49th IEEE Conference on Decision and Control
December 15-17, 2010
Hilton Atlanta Hotel, Atlanta, GA, USA

Geometric Methods for Unknown-State, Unknown-Input Reconstruction in Discrete-Time Nonminimum-Phase Systems with Feedthrough Terms

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Abstract— **Unknown-state, unknown-input reconstruction in systems with invariant zeros is intrinsically limited by the fact that, for any invariant zero, at least one initial state exists, s.t. when the mode of the invariant zero is suitably injected into the system, the output remains identically equal to zero. Nonetheless, the problem has recently attracted considerable interest, mainly due to its connections with fault diagnosis and fault tolerant issues. This paper discusses the synthesis of a system capable of reconstructing the generic initial state and the generic bounded input in discrete-time nonminimum-phase linear systems with feedthrough terms. The procedure described is developed within the geometric approach.**

I. Introduction

The problem of reconstructing the inaccessible inputs from the available measurements when the initial state of the system is unknown was first approached for continous-time systems. Geometric necessary and sufficient conditions for a system to be completely unknown-state, unknown-input reconstructable by measurement differentiation were proved in [1]. Under the left-invertibility assumption, those conditions were equivalent to the absence of invariant zeros. More recently, discrete-time systems have been addressed in [2]. Several algorithms, completely relying on algebraic approaches, have been proposed in [3], [4]. However, those algorithms, using least-square techniques, behave in such a way that the reconstruction error converges asymptotically to zero in minimum-phase systems, is constant in systems with one invariant zero on the unit circle, diverges in nonminimum-phase systems. A solution that can successfully be applied also to nonminimum-phase systems has been lately presented in [5]: an extensive use of notions and results of the geometric approach and duality arguments made it possible to deal with nonminimum-phase systems in the context of reconstruction by resorting to the steering-along-zeros techniques that enabled mastering of nonminimum-phase systems in reference tracking. However, in [5] the system is in minimal form and does not have any feedthrough term. The contribution of this paper is to solve the unknown-state, unknown-input reconstruction problem for systems that are observable and have feedthrough terms. Moreover, the synthesis procedure is optimized with respect to that in [5], since it yields a filter of minimal dynamic order for the cancellation of the zeros in the open unit disc.

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Notation: $\mathbb{Z}$, $\mathbb{Z}_0^+$, $\mathbb{Z}^+$, $\mathbb{R}$ denote the sets of integers, nonnegative integers, positive integers, and real numbers, respectively. $\mathbb{C}^\odot$ and $\mathbb{C}^\otimes$ denote the open unit disc and the complement of the closed unit disc in the complex plane $\mathbb{C}$, respectively. Matrices and linear maps are denoted by slanted capital letters, like A. The spectrum, the image, and the kernel of A are denoted by $\sigma(A)$, $\operatorname{im} A$, and $\ker A$, respectively. The symbols A^{-1}, $A^\dagger$, and $A^\top$ are respectively used to denote the inverse, the Moore-Penrose inverse, and the transpose of A. The restriction of the linear map A to an A-invariant subspace $\mathcal{J} \subseteq \mathbb{R}^n$ is denoted by $A|_\mathcal{J}$. The symbol $\mathcal{J}_c$ denotes a complement of $\mathcal{J}$: i.e., an A-invariant subspace s.t. $\mathcal{J} \oplus \mathcal{J}_c = \mathbb{R}^n$, where $\oplus$ denotes the direct sum of subspaces. The quotient space of a vector space $\mathcal{X}$ over a subspace $\mathcal{V} \subseteq \mathcal{X}$ is denoted by $\mathcal{X}/\mathcal{V}$. The orthogonal complement of $\mathcal{V}$ is denoted by $\mathcal{V}^\perp$. The dimension of $\mathcal{V}$ is denoted by $\dim \mathcal{V}$. The symbols I_n and $O_{m\times n}$ denote the $n \times n$ identity matrix and the $m \times n$ zero matrix (subscripts will be omitted when dimensions are clear from the context).

