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Retrospective Cost Adaptive Control for Nonminimum-Phase Discrete-Time Systems, Part 2: The Adaptive Controller and Stability Analysis

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Abstract— **This paper is the second part of a pair of papers, which together present a direct adaptive controller for discrete-time systems that are possibly nonminimum phase.**

I. Introduction

In this paper and its companion paper [1], we present a discrete-time adaptive controller for stabilization, command following, and disturbance rejection of discrete-time systems that are possibly nonminimum phase. This paper is intended to be read in conjunction with [1], which focuses on the existence and properties of an ideal control law as well as the construction of a closed-loop error system. The results of [1] are needed in the present paper to develop the adaptive law and analyze the closed-loop stability properties

In this second paper, we define the retrospective performance and develop adaptive control laws by minimizing quadratic cost functions of the retrospective performance. We consider an instantaneous retrospective cost, which is a function of the retrospective performance at the current time, as well as a cumulative retrospective cost, which is a function of the retrospective performance at the current time step and all previous time steps. The instantaneous retrospective cost is minimized by a gradient-type algorithm, while the cumulative retrospective cost is minimized by a recursive-least-squares-type algorithm. We then examine the stability properties of these retrospective cost adaptive controllers.

II. Review of the Problem Formulation

Consider the discrete-time system

$$x(k+1) = Ax(k) + Bu(k) + D_1 w(k), \tag{1}$$

$$y(k) = Cx(k) + D_2 w(k), \tag{2}$$

where $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}$, $u(k) \in \mathbb{R}$, $w(k) \in \mathbb{R}^{l_w}$, and $k \geq 0$. Our goal is to develop an adaptive output feedback controller, which generates a control signal u that drives the performance variable y to zero in the presence of the exogenous signal w. The relative degree $d \geq 1$ is the smallest positive integer i such that the ith Markov parameter $H_i \stackrel{\triangle}{=} CA^{i-1}B$ is nonzero. We make the following assumptions.

(A1) The triple (A, B, C) is controllable and observable.

(A2) If $\lambda \in \mathbb{C}$, $|\lambda| \geq 1$, and rank $\begin{bmatrix} A-\lambda I & B \\ C & 0 \end{bmatrix} < n+1$, then λ is known.

(A3) d is known.

(A4) The first nonzero Markov parameter H_d is known.

(A5) There exists an integer $\bar{n}$ such that $n \leq \bar{n}$ and $\bar{n}$ is known.

(A6) $y(k)$ is measured and available for feedback.

(A7) The exogenous signal $w(k)$ is generated by

$$x_w(k+1) = A_w x_w(k), \quad w(k) = C_w x_w(k), \tag{3}$$

where $x_w \in \mathbb{R}^{n_w}$ and A_w has distinct eigenvalues, all of which are on the unit circle, and none of which coincide with a zero of (A, B, C).

(A8) There exists an integer $\bar{n}_w$ such that $n_w \leq \bar{n}_w$ and $\bar{n}_w$ is known.

(A9) A, B, C, D_1, D_2, A_w, C_w, n, and n_w are not known.

Let $\beta_{\rm u}(\mathbf{z})$ be a monic polynomial whose roots are a subset of the zeros of the transfer function $G_{yu}(\mathbf{z}) \stackrel{\triangle}{=} C(\mathbf{z}I - A)^{-1}B$ and include all the zeros of $G_{yu}(\mathbf{z})$ that lie on or outside the unit circle. Furthermore, write $\beta_{\rm u}(\mathbf{z}) = \mathbf{z}^{n_{\rm u}} + \beta_{{\rm u},1}\mathbf{z}^{n_{\rm u}-1} + \cdots + \beta_{{\rm u},n_{\rm u}-1}\mathbf{z} + \beta_{{\rm u},n_{\rm u}}$, where $\beta_{{\rm u},1}, \ldots, \beta_{{\rm u},n_{\rm u}} \in \mathbb{R}$, and $n_{\rm u} \leq n-d$ is the degree of $\beta_{\rm u}(\mathbf{z})$.

III. Brief Review of [1]

In this section, we briefly review select definitions from [1]. First, let $n_{\rm c}$ be a positive integer that satisfies

$$n_{\rm c} \geq 2\bar{n} + 2\bar{n}_w - n_{\rm u} - d. \tag{4}$$

In [1], we show that, for all $k \geq n_{\rm c}$, (1), (2) has the $2n_{\rm c}^{\rm th}$-order nonminimal-state-space realization

$$\phi(k+1) = \mathcal{A}\phi(k) + \mathcal{B}u(k) + \mathcal{D}_1 W(k), \tag{5}$$

$$y(k) = \mathcal{C}\phi(k) + \mathcal{D}_2 W(k), \tag{6}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}_1$, $\mathcal{C}$, $\mathcal{D}_2$, and $W(k)$ are given in [1], and

$$\phi(k) \stackrel{\triangle}{=} \begin{bmatrix} y(k-1) & \cdots & y(k-n_{\rm c}) \\ u(k-1) & \cdots & u(k-n_{\rm c}) \end{bmatrix}^{\rm T}. \tag{7}$$

Next, for all $k \geq n_{\rm c}$, consider the time-varying controller

$$u(k) = \sum_{i=1}^{n_{\rm c}} M_i(k)u(k-i) + \sum_{i=1}^{n_{\rm c}} N_i(k)y(k-i), \tag{8}$$

where, for all $i = 1, \ldots, n_{\rm c}$, $M_i : \mathbb{N} \to \mathbb{R}$ and $N_i : \mathbb{N} \to \mathbb{R}$ are given by the adaptive law presented in the following section. The controller (8) can be expressed as

$$u(k) = \phi^{\rm T}(k)\theta(k), \tag{9}$$

where $\theta(k) \stackrel{\triangle}{=} \begin{bmatrix} N_1(k) & \cdots & N_{n_{\rm c}}(k) & M_1(k) & \cdots & M_{n_{\rm c}}(k) \end{bmatrix}^{\rm T}$.

Next, let $\mathbf{q}$ and $\mathbf{q}^{-1}$ denote the forward-shift and backward-shift operators, respectively. Define $n_* \stackrel{\triangle}{=} n_{\rm c} +$

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$n_\mathrm{u} + d$, and let $d_*(\mathbf{q})$ be an asymptotically stable monic polynomial of degree n_*. Furthermore, let $\theta_* \in \mathbb{R}^{2n_\mathrm{c}}$ be the ideal fixed-gain controller given by [1, Theorem IV.1], and, for all $k \geq n_\mathrm{c}$, let $\phi_*(k)$ be the state of the ideal closed-loop system given by [1, Theorem IV.1], where the initial condition is $\phi_*(n_\mathrm{c}) = \phi(n_\mathrm{c})$. Finally, define $D_*(\mathbf{q}^{-1}) \triangleq \mathbf{q}^{-n_*} d_*(\mathbf{q})$, $\beta_*(\mathbf{q}^{-1}) \triangleq \mathbf{q}^{-n_\mathrm{u}-d}\beta_\mathrm{u}(\mathbf{q})$, and $k_0 \triangleq n_\mathrm{c} + n_*$.

