(k)$ be the state of the ideal closed-loop system (15), (16), where the initial condition is $\phi_*(n_c) = \phi(n_c)$. Finally, define $D_*(\mathbf{q}^{-1}) \triangleq \mathbf{q}^{-n_*} d_*(\mathbf{q})$, $\beta_*(\mathbf{q}^{-1}) \triangleq \mathbf{q}^{-n_u - d}\beta_u(\mathbf{q})$, and $k_0 \triangleq n_c + n_*$.

Next, defining $\tilde{\theta}(k) \triangleq \theta(k) - \theta_*$, and substituting $\theta(k) = \tilde{\theta}(k) + \theta_*$ into (53), for all $k \geq n_c$, the closed-loop system consisting of (5), (6), and (52) becomes

$$\phi(k+1) = \mathcal{A}_*\phi(k) + \mathcal{B}\phi^{\mathrm{T}}(k)\tilde{\theta}(k) + \mathcal{D}_1 W(k), \quad (54)$$
$$y(k) = \mathcal{C}\phi(k) + \mathcal{D}_2 W(k). \quad (55)$$

Now, we construct an error system by combining the ideal closed-loop system (15), (16) with the closed-loop system (54), (55). Define the error state $\tilde{\phi}(k) \triangleq \phi(k) - \phi_*(k)$, and subtract (15), (16) from (54), (55) to obtain, for all $k \geq n_c$,

$$\tilde{\phi}(k+1) = \mathcal{A}_*\tilde{\phi}(k) + \mathcal{B}\phi^{\mathrm{T}}(k)\tilde{\theta}(k), \quad (56)$$
$$\tilde{y}(k) = \mathcal{C}\tilde{\phi}(k), \quad (57)$$

where $\tilde{y}(k) \triangleq y(k) - y_*(k)$.

Now, for all $k \geq 0$, we define the filtered performance $y_f(k) \triangleq D_*(\mathbf{q}^{-1})y(k)$. Note that $y_f(0)$ depends on $y(-1), \ldots, y(-n_*)$, which can be chosen arbitrarily. In addition, for all $k \geq k_0$, define the ideal filtered performance $y_{f,*}(k) \triangleq D_*(\mathbf{q}^{-1})y_*(k)$ and the filtered performance error $\tilde{y}_f(k) \triangleq y_f(k) - y_{f,*}(k) = D_*(\mathbf{q}^{-1})\tilde{y}(k)$. Recall that the control objective is to drive y to zero. Since $d_*(\mathbf{q}) = \mathbf{q}^{n_*}D_*(\mathbf{q}^{-1})$ is an asymptotically stable polynomial it follows that y tends to zero if and only if y_f tends to zero. Furthermore, note that $y_f(k)$ is computable from $y(k)$ and the known asymptotically stable polynomial $d_*(\mathbf{z})$.

Note that the transfer function of the error system (56), (57) is $G_*(\mathbf{z})$, as given by (19). The following result relates $\tilde{y}_f(k)$ and $y_f(k)$ to the estimation error $\tilde{\theta}(k)$.

Lemma V.1. *Consider the error system (56), (57) with initial conditions $\theta(0)$ and $\phi(n_c)$. Then, for all $k \geq k_0$,*

$$y_f(k) = \tilde{y}_f(k) = H_d\beta_*(\mathbf{q}^{-1})\left[\phi^{\mathrm{T}}(k)\tilde{\theta}(k)\right]. \quad (58)$$

Proof. For all $k \geq k_0$, it follows from the error system (56), (57) that $\tilde{y}(k)$ is given by

$$\tilde{y}(k) = \mathcal{C}\mathcal{A}_*^{k-n_c}\tilde{\phi}(n_c) + \sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}\phi^{\mathrm{T}}(k-i)\tilde{\theta}(k-i).$$

Since $\phi_*(n_c) = \phi(n_c)$ it follows that $\tilde{\phi}(n_c) = 0$, and thus, for all $k \geq k_0$,

$$\tilde{y}(k) = \sum_{i=1}^{k-n_c} \mathcal{C}\mathcal{A}_*^{i-1}\mathcal{B}\phi^{\mathrm{T}}(k-i)\tilde{\theta}(k-i),$$