II. Geometric Approach Background

This section reviews notions and results used in the geometric approach (see, e.g., [1] and the references therein). Consider the discrete-time linear time-invariant system

$$x_{t+1} = A\,x_t + B\,u_t, \tag{1}$$
$$y_t = C\,x_t + D\,u_t, \tag{2}$$

where $t \in \mathbb{Z}$ is the time variable and $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$, with $p \le n$ and $q \le n$, are the state, the input, the output, respectively. A, B, C, D are constant real matrices of appropriate dimensions. $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ and $[C\,D]$ are full-rank matrices. $\mathcal{U}_f$ is the set of the admissible input sequences, that are all the sequences with bounded values in $\mathbb{R}^p$. $\mathcal{B}$ and $\mathcal{C}$ stand for $\operatorname{im} B$ and $\ker C$, respectively. Geometric objects extensively used in this paper are $\min \mathcal{J}(A,\mathcal{B})$, the minimal A-invariant subspace containing $\mathcal{B}$, $\max \mathcal{J}(A,\mathcal{C})$, the maximal A-invariant subspace contained in $\mathcal{C}$, $\mathcal{V}^*$ or $\max \mathcal{V}(A,B,C,D)$, the maximal output-nulling controlled invariant subspace of (1), (2), $\mathcal{S}^*$ or $\min \mathcal{S}(A,B,C,D)$, the minimal input-containing conditioned invariant subspace of (1), (2), $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$, the reachability subspace on $\mathcal{V}^*$. Recall that an $(A,\mathcal{B})$-controlled invariant subspace $\mathcal{V}$ is output-nulling if and only if at least one linear map F exists s.t. $(A+BF)\mathcal{V} \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq \ker(C+DF)$. An $(A,\mathcal{C})$-conditioned invariant subspace $\mathcal{S}$ is input-containing if and only if at least one linear map G exists s.t. $(A+GC)\mathcal{S} \subseteq \mathcal{S}$ and $\mathcal{S} \supseteq \operatorname{im}(B+GD)$. Let F be s.t. $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$ and $\mathcal{V}^* \subseteq \ker(C+DF)$, then $(A+BF)\mathcal{R}_{\mathcal{V}^*} \subseteq \mathcal{R}_{\mathcal{V}^*}$

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and $\mathcal{R}_{\mathcal{V}^*} \subseteq \ker(C + DF)$ hold with the same F. The spectrum $\sigma((A + BF)|_{\mathcal{R}_{\mathcal{V}^*}})$ is assignable. The spectrum $\sigma((A + BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}})$ is fixed and its elements are the invariant zeros of (1), (2), so that the same set is also denoted by $\mathcal{Z}(A,B,C,D)$. Let G be s.t. $(A+GC)\mathcal{S}^* \subseteq \mathcal{S}^*$ and $\mathcal{S}^* \supseteq \mathrm{im}\,(B+GD)$. Equivalently, the invariant zeros of (1), (2) are the external unassignable eigenvalues of $\mathcal{S}^*$ regarded as an $(A+GC)$-invariant subspace. The invariant zeros in the open unit disc of the complex plane are called the minimum-phase invariant zeros and their set is denoted by $\mathcal{Z}_{MP}(A,B,C,D)$. The invariant zeros in the complement of the closed unit disc are called the nonminimum-phase invariant zeros and their set is denoted by $\mathcal{Z}_{NMP}(A,B,C,D)$. Further crucial notions are those of right invertibility and left invertibility. A geometric condition equivalent to the property of system (1), (2) of being right invertible is $\mathcal{V}^* + \mathcal{S}^* = \mathbb{R}^n$. A geometric condition equivalent to the property of (1), (2) of being left invertible is $\mathcal{V}^* \cap \mathcal{S}^* = \{0\}$.