Next, it follows from [1] that, for all $k \geq n_\mathrm{c}$, the error system between the ideal closed-loop system (given by [1, Theorem IV.1]) and the closed-loop system (5), (6) with the control (9), is given by

$$\tilde{\phi}(k+1) = \mathcal{A}_*\tilde{\phi}(k) + \mathcal{B}\phi^\mathrm{T}(k)\tilde{\theta}(k), \tag{10}$$

$$\tilde{y}(k) = \mathcal{C}\tilde{\phi}(k), \tag{11}$$

where $\tilde{\theta}(k) \triangleq \theta(k) - \theta_*$, $\tilde{\phi}(k) \triangleq \phi(k) - \phi_*(k)$, $\tilde{y}(k) \triangleq y(k) - y_*(k)$, and $\mathcal{A}_* \triangleq \mathcal{A} + \mathcal{B}\theta_*^\mathrm{T}$. Note that it follows from [1, Theorem IV.1] that $\mathcal{A}_*$ is asymptotically stable.

Finally, for all $k \geq 0$, we define the filtered performance $y_\mathrm{f}(k) \triangleq D_*(\mathbf{q}^{-1})y(k)$. In addition, for all $k \geq k_0$, define the ideal filtered performance $y_{\mathrm{f},*}(k) \triangleq D_*(\mathbf{q}^{-1})y_*(k)$ and the filtered performance error $\tilde{y}_\mathrm{f}(k) \triangleq y_\mathrm{f}(k) - y_{\mathrm{f},*}(k) = D_*(\mathbf{q}^{-1})\tilde{y}(k)$.

IV. Retrospective Performance and Adaptation

In this section, we define the retrospective performance and present two adaptive laws for the controller (9). Let $\hat{\theta} \in \mathbb{R}^{2n_\mathrm{c}}$ be an optimization variable, and, for all $k \geq 0$, define the retrospective performance

$$\begin{aligned}\hat{y}_\mathrm{f}(\hat{\theta}, k) &\triangleq y_\mathrm{f}(k) + H_d\left[\beta_*(\mathbf{q}^{-1})\phi^\mathrm{T}(k)\right]\hat{\theta} - H_d\beta_*(\mathbf{q}^{-1})u(k) \\ &= y_\mathrm{f}(k) + \Phi^\mathrm{T}(k)\hat{\theta} - H_d\beta_*(\mathbf{q}^{-1})\left[\phi^\mathrm{T}(k)\theta(k)\right],\end{aligned} \tag{12}$$

where the filtered regressor is defined by $\Phi(k) \triangleq H_d\beta_*(\mathbf{q}^{-1})\phi(k) = H_d\sum_{i=d}^{n_\mathrm{u}+d}\beta_{\mathrm{u},i-d}\phi(k-i)$, where $\beta_{\mathrm{u},0} = 1$ and, for $k < 0$, $u(k) = 0$ and $\phi(k) = 0$. Note that the retrospective performance (12) modifies $y_\mathrm{f}(k)$ based on the difference between the actual past control inputs $u(k-d), \ldots, u(k-n_\mathrm{u}-d)$ and the recomputed control inputs $\hat{u}(\hat{\theta}, k-d) \triangleq \phi^\mathrm{T}(k-d)\hat{\theta}, \ldots, \hat{u}(\hat{\theta}, k-n_\mathrm{u}-d) \triangleq \phi^\mathrm{T}(k-n_\mathrm{u}-d)\hat{\theta}$, assuming that the controller parameter vector $\hat{\theta}$ had been used in the past.

For all $k \geq 0$, we also define the retrospective performance measurement

$$y_{\mathrm{f,r}}(k) \triangleq \hat{y}_\mathrm{f}(\theta(k), k). \tag{13}$$

Although $y_{\mathrm{f,r}}(k)$ is not a measurement, it can be computed from $y_\mathrm{f}(k)$, $\theta(k)$, $\theta(k-d), \ldots, \theta(k-n_\mathrm{u}-d)$, $\phi(k-d), \ldots, \phi(k-n_\mathrm{u}-d)$, and knowledge of $\beta_*(\mathbf{q}^{-1})$ by using (12). Now, we develop two adaptive laws using $\hat{y}_\mathrm{f}(\hat{\theta}, k)$.

A. Instantaneous Retrospective Cost Function

Define the instantaneous retrospective cost function

$$J_\mathrm{I}(\hat{\theta}, k) \triangleq \hat{y}_\mathrm{f}^2(\hat{\theta}, k) + \zeta(k)\left[\hat{\theta} - \theta(k)\right]^\mathrm{T} R\left[\hat{\theta} - \theta(k)\right],$$

where $R \in \mathbb{R}^{2n_\mathrm{c}\times 2n_\mathrm{c}}$ is positive definite, $\zeta : \mathbb{N} \to (0, \infty)$, $\zeta_\mathrm{L} \triangleq \inf_{k\geq 0}\zeta(k)$, and $\zeta_\mathrm{U} \triangleq \sup_{k\geq 0}\zeta(k)$. We assume that $\zeta_\mathrm{L} > 0$ and $\zeta_\mathrm{U} < \infty$.

Lemma IV.1. *Let $\theta(0) \in \mathbb{R}^{2n_\mathrm{c}}$. Then, for each $k \geq 0$, the unique global minimizer of the instantaneous retrospective cost function $J_\mathrm{I}(\hat{\theta}, k)$ is given by*

$$\theta(k+1) = \theta(k) - \eta(k)R^{-1}\Phi(k)y_{\mathrm{f,r}}(k), \tag{14}$$

where

$$\eta(k) \triangleq \frac{1}{\zeta(k) + \Phi^\mathrm{T}(k)R^{-1}\Phi(k)}. \tag{15}$$

The proof of this result has been omitted due to space considerations.

B. Cumulative Retrospective Cost Function

As an alternative to $J_\mathrm{I}(\hat{\theta}, k)$, we define the cumulative retrospective cost function

$$J_\mathrm{C}(\hat{\theta}, k) \triangleq \sum_{i=0}^{k}\lambda^{k-i}\hat{y}_\mathrm{f}^2(\hat{\theta}, i) + \lambda^k\left[\hat{\theta} - \theta(0)\right]^\mathrm{T} R\left[\hat{\theta} - \theta(0)\right],$$

where $\lambda \in (0, 1]$ and $R \in \mathbb{R}^{2n_\mathrm{c}\times 2n_\mathrm{c}}$ is positive definite. The next result follows from standard recursive-least-squares theory [2].