which is the forced solution of (56), (57), and thus equivalent to $\tilde{y}(k) = G_*(\mathbf{z})\left[\phi^{\mathrm{T}}(k)\tilde{\theta}(k)\right]$. Next, it follows from *(iv)* of Theorem IV.1 that, for all $k \geq k_0$, $\tilde{y}(k) = \frac{H_d\beta_u(\mathbf{z})\mathbf{z}^{n_c}}{d_*(\mathbf{z})}\left[\phi^{\mathrm{T}}(k)\tilde{\theta}(k)\right]$, which implies $d_*(\mathbf{q})\tilde{y}(k) = H_d\beta_u(\mathbf{q})\mathbf{q}^{n_c}\left[\phi^{\mathrm{T}}(k)\tilde{\theta}(k)\right]$. Multiplying both sides by $\mathbf{q}^{-n_*}$ yields $\tilde{y}_f(k) = H_d\beta_*(\mathbf{q}^{-1})\left[\phi^{\mathrm{T}}(k)\tilde{\theta}(k)\right]$. Furthermore, it follows from *(iii)* of Theorem IV.1 that, for all $k \geq k_0$, $y_{f,*}(k) = 0$, which implies that $y_f(k) = \tilde{y}_f(k) + y_{f,*}(k) = \tilde{y}_f(k)$, thus verifying (58). □

Lemma V.1 relates the filtered performance $y_f(k)$ to the estimation error $\tilde{\theta}(k)$. This relationship, given by (58), is not a linear regression in the estimation error $\tilde{\theta}(k)$; however, it is possible to define a surrogate measure of the filtered performance, which is a linear regression in $\tilde{\theta}(k)$. This surrogate measure is called the retrospective performance measure. We define the retrospective performance in [7], and use it to develop adaptive laws for the controller (52). The results from the present paper, specifically, Theorem IV.1 and Lemma V.1, are used in the construction and analysis of the retrospective cost adaptive controllers presented in [7].

REFERENCES

[1] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*. Prentice Hall, 1989.

[2] K. J. Åström and B. Wittenmark, *Adaptive Control*, 2nd ed. Addison-Wesley, 1995.

[3] P. Ioannou and J. Sun, *Robust Adaptive Control*. Prentice Hall, 1996.

[4] G. C. Goodwin and K. S. Sin, *Adaptive Filtering, Prediction, and Control*. Prentice Hall, 1984.

[5] G. Tao, *Adaptive Control Design and Analysis*. Wiley, 2003.

[6] R. J. Fuentes and M. J. Balas, "Direct adaptive disturbance accommodation," in *Proc. IEEE Conf. Dec. Contr.*, Sydney, Australia, December 2000, pp. 4921–4925.

[7] J. B. Hoagg and D. S. Bernstein, "Retrospective cost adaptive control for nonminimum-phase discrete-time systems, Part 2: The adaptive controller and stability analysis," in *Proc. Conf. Dec. Contr.*, Atlanta, GA, December 2010.

[8] R. Johansson, "Global Lyapunov stability and exponential convergence of direct adaptive control," *Int. J. Contr.*, vol. 50, pp. 859–869, 1989.

[9] G. C. Goodwin, P. J. Ramadge, and P. E. Caines, "Discrete-time multivariable adaptive control," *IEEE Trans. Autom. Contr.*, vol. 25, pp. 449–456, 1980.

[10] S. Akhtar and D. S. Bernstein, "Lyapunov-stable discrete-time model reference adaptive control," *Int. J. Adaptive Contr. Signal Proc.*, vol. 19, pp. 745–767, 2005.

[11] J. B. Hoagg, M. A. Santillo, and D. S. Bernstein, "Discrete-time adaptive command following and disturbance rejection with unknown exogenous dynamics," *IEEE Trans. Autom. Contr.*, vol. 53, pp. 912–928, 2008.

[12] D. S. Bayard, "Extended horizon liftings for stable inversion of nonminimum-phase systems," *IEEE Trans. Autom. Contr.*, vol. 39, pp. 1333–1338, 1994.

[13] R. Venugopal and D. S. Bernstein, "Adaptive disturbance rejection using ARMARKOV/Toeplitz models," *IEEE Trans. Contr. Sys. Tech.*, vol. 8, pp. 257–269, 2000.

[14] M. A. Santillo and D. S. Bernstein, "Adaptive control based on retrospective cost optimization," *AIAA J. Guid. Contr. Dyn.*, vol. 33, pp. 289–304, 2010.

[15] J. B. Hoagg and D. S. Bernstein, "Cumulative retrospective cost adaptive control with RLS-based optimization," in *Proc. Amer. Contr. Conf.*, Baltimore, MD, June 2010, pp. 4016–4021.

[16] L. Praly, S. T. Hung, and D. S. Rhode, "Towards a direct adaptive scheme for a discrete-time control of a minimum-phase continuous-time system," in *Proc. Conf. Dec. Contr.*, Ft. Lauderdale, FL, December 1985, pp. 1188–1191.

[17] J. B. Hoagg, M. A. Santillo, and D. S. Bernstein, "Internal model control in the shift and delta domains," *IEEE Trans. Autom. Contr.*, vol. 53, pp. 1066–1072, 2008.