III. Problem Statement

Consider (1), (2), where u is the unknown input and y the measured output. The initial state, denoted by x_0, is unknown. The objective of this work is to provide a procedure for synthesizing a discrete-time linear time-invariant system, henceforth called the unknown-state, unknown-input reconstructor, which, having the measured output of the original system as input, produces as output both the state and the input of the original system, with an admissible delay of a finite number of steps. The solution to the unknown-state, unknown-input reconstruction problem is presented under the following assumptions, concerning (1), (2):

- $\mathcal{A}$1. observability: i.e., $\mathcal{Q} = \max\mathcal{J}(A,\mathcal{C}) = \{0\}$, where $\mathcal{Q}$ is the unobservable subspace of (A,C);
- $\mathcal{A}$2. left invertibility: i.e., $\mathcal{V}^* \cap \mathcal{S}^* = \{0\}$;
- $\mathcal{A}$3. no invariant zeros with unit absolute value: i.e., $\mathcal{Z}(A,B,C,D) \subset \mathbb{C}^{\odot} \cup \mathbb{C}^{\otimes}$.

IV. Outline of the Problem Solution

This section describes the structure of the device that solves the unknown-state, unknown-input reconstruction problem. If the original system has any minimum-phase invariant zeros, these are cancelled by a filter cascaded to the system. The filter is required to be permanently connected to the system so as to guarantee the synchronization of the respective, though unknown, states. Then, a finite impulse response (FIR) system processes the output of the filter in order to reconstruct the initial state and the subsequent state trajectory, with a possible delay related to properties of the original system, i.e., the number of steps required by the algorithm of a certain input-containing conditioned invariant subspace to converge as well as the time constants of the nonminimum-phase invariant zeros. The FIR system is directly fed with the system output if there are no minimum-phase invariant zeros. Then, a dynamic system processes the reconstructed state trajectory and the delayed output so as to reproduce the unknown input with a further, one-step delay.

V. Duality Arguments

The synthesis of the filter and the state reconstructor will be carried out in the dual context of control, where a more intuitive interpretation of the problems and their solutions in terms of subspaces and state trajectories is possible. The filter cascaded to the original system, cancelling its minimum-phase zeros, will be derived as the dual counterpart of a feedforward compensator achieving the same target for the dual system. The FIR system solving the problem of reconstructing the state trajectory in the presence of the unknown inputs will be obtained as the dual counterpart of that solving the problem of steering the state of the dual system from the origin to any assigned final state with zero output until the last step but one. In the following, Property 1 reviews some useful properties of dual quadruples and Theorem 1 explains the exact terms of the duality between the reconstruction problem and the control problem by relating the sequences of inputs and outputs of suitably defined systems. Proofs will be omitted for the sake of brevity.

Property 1: Consider a system like (1), (2). The following propositions hold: (i) (A,C) is observable if and only if $(A^\top, C^\top)$ is controllable; (ii) (A,B,C,D) is left-invertible if and only if $(A^\top, C^\top, B^\top, D^\top)$ is right-invertible; (iii) $\mathcal{Z}(A,B,C,D) = \mathcal{Z}(A^\top, C^\top, B^\top, D^\top)$.

In order to state Theorem 1, that follows, the quadruple (A_e, B_e, C_e, D_e) is first introduced. If the original system, like (1), (2), has any minimum-phase invariant zero, (A_e, B_e, C_e, D_e) denotes a particular IO-equivalent description (where the cancellation of the minimum-phase zeros has been taken into account) of the cascade of (1), (2) and the filter. Details on the IO-equivalent description will be given in Section VI. However, it is worth mentioning that $A_e = A$, $B_e = B$, and that assuming $\mathcal{A}$1, $\mathcal{A}$2, $\mathcal{A}$3 on (1), (2) implies that the same properties hold for (A_e, B_e, C_e, D_e). Otherwise, if (1), (2) does not have any minimum-phase zero, (A_e, B_e, C_e, D_e) denotes the same (1), (2). Then, the quadruple $(A_{fir}, B_{fir}, C_{fir}, D_{fir})$ is introduced to denote the FIR system that reconstructs the initial state and the subsequent state trajectory of (A_e, B_e, C_e, D_e), in the presence of the unknown input. Moreover, the symbol $\mu \in \mathbb{R}^n$ denotes a further input to (A_e, B_e, C_e, D_e), with identity distribution matrix, and $\tilde{x} \in \mathbb{R}^n$ denotes the output of the FIR system: i.e., the possibly-delayed reconstruction of the state trajectory. Then, the cascade of (A_e, B_e, C_e, D_e) and $(A_{fir}, B_{fir}, C_{fir}, D_{fir})$ is