Lemma IV.2. *Let $P(0) = R^{-1}$ and $\theta(0) \in \mathbb{R}^{2n_\mathrm{c}}$. Then, for each $k \geq 0$, the unique global minimizer of the cumulative retrospective cost function $J_\mathrm{C}(\hat{\theta}, k)$ is given by*

$$\theta(k+1) = \theta(k) - \frac{P(k)\Phi(k)y_{\mathrm{f,r}}(k)}{\lambda + \Phi^\mathrm{T}(k)P(k)\Phi(k)}, \tag{16}$$

where

$$P(k+1) = \frac{1}{\lambda}\left[P(k) - \frac{P(k)\Phi(k)\Phi^\mathrm{T}(k)P(k)}{\lambda + \Phi^\mathrm{T}(k)P(k)\Phi(k)}\right]. \tag{17}$$

V. Adaptive System and Stability Analysis

We now analyze the closed-loop stability of the instantaneous retrospective cost adaptive controller presented in Lemma IV.1 as well as the cumulative retrospective cost adaptive controller presented in Lemma IV.2.

For all $k \geq k_0$, we define the ideal filtered regressor

$$\Phi_*(k) \triangleq H_d\beta_*(\mathbf{q}^{-1})\phi_*(k), \tag{18}$$

and the filtered regressor error

$$\tilde{\Phi}(k) \triangleq \Phi(k) - \Phi_*(k) = H_d\beta_*(\mathbf{q}^{-1})\tilde{\phi}(k). \tag{19}$$

Next, we apply the operator $H_d\beta_*(\mathbf{q}^{-1})$ to (10) and use [1, Lemma V.1] to obtain the filtered error system

$$\begin{aligned}\tilde{\Phi}(k+1) &= \mathcal{A}_*\tilde{\Phi}(k) + \mathcal{B}H_d\beta_*(\mathbf{q}^{-1})\left[\phi^\mathrm{T}(k)\tilde{\theta}(k)\right] \\ &= \mathcal{A}_*\tilde{\Phi}(k) + \mathcal{B}\tilde{y}_\mathrm{f}(k),\end{aligned} \tag{20}$$

which is defined for all $k \geq k_0$.

Next, for all $k \geq k_0$, define the retrospective performance measure error $\tilde{y}_{\mathrm{f,r}}(k) \triangleq y_{\mathrm{f,r}}(k) - y_{\mathrm{f},*}(k)$. The following result relates $\tilde{y}_{\mathrm{f,r}}(k)$ and $y_{\mathrm{f,r}}(k)$ to the estimation error $\tilde{\theta}(k)$.

Proposition V.1. *Consider the error system (10), (11) with initial conditions* $\theta(0)$ *and* $\phi(n_{\rm c})$*. Then, for all* $k \geq k_0$*,*

$$y_{\rm f,r}(k) = \tilde{y}_{\rm f,r}(k) = \Phi^{\rm T}(k)\tilde{\theta}(k). \tag{21}$$

Proof. It follows from (12) and (13) that, for all $k \geq k_0$, $\tilde{y}_{\rm f,r}(k) = \tilde{y}_{\rm f}(k) - H_d\beta_*(\mathbf{q}^{-1})\left[\phi^{\rm T}(k)\theta(k)\right] + \Phi^{\rm T}(k)\theta(k)$. Next, it follows from [1, Lemma V.1] that, for all $k \geq k_0$, $\tilde{y}_{\rm f,r}(k) = \Phi^{\rm T}(k)\theta(k) - \Phi^{\rm T}(k)\theta_* = \Phi^{\rm T}(k)\tilde{\theta}(k)$. Furthermore, it follows from [1, Theorem IV.1] that, for all $k \geq k_0$, $y_{\rm f,*}(k) = 0$, which implies that $y_{\rm f,r}(k) = \tilde{y}_{\rm f,r}(k) + y_{\rm f,*}(k) = \tilde{y}_{\rm f,r}(k)$, thus verifying (21). □

A. Instantaneous Retrospective Cost Adaptive Control

In this section, we analyze the stability properties of the instantaneous retrospective cost adaptive controller (9), (14), and (15), as well as the stability properties of the closed-loop system. The following lemma provides the stability properties of the instantaneous retrospective cost adaptive controller.

Lemma V.1. *Consider the open-loop system (1), (2) satisfying assumptions (A1)-(A9), and the instantaneous retrospective cost adaptive controller (9), (14), and (15), where* $n_{\rm c}$ *satisfies (4). Then, for all initial conditions* $x(0)$*,* $x_w(0)$*, and* $\theta(0)$*, the following properties hold:*

(i) $\theta(k)$ *is bounded.*
(ii) $\lim_{k\to\infty}\sum_{j=0}^{k}\eta(j)\tilde{y}_{\rm f,r}^2(j)$ *exists.*
(iii) For all $N > 0$*,* $\lim_{k\to\infty}\sum_{j=N}^{k}\|\theta(j) - \theta(j-N)\|^2$ *exists.*

Proof. Subtracting θ_* from both sides of (14) yields the estimator-error update equation

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \eta(k)R^{-1}\Phi(k)y_{\rm f,r}(k). \tag{22}$$

Define the positive-definite, radially unbounded Lyapunov-like function $V_{\tilde{\theta}}(\tilde{\theta}(k)) \triangleq \tilde{\theta}^{\rm T}(k)R\tilde{\theta}(k)$, and the Lyapunov-like difference $\Delta V_{\tilde{\theta}}(k) \triangleq V_{\tilde{\theta}}(\tilde{\theta}(k+1)) - V_{\tilde{\theta}}(\tilde{\theta}(k))$. Evaluating $\Delta V_{\tilde{\theta}}(k)$ along the trajectories of the estimator-error system (22) yields $\Delta V_{\tilde{\theta}}(k) = -2\eta(k)y_{\rm f,r}(k)\Phi^{\rm T}(k)\tilde{\theta}(k) + \eta^2(k)y_{\rm f,r}^2(k)\Phi^{\rm T}(k)R^{-1}\Phi(k)$. Next, it follows from Proposition V.1 and (15) that, for all $k \geq k_0$,

$$\begin{aligned}\Delta V_{\tilde{\theta}}(k) &= -2\eta(k)\tilde{y}_{\rm f,r}^2(k) + \eta^2(k)\tilde{y}_{\rm f,r}^2(k)\Phi^{\rm T}(k)R^{-1}\Phi(k)\\ &\leq -\eta(k)\tilde{y}_{\rm f,r}^2(k).\end{aligned} \tag{23}$$

Since $V_{\tilde{\theta}}$ is a positive-definite radially unbounded function of $\tilde{\theta}(k)$ and, for $k \geq k_0$, $\Delta V_{\tilde{\theta}}(k)$ is non-positive, it follows that $\tilde{\theta}(k)$ is bounded and thus $\theta(k)$ is bounded. Thus, we have verified *(i)*.