$$\hat{x}_{t+1} = \hat{A}\hat{x}_t + \hat{B}u_t + \hat{E}\mu_t, \qquad (3)$$

$$\tilde{x}_t = \hat{C}\hat{x}_t + \hat{D}u_t, \qquad (4)$$

with

$$\hat{A} = \begin{bmatrix} A_e & O \\ B_{fir}C_e & A_{fir} \end{bmatrix} \; \hat{B} = \begin{bmatrix} B_e \\ B_{fir}D_e \end{bmatrix} \; \hat{E} = \begin{bmatrix} I \\ O \end{bmatrix} \qquad (5)$$

$$\hat{C} = \begin{bmatrix} D_{fir}C_e & C_{fir} \end{bmatrix} \; \hat{D} = D_{fir}D_e \qquad (6)$$

In the dual context, the cascade of $(A_{fir}^\top, C_{fir}^\top, B_{fir}^\top, D_{fir}^\top)$ and $(A_e^\top, C_e^\top, B_e^\top, D_e^\top)$ must be considered. Let $h \in \mathbb{R}^n$

denote the input to the FIR system and let $\eta \in \mathbb{R}^n$ denote of a further output of the controlled system, with identity distribution matrix. Then, the cascade is

$$\hat{x}_{t+1} = \hat{A}^\top \hat{x}_t + \hat{C}^\top h_t, \quad (7)$$
$$y_t = \hat{B}^\top \hat{x}_t + \hat{D}^\top h_t, \quad (8)$$
$$\eta_t = \hat{E}^\top \hat{x}_t. \quad (9)$$

Theorem 1: Consider system (3), (4) with (5), (6). Let the initial state $\hat{x}_0$ of (3), (4) be the zero state. Let the unknown input be a sequence u_t, $t \in \mathbb{Z}_0^+$, with bounded values in $\mathbb{R}^p$ and $u_0 = 0$. Let the unknown state x_0 of (A_e, B_e, C_e, D_e) be injected into (3), (4) as the discrete pulse signal $\mu_t = x_0 \delta_t$, where $\delta_t = 1$ with $t = 0$ and $\delta_t = 0$ otherwise. Consider system (7), (8), (9). Let the initial state $\hat{x}_0$ of (7), (8), (9) be the zero state. Let $N \in \mathbb{Z}^+$ denote the FIR system time window. Let the desired state x_f of $(A_e^\top, C_e^\top, B_e^\top, D_e^\top)$ at $t = N$ be injected into (7), (8), (9) as the discrete pulse signal $h_t = x_f \delta_t$. Then, the output sequence $\tilde{x}_t$, $t \in \mathbb{Z}_0^+$, of (3), (4) is s.t.

$$\tilde{x}_t = A_e^{t-N} x_0 + \sum_{\ell=0}^{t-N} A_e^{t-N-\ell} B_e u_\ell, \quad t = N, N+1, \ldots$$

if and only if the output sequences η_t and y_t, $t \in \mathbb{Z}_0^+$, of (7), (8), (9) are s.t.

$$\eta_t = (A_e^\top)^{t-N} x_f, \qquad t = N, N+1, \ldots,$$
$$y_t = \begin{cases} 0, & t = 0, 1, \ldots, N-1, \\ B_e^\top (A_e^\top)^{t-N} x_f, & t = N, N+1, \ldots. \end{cases}$$

VI. Synthesis of the Minimal-Order Filter

In light of the previous section, the minimal-order filter cancelling the minimum-phase invariant zeros of the original system will be obtained as the dual counterpart of a feedforward compensator cancelling the minimum-phase invariant zeros of the dual system. In order to avoid notation clutter, this section will directly refer to a system like (1), (2), that satisfies assumptions

$\mathcal{A}\,1'$. controllability: i.e., $\mathcal{R} = \min \mathcal{J}(A, \mathcal{B}) = \mathbb{R}^n$, where $\mathcal{R}$ denotes the reachable subspace of (A, B),

$\mathcal{A}\,2'$. right invertibility: i.e., $\mathcal{V}^* + \mathcal{S}^* = \mathbb{R}^n$,

and $\mathcal{A}\,3$ if and only if the original system satisfies $\mathcal{A}\,1$, $\mathcal{A}\,2$, $\mathcal{A}\,3$, respectively (Property 1). The following lemmas and theorems are aimed at pointing out the key subspace, that will be denoted by $\mathcal{V}_S^*$, for the synthesis of the feedforward compensator. Proofs, which exploit basic geometric properties, are omitted.

Lemma 1: Consider (1), (2). Let $\mathcal{A}\,2'$ hold. Let F be s.t. $(A + BF)\mathcal{V}^* \subseteq \mathcal{V}^*$ and $\mathcal{V}^* \subseteq \ker(C + DF)$. Perform the similarity transformation $T = [T_1\ T_2\ T_3]$, where $\operatorname{im} T_1 = \mathcal{R}_{\mathcal{V}^*}$, $\operatorname{im}\,[T_1\ T_2] = \mathcal{V}^*$, $\operatorname{im}\,[T_1\ T_3] = \mathcal{S}^*$. Then,

$$A'_F = T^{-1}(A + BF)T = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ O & A'_{22} & A'_{23} \\ O & O & A'_{33} \end{bmatrix}, \quad (10)$$
$$C'_F = (C + DF)T = \begin{bmatrix} O & O & C'_3 \end{bmatrix}. \quad (11)$$

Remark 1: The set of the internal eigenvalues of $\mathcal{R}_{\mathcal{V}^*}$ is the set of the eigenvalues of A'_{11}: i.e., $\sigma((A + BF)|_{\mathcal{R}_{\mathcal{V}^*}}) = \sigma(A'_{11})$. The set of the internal unassignable eigenvalues of $\mathcal{V}^*$ (or, equivalently the set of the system invariant zeros) is the set of the eigenvalues of A'_{22}: i.e., $\sigma((A + BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \mathcal{Z}(A, B, C, D) = \sigma(A'_{22})$. □

Let $n_R = \dim \mathcal{R}_{\mathcal{V}^*}$, $n_V = \dim \mathcal{V}^* - n_R$, $n_S = \dim \mathcal{S}^* - n_R$,

$$T'_1 = \begin{bmatrix} I_{n_R} & O & O \end{bmatrix}^\top \quad T'_3 = \begin{bmatrix} O & O & I_{n_S} \end{bmatrix}^\top \quad (12)$$

Lemma 2: Consider (1), (2). Let $\mathcal{A}\,2'$ hold. Let F be s.t. $(A + BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C + DF)$, and $\sigma((A + BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A + BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$. Consider (10), (11) and perform the similarity transformation $T' = [T'_1\ T'_2\ T'_3]$, where T'_1, T'_3 are defined by (12) and $T'_2 = [X^\top\ I_{n_V}\ O]^\top$, where X is the solution of the Sylvester equation

$$A'_{11} X - X A'_{22} = -A'_{12}. \quad (13)$$

Then,

$$A''_F = T'^{-1} A'_F T' = \begin{bmatrix} A'_{11} & O & A''_{13} \\ O & A'_{22} & A'_{23} \\ O & O & A'_{33} \end{bmatrix}, \quad (14)$$
$$C''_F = C'_F T' = C'_F. \quad (15)$$

Lemma 3: Consider (1), (2). Let $\mathcal{A}\,2'$, $\mathcal{A}\,3$ hold. Let F be s.t. $(A + BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C + DF)$, and $\sigma((A + BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A + BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$. Consider (14), (15) and perform the similarity transformation $T'' = [T'_1\ T''_2\ T'_3]$, where T'_1, T'_3 are defined by (12) and $T''_2 = [O\ V^\top\ O]^\top$, where $V = [V_S\ V_U]$, with V_S and V_U basis matrices of the subspaces $\mathcal{V}_S$ and $\mathcal{V}_U$ of the stable and the unstable modes of A'_{22}. Then,

$$A'''_F = T''^{-1} A''_F T'' = \begin{bmatrix} A'_{11} & O & A''_{13} \\ O & A''_{22} & A''_{23} \\ O & O & A'_{33} \end{bmatrix},$$
$$C'''_F = C''_F T'' = C'_F,$$

with

$$A''_{22} = V^{-1} A'_{22} V = \begin{bmatrix} A''_{22s} & O \\ O & A''_{22u} \end{bmatrix}.$$