To show *(ii)*, first we show that $\lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j)$ exists. Since $V_{\tilde{\theta}}$ is positive definite, and, for all $k \geq k_0$, $\Delta V_{\tilde{\theta}}(k)$ is non-positive, it follows that $0 \leq -\lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j) = V_{\tilde{\theta}}(\tilde{\theta}(k_0)) - \lim_{k\to\infty}V_{\tilde{\theta}}(\tilde{\theta}(k)) \leq V_{\tilde{\theta}}(\tilde{\theta}(k_0))$, where the upper and lower bounds imply that both limits exist. Since $\lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j)$ exists, (23) implies that $\lim_{k\to\infty}\sum_{j=k_0}^{k}\eta(j)\tilde{y}_{\rm f,r}^2(j)$ exists, and thus $\lim_{k\to\infty}\sum_{j=0}^{k}\eta(j)\tilde{y}_{\rm f,r}^2(j)$ exists, which verifies *(ii)*.

To show *(iii)*, we first show that $\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1) - \theta(j)\|^2$ exists. It follows from (14) that

$$\begin{aligned}\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1)-\theta(j)\|^2 &= \lim_{k\to\infty}\sum_{j=0}^{k}\eta^2(j)y_{\rm f,r}^2(j)\Phi^{\rm T}(j)R^{-2}\Phi(j)\\ &\leq \|R^{-1}\|_{\rm F}\lim_{k\to\infty}\sum_{j=0}^{k}\eta^2(j)y_{\rm f,r}^2(j)\Phi^{\rm T}(j)R^{-1}\Phi(j),\end{aligned}$$

where $\|\cdot\|_{\rm F}$ denotes the Frobenius norm. Next, it follows from (15) that, for all $k \geq 0$, $\eta(k)\Phi^{\rm T}(k)R^{-1}\Phi(k) \leq 1$, which implies that $\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1) - \theta(j)\|^2 \leq \|R^{-1}\|_{\rm F}\lim_{k\to\infty}\sum_{j=0}^{k}\eta(j)y_{\rm f,r}^2(j)$. Furthermore, since by *(ii)*, $\lim_{k\to\infty}\sum_{j=0}^{k}\eta(j)\tilde{y}_{\rm f,r}^2(j)$ exists, and, for all $k \geq k_0$, $y_{\rm f,r}(k) = \tilde{y}_{\rm f,r}(k)$, it follows that $\lim_{k\to\infty}\sum_{j=0}^{k}\eta(j)y_{\rm f,r}^2(j)$ exists. Therefore, since $\lim_{k\to\infty}\sum_{j=0}^{k}\eta(j)y_{\rm f,r}^2(j)$ exists, it follows that $\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1) - \theta(j)\|^2$ exists. Next, let $N > 0$ and note that

$$\begin{aligned}\lim_{k\to\infty}\sum_{j=N}^{k}\|\theta(j)-\theta(j-N)\|^2 &\leq \lim_{k\to\infty}\sum_{j=N}^{k}(\|\theta(j)-\theta(j-1)\|\\ &\quad+\cdots+\|\theta(j-N+1)-\theta(j-N)\|)^2\\ &\leq \lim_{k\to\infty}2^{N-1}\sum_{j=N}^{k}\left(\|\theta(j)-\theta(j-1)\|^2\right.\\ &\quad\left.+\cdots+\|\theta(j-N+1)-\theta(j-N)\|^2\right).\end{aligned} \tag{24}$$

Since all of the limits on the right hand side of (24) exist, it follows that $\lim_{k\to\infty}\sum_{j=N}^{k}\|\theta(j) - \theta(j-N)\|^2$ exists. This verifies *(iii)*. □

The following theorem is the main result of the paper regarding the instantaneous retrospective cost adaptive controller. Let $\xi_1, \ldots, \xi_{n_{\rm u}} \in \mathbb{C}$ denote the $n_{\rm u}$ roots of $\beta_{\rm u}(\mathbf{z})$, and define $M(\mathbf{z}, k) \triangleq \mathbf{z}^{n_{\rm c}} - M_1(k)\mathbf{z}^{n_{\rm c}-1} - \cdots - M_{n_{\rm c}-1}(k)\mathbf{z} - M_{n_{\rm c}}(k)$, which can be interpreted as the denominator polynomial of the controller (9) at frozen time k.

Theorem V.1. *Consider the open-loop system (1), (2) satisfying assumptions (A1)-(A9), and the instantaneous retrospective cost adaptive controller (9), (14), and (15), where* $n_{\rm c}$ *satisfies (4). Assume that there exist* $\epsilon > 0$ *and* $k_1 > 0$ *such that, for all* $k \geq k_1$ *and for all* $i = 1, \ldots, n_{\rm u}$*,* $|M(\xi_i, k)| \geq \epsilon$*. Then, for all initial conditions* $x(0)$*,* $x_w(0)$*, and* $\theta(0)$*,* $\theta(k)$ *is bounded,* $u(k)$ *is bounded, and* $\lim_{k\to\infty} y(k) = 0$*.*

Proof. It follows immediately from *(i)* of Lemma V.1 that $\theta(k)$ is bounded. To prove the remaining properties, define the quadratic function $J(\tilde{\Phi}(k)) \triangleq \tilde{\Phi}^{\rm T}(k)\mathcal{P}\tilde{\Phi}(k)$, where $\mathcal{P} > 0$ satisfies the discrete-time Lyapunov equation $\mathcal{P} = \mathcal{A}_*^{\rm T}\mathcal{P}\mathcal{A}_* + \mathcal{Q} + \alpha I$, where $\mathcal{Q} > 0$ and $\alpha > 0$. Note that $\mathcal{P}$ exists since $\mathcal{A}_*$ is asymptotically stable. Defining $\Delta J(k) \triangleq J(\tilde{\Phi}(k+1)) - J(\tilde{\Phi}(k))$, it follows from (20) that, for all $k \geq k_0$,

$$\begin{aligned}\Delta J(k) =& -\tilde{\Phi}^{\rm T}(k)\left(\mathcal{Q}+\alpha I\right)\tilde{\Phi}(k) + \tilde{\Phi}^{\rm T}(k)\mathcal{A}_*^{\rm T}\mathcal{P}\mathcal{B}\tilde{y}_{\rm f}(k)\\ &+ \tilde{y}_{\rm f}(k)\mathcal{B}^{\rm T}\mathcal{P}\tilde{\mathcal{A}}^*\tilde{\Phi}(k) + \tilde{y}_{\rm f}^2(k)\mathcal{B}^{\rm T}\mathcal{P}\mathcal{B}\\ \leq& -\tilde{\Phi}^{\rm T}(k)\mathcal{Q}\tilde{\Phi}(k) + \sigma_1\tilde{y}_{\rm f}^2(k), \end{aligned} \tag{25}$$

where $\sigma_1 \triangleq \mathcal{B}^{\rm T}\mathcal{P}\mathcal{B} + \frac{1}{\alpha}\mathcal{B}^{\rm T}\mathcal{P}\mathcal{A}_*\mathcal{A}_*^{\rm T}\mathcal{P}\mathcal{B}$.

Now, consider the positive-definite, radially unbounded Lyapunov-like function $V_{\tilde{\Phi}}(\tilde{\Phi}(k)) \triangleq \ln\left(1 + a_1 J(\tilde{\Phi}(k))\right)$, where $a_1 > 0$ is specified below. The Lyapunov-like difference is thus given by $\Delta V_{\tilde{\Phi}}(k) \triangleq V_{\tilde{\Phi}}(\tilde{\Phi}(k+1)) - V_{\tilde{\Phi}}(\tilde{\Phi}(k))$. For all $k \geq k_0$, evaluating $\Delta V_{\tilde{\Phi}}(k)$ along the trajectories of (20) yields

$$\Delta V_{\tilde{\Phi}}(k) = \ln\left(1 + \frac{a_1\Delta J(k)}{1 + a_1 J(\tilde{\Phi}(k))}\right), \tag{26}$$

Since, for all $x > 0$, $\ln x \leq x - 1$, and using (25) we have

$$\begin{aligned}\Delta V_{\tilde{\Phi}}(k) &\leq a_1\frac{\Delta J(k)}{1 + a_1 J(\tilde{\Phi}(k))}\\ &\leq -W(\tilde{\Phi}(k)) + a_1\sigma_1\frac{\tilde{y}_{\rm f}^2(k)}{1 + a_1\tilde{\Phi}^{\rm T}(k)\mathcal{P}\tilde{\Phi}(k)}\\ &\leq -W(\tilde{\Phi}(k)) + a_1\sigma_1\epsilon^2(k), \end{aligned} \tag{27}$$

where

$$W(\tilde{\Phi}(k)) \triangleq a_1\frac{\tilde{\Phi}^{\rm T}(k)\mathcal{Q}\tilde{\Phi}(k)}{1 + a_1\tilde{\Phi}^{\rm T}(k)\mathcal{P}\tilde{\Phi}(k)}, \tag{28}$$

$$\epsilon(k) \triangleq \frac{\tilde{y}_{\rm f}(k)}{\sqrt{1 + a_1\lambda_{\min}\left(\mathcal{P}\right)\tilde{\Phi}^{\rm T}(k)\tilde{\Phi}(k)}}. \tag{29}$$

Now, we show that $\lim_{k\to\infty}\sum_{j=0}^{k}\epsilon^2(j)$ exists. First, it follows from [1, Lemma V.1] and Proposition V.1 that, for all $k \geq k_0$,

$$\begin{aligned}\tilde{y}_{\rm f}(k) =& -H_d\sum_{i=d}^{n_{\rm u}+d}\beta_{{\rm u},i-d}\phi^{\rm T}(k-i)\left[\theta(k) - \theta(k-i)\right]\\ &+ \tilde{y}_{\rm f,r}(k). \end{aligned} \tag{30}$$

Using (29) and (30) yields, for all $k \geq k_0$,

$$\begin{aligned}|\epsilon(k)| \leq& \frac{|\tilde{y}_{\rm f,r}(k)|}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\rm T}(k)\tilde{\Phi}(k)}}\\ &+ \frac{|H_d|\sum_{i=d}^{n_{\rm u}+d}|\beta_{{\rm u},i-d}|\|\phi(k-i)\|\|\theta(k)-\theta(k-i)\|}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\rm T}(k)\tilde{\Phi}(k)}}. \end{aligned}$$

It follows from Lemma V.1 that $\theta(k)$ is bounded and $\lim_{k\to\infty}\|\theta(k) - \theta(k-1)\| = 0$. Therefore, Lemma A.1 implies that there exist $k_2 \geq k_0 > 0$, $c_1 > 0$, and $c_2 > 0$, such that, for all $k \geq k_2$, and, for all $i = d, \ldots, n_{\rm u} + d$, $\|\phi(k-i)\| \leq c_1 + c_2\|\Phi(k)\|$. In addition, note that $\|\Phi(k)\| = \|\tilde{\Phi}(k) + \Phi_*(k)\| \leq \|\tilde{\Phi}(k)\| + \|\Phi_*(k)\| \leq \|\tilde{\Phi}(k)\| + \Phi_{*,\max}$, where $\Phi_{*,\max} \triangleq \sup_{k\geq 0}\|\Phi_*(k)\|$ exists because Φ_* is bounded. Therefore, for all $k \geq k_2$, $\|\phi(k-i)\| \leq c_1 + c_2\Phi_{*,\max} + c_2\|\tilde{\Phi}(k)\|$, which implies that

$$\begin{aligned}|\epsilon(k)| \leq& \frac{|\tilde{y}_{\rm f,r}(k)|}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\rm T}(k)\tilde{\Phi}(k)}}\\ &+ \frac{c_3 + c_4\|\tilde{\Phi}(k)\|}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\rm T}(k)\tilde{\Phi}(k)}}\\ &\times\left(\sum_{i=d}^{n_{\rm u}+d}\|\theta(k) - \theta(k-i)\|\right), \end{aligned}$$

where $c_3 \triangleq (c_1 + c_2\Phi_{*,\max})|H_d|(\sum_{i=d}^{n_{\rm u}+d}|\beta_{{\rm u},i-d}|) > 0$ and $c_4 \triangleq c_2|H_d|(\sum_{i=d}^{n_{\rm u}+d}|\beta_{{\rm u},i-d}|) > 0$. Next, note that $\frac{1}{\sqrt{1+a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\rm T}(k)\tilde{\Phi}(k)}} \leq 1$ and $\frac{\|\tilde{\Phi}(k)\|}{\sqrt{1+a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\rm T}(k)\tilde{\Phi}(k)}} \leq \max\left(1, 1/\sqrt{a_1\lambda_{\min}(\mathcal{P})}\right)$, which implies that

$$\begin{aligned}|\epsilon(k)| \leq& \frac{|\tilde{y}_{\rm f,r}(k)|}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\rm T}(k)\tilde{\Phi}(k)}}\\ &+ c_5\sum_{i=d}^{n_{\rm u}+d}\|\theta(k) - \theta(k-i)\|, \end{aligned} \tag{31}$$

where $c_5 \triangleq c_3 + c_4\max\left(1, 1/\sqrt{a_1\lambda_{\min}(\mathcal{P})}\right) > 0$.