In light of Lemma 3, a representation of A'''_F and C'''_F is

$$A'''_F = \begin{bmatrix} A'_{11} & O & O & A''_{13} \\ O & A''_{22s} & O & A''_{23s} \\ O & O & A''_{22u} & A''_{23u} \\ O & O & O & A'_{33} \end{bmatrix}, \quad (16)$$
$$C'''_F = \begin{bmatrix} O & O & O & C'_3 \end{bmatrix}. \quad (17)$$

Theorem 2: Consider (1), (2). Let $\mathcal{A}\,2'$, $\mathcal{A}\,3$ hold. Let F be s.t. $(A + BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C + DF)$, and $\sigma((A + BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A + BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$. Let the subspace $\mathcal{V}_S^*$ be defined by

$$\mathcal{V}_S^* = \operatorname{im} V_S^{*'''} = \operatorname{im} \begin{bmatrix} O & I_{n_{SS}} & O & O \end{bmatrix}^\top, \quad (18)$$

where basis matrix $V_S^{*'''}$ refers to the coordinates introduced in Lemma 3 and is partitioned according to (16). Then: (i) $\mathcal{V}_S^*$ in an output-nulling controlled invariant subspace; (ii) $\sigma((A + BF)|_{\mathcal{V}_S^*}) = \mathcal{Z}_{MP}(A, B, C, D)$.

Corollary 1: Consider (1), (2). Let $\mathcal{A}2'$, $\mathcal{A}3$ hold. Let F be s.t. $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C+DF)$, and $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$. Let $\mathcal{V}_S^*$ be the output-nulling controlled invariant subspace defined by (18). Let $\bar{T} = TT'T''$, where T, T', T'' are the similarity transformations respectively considered in Lemmas 1, 2, 3, and let $V_S^* = \bar{T} V_S^{*'''}$ be the consistent basis matrix of $\mathcal{V}_S^*$ with respect to the original coordinates. Then,

$$\begin{aligned} AV_S^* - V_S^* W &= -BL, \\ CV_S^* &= -DL, \end{aligned}$$

hold with $W = A''_{22s}$ and $L = FV_S^*$.

The subspace $\mathcal{V}_S^*$ introduced in Theorem 2 is the key subspace for the synthesis of the feedforward compensator cancelling the minimum-phase invariant zeros of (1), (2). Matrices W, L defined in Corollary 1 will henceforth be used to that aim. The synthesis of the feedforward compensator and the properties of a specific ISO description of the cascade of the feedforward compensator and the system are the object of the remainder of this section. Let the feedforward compensator be

$$x_{f,t+1} = A_f x_{f,t} + B_f w_t, \tag{19}$$
$$u_t = C_f x_{f,t} + D_f w_t, \tag{20}$$

where

$$A_f = W, \quad B_f = \left[\begin{array}{cc} I_{n_f} & O_{n_f \times p} \end{array}\right], \tag{21}$$
$$C_f = L, \quad D_f = \left[\begin{array}{cc} O_{p \times n_f} & I_p \end{array}\right], \tag{22}$$

with $n_f = n_{SS} = \dim \mathcal{V}_S^*$. The cascade of (19), (20) and (1), (2) is

$$\hat{x}_{t+1} = \hat{A}\hat{x}_t + \hat{B} w_t, \tag{23}$$
$$y_t = \hat{C}\hat{x}_t + \hat{D} w_t, \tag{24}$$

where

$$\hat{A} = \left[\begin{array}{cc} A & BC_f \\ O & A_f \end{array}\right], \quad \hat{B} = \left[\begin{array}{c} BD_f \\ B_f \end{array}\right], \tag{25}$$
$$\hat{C} = \left[\begin{array}{cc} C & DC_f \end{array}\right], \quad \hat{D} = DD_f. \tag{26}$$

Lemma 4: Consider system (23), (24) with (25), (26). Let

$$\mathcal{J} = \operatorname{im} J = \operatorname{im} \left[\begin{array}{c} I_n \\ O_{n_f \times n} \end{array}\right], \ \mathcal{J}_c = \operatorname{im} J_c = \operatorname{im} \left[\begin{array}{c} V_S^* \\ I_{n_f} \end{array}\right].$$

Then: (i) $\mathcal{J}$ is an $\hat{A}$-invariant subspace; (ii) $\mathcal{J}_c$ is an $\hat{A}$-invariant subspace; (iii) $\mathcal{J} \oplus \mathcal{J}_c = \hat{\mathcal{X}}$, where $\hat{\mathcal{X}}$ is the state space of (23), (24).

Theorem 3: Consider system (23), (24) with (25), (26), where (A,B,C,D) satisfies $\mathcal{A}1'$ and (A_f, B_f, C_f, D_f) is defined according to (21), (22). System

$$x_{c,t+1} = A_c x_{c,t} + B_c w_t, \tag{27}$$
$$y_t = C_c x_{c,t} + D_c w_t, \tag{28}$$

with

$$A_c = A, \quad B_c = \left[\begin{array}{cc} -V_S^* & B \end{array}\right], \tag{29}$$
$$C_c = C, \quad D_c = \left[\begin{array}{cc} O & D \end{array}\right], \tag{30}$$

is an IO-equivalent realization of (23), (24): i.e., its controllable and observable subsystem matches that of (23), (24).

Lemma 5: Consider systems (1), (2) and (27), (28) with (29), (30). Let $\mathcal{V}^* = \max \mathcal{V}(A,B,C,D)$ and $\mathcal{V}_c^* = \max \mathcal{V}(A_c, B_c, C_c, D_c)$. Then, $\mathcal{V}_c^* = \mathcal{V}^*$.

Lemma 6: Consider systems (1), (2) and (27), (28) with (29), (30). Let $\mathcal{S}^* = \min \mathcal{S}(A,B,C,D)$ and $\mathcal{S}_c^* = \min \mathcal{S}(A_c, B_c, C_c, D_c)$. Then, $\mathcal{S}_c^* = \mathcal{S}^* + \mathcal{V}_S^*$.

Theorem 4: Consider systems (1), (2) and (27), (28) with (29), (30). Let (1), (2) satisfy $\mathcal{A}2'$. Then, (27), (28) satisfies $\mathcal{A}2'$.

Lemma 7: Consider systems (1), (2) and (27), (28) with (29), (30). Let $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$ and $\mathcal{R}_{\mathcal{V}_c^*} = \mathcal{V}_c^* \cap \mathcal{S}_c^*$. Then, $\mathcal{R}_{\mathcal{V}_c^*} = \mathcal{R}_{\mathcal{V}^*} + \mathcal{V}_S^*$.

Theorem 5: Consider systems (1), (2) and (27), (28) with (29), (30). Let $\mathcal{Z}_{NMP}(A,B,C,D)$ be the set of the nonminimum-phase invariant zeros of (1), (2). Let $\mathcal{Z}(A_c, B_c, C_c, D_c)$ be the set of the invariant zeros of (27), (28). Then, $\mathcal{Z}(A_c, B_c, C_c, D_c) = \mathcal{Z}_{NMP}(A,B,C,D)$.

It is worth noting that the quadruple (A_e, B_e, C_e, D_e) used in Section V is the dual counterpart of the quadruple (A_c, B_c, C_c, D_c) defined in this section.

VII. Synthesis of the FIR System for State Reconstruction and of the Dynamic System for Input Reconstruction

In light of Section V, the FIR system for state reconstruction in the presence of the unknown input will be derived as the dual counterpart of the FIR system driving the state of the dual system from the origin to any assigned final state along a trajectory invisible to the output until the last step but one. In light of Section VI, there is no loss of generality in restricting the discussion to systems without invariant zeros in the closed unit disc of the complex plane. Hence, from now on, we will refer to a system like (1), (2), satisfying $\mathcal{A}1'$, $\mathcal{A}2'$, and

$\mathcal{A}3'$. no invariant zeros in the closure of the unit disc of the complex plane: i.e., $\mathcal{Z}(A,B,C,D) \subset \mathbb{C}^{\otimes}$.