Next, we show that we can choose $a_1 > 0$ such that the first term of (31) is bounded by $\sqrt{\eta(k)}|\tilde{y}_{\rm f,r}(k)|$, which is square summable according to *(ii)* of Lemma V.1. Note that $\Phi^{\rm T}(k)\Phi(k) \leq 2\tilde{\Phi}^{\rm T}(k)\tilde{\Phi}(k) + 2\Phi_*^{\rm T}(k)\Phi_*(k)$. Therefore, it follows from (15) that

$$\begin{aligned}\frac{1}{\eta(k)} &= \zeta(k) + \Phi^{\rm T}(k)R^{-1}\Phi(k)\\ &\leq \zeta_{\rm U} + \lambda_{\max}(R^{-1})\Phi^{\rm T}(k)\Phi(k)\\ &\leq \zeta_{\rm U} + 2\lambda_{\max}(R^{-1})\Phi_{*,\max}^2 + 2\lambda_{\max}(R^{-1})\tilde{\Phi}^{\rm T}(k)\tilde{\Phi}(k)\\ &= c_6\left[1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\rm T}(k)\tilde{\Phi}(k)\right], \end{aligned}$$

where $a_1 \triangleq \frac{2\lambda_{\max}(R^{-1})}{\lambda_{\min}(\mathcal{P})\left[\zeta_{\rm U} + 2\lambda_{\max}(R^{-1})\Phi_{*,\max}^2\right]} > 0$ and $c_6 \triangleq \zeta_{\rm U} + 2\lambda_{\max}(R^{-1})\Phi_{*,\max}^2 > 0$. Therefore,

$$\frac{1}{\sqrt{1 + a_1\lambda_{\min}(\mathcal{P})\tilde{\Phi}^{\rm T}(k)\tilde{\Phi}(k)}} \leq \sqrt{c_6}\sqrt{\eta(k)},$$

which combining with (31) implies that, for all $k \geq k_2$, $|\epsilon(k)| \leq \sqrt{c_6}\sqrt{\eta(k)}|\tilde{y}_{\rm f,r}(k)| + c_5\sum_{i=d}^{n_{\rm u}+d}\|\theta(k) - \theta(k-i)\|$. Therefore, for all $k \geq k_2$,

$$\begin{aligned}\epsilon^2(k) &\leq \left[\sqrt{c_6}\sqrt{\eta(k)}|\tilde{y}_{\rm f,r}(k)| + c_5\sum_{i=d}^{n_{\rm u}+d}\|\theta(k) - \theta(k-i)\|\right]^2\\ &\leq 2c_6\eta(k)\tilde{y}_{\rm f,r}^2(k) + 2c_5^2\left[\sum_{i=d}^{n_{\rm u}+d}\|\theta(k) - \theta(k-i)\|\right]^2\\ &\leq 2c_6\eta(k)\tilde{y}_{\rm f,r}^2(k) + 2^{n_{\rm u}+1}c_5^2\sum_{i=d}^{n_{\rm u}+d}\|\theta(k) - \theta(k-i)\|^2. \end{aligned} \tag{32}$$

It follows from *(ii)* of Lemma V.1 that $\lim_{k\to\infty}\sum_{j=0}^{k}\eta(j)\tilde{y}_{\mathrm{f,r}}^2(j)$ exists. Furthermore, it follows from *(iii)* of Lemma V.1 that, for all $i=d,\ldots,n_{\mathrm{u}}+d$, $\lim_{k\to\infty}\sum_{j=i}^{k}\|\theta(j)-\theta(j-i)\|^2$ exists. Thus, (32) implies that $\lim_{k\to\infty}\sum_{j=0}^{k}\epsilon^2(j)$ exists.

Now, we show that $\lim_{k\to\infty}W(\tilde{\Phi}(k))=0$. Since W and V are positive definite, it follows from (27) that

$$\begin{aligned}0 &\le \lim_{k\to\infty}\sum_{j=0}^{k}W(\tilde{\Phi}(j))\\ &\le \lim_{k\to\infty}\sum_{j=0}^{k}-\Delta V(j)+a_1\sigma_1\lim_{k\to\infty}\sum_{j=0}^{k}\epsilon^2(j)\\ &= V(\tilde{\Phi}(0))-\lim_{k\to\infty}V(\tilde{\Phi}(k))+a_1\sigma_1\lim_{k\to\infty}\sum_{j=0}^{k}\epsilon^2(j)\\ &\le V(\tilde{\Phi}(0))+a_1\sigma_1\lim_{k\to\infty}\sum_{j=0}^{k}\epsilon^2(j),\end{aligned}$$

where the upper and lower bound imply that all limits exist. Thus, $\lim_{k\to\infty}W(\tilde{\Phi}(k))=0$, which implies that $\lim_{k\to\infty}\|\tilde{\Phi}(k)\|=0$.

To prove that $u(k)$ is bounded, first note that since $\lim_{k\to\infty}\|\tilde{\Phi}(k)\|=0$ and $\Phi_*(k)$ is bounded, it follows that $\Phi(k)$ is bounded. Next, since $\Phi(k)$ is bounded, it follows from Lemma A.1 that $\phi(k)$ is bounded. Furthermore, since $y(k)$ and $u(k)$ are components of $\phi(k+1)$, it follows that $y(k)$ and $u(k)$ are bounded.

To prove that $\lim_{k\to\infty}\tilde{y}(k)=0$, note that it follows from (20) and the fact that $\|\mathcal{B}\tilde{y}_{\mathrm{f}}(k)\|=|\tilde{y}_{\mathrm{f}}(k)|$ that

$$\lim_{k\to\infty}|\tilde{y}_{\mathrm{f}}(k)|\le\lim_{k\to\infty}\|\tilde{\Phi}(k+1)\|+\|\mathcal{A}_*\|_{\mathrm{F}}\lim_{k\to\infty}\|\tilde{\Phi}(k)\|=0.$$

Since $\lim_{k\to\infty}\tilde{y}_{\mathrm{f}}(k)=0$, $\tilde{y}_{\mathrm{f}}(k)=D_*(\mathbf{q}^{-1})\tilde{y}(k)$, and $d_*(\mathbf{q})=\mathbf{q}^{n_*}D_*(\mathbf{q}^{-1})$ is an asymptotically stable polynomial, it follows that $\lim_{k\to\infty}\tilde{y}(k)=0$. Lastly, since $\lim_{k\to\infty}\tilde{y}(k)=0$ and $\lim_{k\to\infty}y_*(k)=0$, it follows that $\lim_{k\to\infty}y(k)=0$. □

Theorem V.1 invokes the assumption that there exist $\epsilon>0$ and $k_1>0$ such that, for all $k\ge k_1$ and for all $i=1,\ldots,n_{\mathrm{u}}$, $|M(\xi_i,k)|\ge\epsilon$. This assumption asymptotically bounds the frozen time controller poles (i.e., the roots of $M(\mathbf{z},k)$) away from the nonminimum-phase zeros of (A,B,C), and thus, asymptotically prevents unstable pole-zero cancellation between the plant zeros and the controller poles.

The condition $|M(\xi_i,k)|\ge\epsilon$ for some arbitrarily small $\epsilon>0$ can be checked at each time step since $M(\xi_i,k)$ can be computed from known values (i.e., the roots of $\beta_{\mathrm{u}}(\mathbf{z})$ and the controller parameter $\theta(k)$). In fact, if, for some arbitrarily small $\epsilon>0$, the condition $|M(\xi_i,k)|\ge\epsilon$ is violated at a particular time step, then the controller parameter $\theta(k)$ can be perturbed to ensure $|M(\xi_i,k)|\ge\epsilon$. In particular, $\theta(k)$ can be projected orthogonally a distance ϵ away from the hyperplane in θ-space defined by the equation $M(\xi_i,k)=0$. Future work will include a stability analysis of the adaptive control algorithm with this projection.

B. Cumulative Retrospective Cost Adaptive Control

In this section, we present the analogous results to Lemma V.1 and Theorem V.1 for the cumulative retrospective cost adaptive controller (9), (16), and (17).

Lemma V.2. *Consider the open-loop system (1), (2) satisfying assumptions (A1)-(A9), and the cumulative retrospective cost adaptive controller (9), (16), and (17), where n_{c} satisfies (4). Furthermore, define $\eta_{\mathrm{C}}(k)\triangleq\frac{1}{1+\Phi^{\mathrm{T}}(k)P(0)\Phi(k)}$. Then, for all initial conditions $x(0)$, $x_w(0)$, and $\theta(0)$, the following properties hold:*

(i) *$\theta(k)$ is bounded.*
(ii) *$\lim_{k\to\infty}\sum_{j=0}^{k}\eta_{\mathrm{C}}(j)\tilde{y}_{\mathrm{f,r}}^2(j)$ exists.*
(iii) *For all $N>0$, $\lim_{k\to\infty}\sum_{j=N}^{k}\|\theta(j)-\theta(j-N)\|^2$ exists.*

Proof. Subtracting θ_* from both sides of (16) yields the estimator-error update equation

$$\tilde{\theta}(k+1)=\tilde{\theta}(k)-\frac{P(k)\Phi(k)y_{\mathrm{f,r}}(k)}{\lambda+\Phi^{\mathrm{T}}(k)P(k)\Phi(k)}.\quad(33)$$

Next, note from (17) that

$$P(k+1)\Phi(k)=\frac{P(k)\Phi(k)}{\lambda+\Phi^{\mathrm{T}}(k)P(k)\Phi(k)},\quad(34)$$

and thus,

$$\tilde{\theta}(k+1)=\tilde{\theta}(k)-P(k+1)\Phi(k)y_{\mathrm{f,r}}(k).\quad(35)$$

Furthermore, note the following RLS identity (see, for example [2])

$$P^{-1}(k+1)=\lambda P^{-1}(k)+\Phi(k)\Phi^{\mathrm{T}}(k).\quad(36)$$

Define $V_P(P(k),k)\triangleq\lambda^{-k}P^{-1}(k)$, and $\Delta V_P(k)\triangleq V_P(P(k+1),k+1)-V_P(P(k),k)$. Evaluating $\Delta V_P(k)$ along the trajectories of (36) yields

$$\Delta V_P(k)=\lambda^{-k-1}\Phi(k)\Phi^{\mathrm{T}}(k).\quad(37)$$

Since $P(0)$ is positive definite and ΔV_P is positive semidefinite, it follows that, for all $k\ge0$, $V_P(P(k),k)$ is positive definite and $V_P(P(k),k)\ge V_P(P(k-1),k-1)$. Therefore, for all $k\ge0$, $V_P(P(0),0)\le V_P(P(k),k)$, which implies that $\lambda^kP(k)\le P(0)$.

Next, define the positive-definite Lyapunov-like function $V_{\tilde{\theta}}(\tilde{\theta}(k),P(k),k)\triangleq\tilde{\theta}^{\mathrm{T}}(k)V_P(P(k),k)\tilde{\theta}(k)$, and define the Lyapunov-like difference $\Delta V_{\tilde{\theta}}(k)\triangleq V_{\tilde{\theta}}(\tilde{\theta}(k+1),P(k+1),k+1)-V_{\tilde{\theta}}(\tilde{\theta}(k),P(k),k)$. Evaluating $\Delta V_{\tilde{\theta}}(k)$ along the trajectories of the estimator-error system (35) and using (37) yields

$$\begin{aligned}\Delta V_{\tilde{\theta}}(k)&=\tilde{\theta}^{\mathrm{T}}(k)\Delta V_P(k)\tilde{\theta}(k)-2\lambda^{-k-1}y_{\mathrm{f,r}}(k)\Phi^{\mathrm{T}}(k)\tilde{\theta}(k)\\&\quad+\lambda^{-k-1}y_{\mathrm{f,r}}^2(k)\Phi^{\mathrm{T}}(k)P(k+1)\Phi(k)\\&=\lambda^{-k-1}\Big[\tilde{\theta}^{\mathrm{T}}(k)\Phi(k)\Phi^{\mathrm{T}}(k)\tilde{\theta}(k)-2y_{\mathrm{f,r}}(k)\Phi^{\mathrm{T}}(k)\tilde{\theta}(k)\\&\quad+y_{\mathrm{f,r}}^2(k)\Phi^{\mathrm{T}}(k)P(k+1)\Phi(k)\Big].\end{aligned}$$

Next, it follows from Proposition V.1 and (34) that, for all $k \geq k_0$,

$$\begin{aligned}\Delta V_{\tilde{\theta}}(k) &= -\lambda^{-k-1}\tilde{y}_{\mathrm{f,r}}^2(k)\left(1-\Phi^{\mathrm{T}}(k)P(k+1)\Phi(k)\right)\\ &= -\lambda^{-k-1}\tilde{y}_{\mathrm{f,r}}^2(k)\left(1-\frac{\Phi^{\mathrm{T}}(k)P(k)\Phi(k)}{\lambda+\Phi^{\mathrm{T}}(k)P(k)\Phi(k)}\right)\\ &= -\lambda^{-k-1}\tilde{y}_{\mathrm{f,r}}^2(k)\frac{\lambda}{\lambda+\Phi^{\mathrm{T}}(k)P(k)\Phi(k)}\\ &= -\bar{\eta}_{\mathrm{C}}(k)\tilde{y}_{\mathrm{f,r}}^2(k). \end{aligned} \tag{38}$$

where $\bar{\eta}_{\mathrm{C}}(k) \triangleq \frac{1}{\lambda^{k+1}+\lambda^k\Phi^{\mathrm{T}}(k)P(k)\Phi(k)}$. Since $V_{\tilde{\theta}}$ is a positive-definite radially unbounded function of $\tilde{\theta}(k)$ and, for $k \geq k_0$, $\Delta V_{\tilde{\theta}}(k)$ is non-positive, it follows that $\tilde{\theta}(k)$ is bounded and thus $\theta(k)$ is bounded, which verifies *(i)*.

To show *(ii)*, first we show that $\lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j)$ exists. Since $V_{\tilde{\theta}}$ is positive definite, and, for all $k \geq k_0$, $\Delta V_{\tilde{\theta}}(k)$ is non-positive, it follows that $0 \leq \lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j) = V_{\tilde{\theta}}(\tilde{\theta}(k_0),P(k_0),k_0) - \lim_{k\to\infty}V_{\tilde{\theta}}(\tilde{\theta}(k),P(k),k) \leq V_{\tilde{\theta}}(\tilde{\theta}(k_0),P(k_0),k_0)$, where the upper and lower bounds imply that both limits exist. Since $\lim_{k\to\infty}\sum_{j=k_0}^{k}\Delta V_{\tilde{\theta}}(j)$ exists, (38) implies that $\lim_{k\to\infty}\sum_{j=k_0}^{k}\bar{\eta}_{\mathrm{C}}(j)\tilde{y}_{\mathrm{f,r}}^2(j)$ exists, and thus $\lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}_{\mathrm{C}}(j)\tilde{y}_{\mathrm{f,r}}^2(j)$ exists. Since, for all $k \geq 0$, $\lambda^{k+1} \leq 1$ and $\lambda^k P(k) \leq P(0)$, it follows that, for all $k \geq 0$, $\eta_{\mathrm{C}}(k) \leq \bar{\eta}_{\mathrm{C}}(k)$, which implies that $\lim_{k\to\infty}\sum_{j=0}^{k}\eta_{\mathrm{C}}(j)\tilde{y}_{\mathrm{f,r}}^2(j) \leq \lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}_{\mathrm{C}}(j)\tilde{y}_{\mathrm{f,r}}^2(j)$. Thus, $\lim_{k\to\infty}\sum_{j=0}^{k}\eta_{\mathrm{C}}(j)\tilde{y}_{\mathrm{f,r}}^2(j)$ exists, which verifies *(ii)*.

To show *(iii)*, we first show that $\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1)-\theta(j)\|^2$ exists. Since $\lambda^k P(k) \leq P(0)$, it follows from (33) that

$$\begin{aligned}&\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1)-\theta(j)\|^2\\ &= \lim_{k\to\infty}\sum_{j=0}^{k}y_{\mathrm{f,r}}^2(j)\frac{\Phi^{\mathrm{T}}(j)P^2(j)\Phi(j)}{[\lambda+\Phi^{\mathrm{T}}(j)P(j)\Phi(j)]^2}\\ &= \lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}_{\mathrm{C}}(j)y_{\mathrm{f,r}}^2(j)\left(\frac{\lambda^j\Phi^{\mathrm{T}}(j)P^2(j)\Phi(j)}{\lambda+\Phi^{\mathrm{T}}(j)P(j)\Phi(j)}\right)\\ &\leq \lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}_{\mathrm{C}}(j)y_{\mathrm{f,r}}^2(j)\|\lambda^j P(j)\|_F\left(\frac{\Phi^{\mathrm{T}}(j)P(j)\Phi(j)}{\lambda+\Phi^{\mathrm{T}}(j)P(j)\Phi(j)}\right)\\ &\leq \|P(0)\|_F\lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}_{\mathrm{C}}(j)y_{\mathrm{f,r}}^2(j)\left(\frac{\Phi^{\mathrm{T}}(j)P(j)\Phi(j)}{\lambda+\Phi^{\mathrm{T}}(j)P(j)\Phi(j)}\right),\end{aligned}$$

where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm. Next, note that, for all $k \geq 0$, $\frac{\Phi^{\mathrm{T}}(k)P(k)\Phi(k)}{\lambda+\Phi^{\mathrm{T}}(k)P(k)\Phi(k)} \leq 1$, which implies that $\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1)-\theta(j)\|^2 \leq \|P(0)\|_F\lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}_{\mathrm{C}}(j)y_{\mathrm{f,r}}^2(j)$. Since $\lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}_{\mathrm{C}}(j)\tilde{y}_{\mathrm{f,r}}^2(j)$ exists, it follows that $\lim_{k\to\infty}\sum_{j=0}^{k}\bar{\eta}_{\mathrm{C}}(j)y_{\mathrm{f,r}}^2(j)$ exists, and thus, it follows that $\lim_{k\to\infty}\sum_{j=0}^{k}\|\theta(j+1)-\theta(j)\|^2$ exists. The remainder of the proof is identical to the proof of Lemma V.1. □

The following theorem is the main result of the paper regarding the cumulative retrospective cost adaptive controller.

Theorem V.2. *Consider the open-loop system (1), (2) satisfying assumptions (A1)-(A9), and the cumulative retrospective cost adaptive controller (9), (16), and (17), where* n_{c} *satisfies (4). Assume that there exist* $\epsilon > 0$ *and* $k_1 > 0$ *such that, for all* $k \geq k_1$ *and for all* $i = 1,\ldots,n_{\mathrm{u}}$, $|M(\xi_i,k)| \geq \epsilon$. *Then, for all initial conditions* $x(0)$, $x_w(0)$, *and* $\theta(0)$, $\theta(k)$ *is bounded,* $u(k)$ *is bounded, and* $\lim_{k\to\infty} y(k) = 0$.

The proof of Theorem V.2 is identical to the proof of Theorem V.1 with $\eta(k)$ replaced by $\eta_{\mathrm{C}}(k)$ and $a_1 \triangleq \frac{2\lambda_{\max}(P(0))}{\lambda_{\min}(\mathcal{P})\left[1+2\lambda_{\max}(P(0))\Phi_{*,\max}^2\right]} > 0$.

VI. Conclusions

This paper, in conjunction with its companion paper [1], presented a direct adaptive controller for discrete-time (including sampled-data) systems that are possibly nonminimum phase. The adaptive controller requires knowledge of the first nonzero Markov parameter and the nonminimum-phase zeros of the transfer function from the control to the performance. The present paper and its companion paper [1] together provided the construction and stability analysis of the retrospective cost adaptive controller. Future work will include a stability analysis of the retrospective cost adaptive control algorithm with an ϵ projection (as discussed in Section V of this paper) to ensure that the frozen time controller poles are asymptotically bounded away from the nonminimum-phase zeros of (A, B, C).

Appendix A:

The following lemma is used in the proofs of Theorems V.1 and V.2. This lemma is presented for an arbitrary feedback controller given by (9), where the controller parameter vector $\theta(k)$ is time varying. More precisely, the following lemma does not depend on the adaptive law used to update $\theta(k)$ provided that such an adaptive law satisfies the assumptions in the lemma.

Lemma A.1. *Consider the open-loop system (1), (2) satisfying assumptions (A1)-(A9). In addition, consider a feedback controller given by (9) that satisfies the following assumptions:*

(i) $\theta(k)$ *is bounded.*
(ii) $\lim_{k\to\infty}\|\theta(k)-\theta(k-1)\| = 0$.
(iii) There exist $\epsilon > 0$ *and* $k_1 > 0$ *such that, for all* $k \geq k_1$ *and all* $i = 1,\ldots,n_{\mathrm{u}}$, $|M(\xi_i,k)| \geq \epsilon$.

Then, for all initial conditions $x(0)$, $x_w(0)$, *and* $\theta(0)$, *there exist* $k_2 > 0$, $c_1 > 0$, *and* $c_2 > 0$, *such that, for all* $k \geq k_2$ *and all* $N = 0,\ldots,n_{\mathrm{u}}$, $\|\phi(k-d-N)\| \leq c_1 + c_2\|\Phi(k)\|$.

The proof has been omitted due to space considerations.

References

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[2] G. C. Goodwin and K. S. Sin, *Adaptive Filtering, Prediction, and Control*. Prentice Hall, 1984.