The system we will refer to represents either the original system or the cascade (A_c, B_c, C_c, D_c) of Section VI if the original system has any minimum-phase invariant zeros. Note that $\mathcal{A}1'$, $\mathcal{A}2'$, $\mathcal{A}3'$ still hold in the latter case by virtue of Theorems 3, 4, 5. The purpose of the following lemmas and theorems is to outline the control strategy in terms of subspaces and state trajectories. Then, the FIR system gain matrix will be obtained through a suitable generalization.

Theorem 6: Consider (1), (2). Let $\mathcal{A}2'$, $\mathcal{A}3'$ hold. Let F be s.t. $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C+DF)$, and $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$. Let the subspace $\mathcal{V}_U^*$ be defined by $\mathcal{V}_U^* = \operatorname{im} V_U^{*''} = \operatorname{im} [O \ I_{n_V} \ O]^\top$, where basis matrix $V_U^{*''}$ refers to the coordinates introduced in Lemma 2 and is partitioned according to (14). Then: (i) $\mathcal{V}_U^*$ is an output-nulling controlled invariant subspace; (ii) $\mathcal{V}_U^* \oplus \mathcal{S}^* = \mathbb{R}^n$; (iii) $\sigma((A+BF)|_{\mathcal{V}_U^*}) = \mathcal{Z}_{NMP}(A,B,C,D)$.

Proof: Proposition (i) is implied by $A''_F V_U^{*''} = V_U^{*''} A'_{22}$ and $C''_F V_U^{*''} = 0$. Proposition (ii) follows from direct inspection of basis matrices $V_U^{*''}$ and $S^{*''} = (T'')^{-1}[T'_1 \ T'_3] =$

$[T_1' \; T_3']$. Proposition (iii) follows from Proposition (i) and $\mathcal{A}3'$, in light of Remark 1. ■

Theorem 6 has indicated the resolving subspace $\mathcal{V}_U^*$ to be considered along with $\mathcal{S}^*$. In particular, proposition (ii) implies that any state $x_f \in \mathbb{R}^n$ can be written as $x_f = x_{\mathcal{V}_U^*} + x_{\mathcal{S}}$, where $x_{\mathcal{V}_U^*} \in \mathcal{V}_U^*$ and $x_{\mathcal{S}} \in \mathcal{S}^*$. Theorem 7, that follows, gives the control sequence (with infinite length, in theory) that steers the state from the origin to $x_{\mathcal{V}_U^*}$ while maintaining the output identically equal to zero.

Theorem 7: Consider (1), (2). Let $\mathcal{A}2'$, $\mathcal{A}3'$ hold. Let F be s.t. $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C+DF)$, and $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$. Let $\mathcal{V}_U^*$ be the subspace introduced in Theorem 6 and $V_U^{*''}$ its basis matrix with respect to the coordinates introduced in Lemma 2. Let T, T' be the similarity transformations introduced in Lemmas 1, 2. Let $x_{\mathcal{V}_U^*} \in \mathcal{V}_U^*$. Let $\beta \in \mathbb{R}^{n_V}$ be s.t. $x''_{\mathcal{V}_U^*} = (TT')^{-1} x_{\mathcal{V}_U^*} = V_U^{*''} \beta$. Let $F'' = FTT' = [F_1'' \; F_2'' \; F_3'']$ be partitioned according to (14). Then, the control sequence $u_{\mathcal{V}_U^*,-\infty}$, ..., $u_{\mathcal{V}_U^*,-2}$, $u_{\mathcal{V}_U^*,-1}$ that drives the state from the origin, as t goes to $-\infty$, to $x_{\mathcal{V}_U^*}$, at $t=0$, is

$$u_{\mathcal{V}_U^*,-t} = F_2'' \left(A_{22}'\right)^{-t} \beta, \quad t = 1, 2, \ldots. \tag{31}$$

Proof: It is an immediate consequence of Theorem 6. ■

Remark 2: Theorem 7 has shown that the state of (1), (2) can be steered from the origin to $x_{\mathcal{V}_U^*} \in \mathcal{V}_U^*$ along a trajectory in $\mathcal{V}_U^*$, hence invisible to the output, by means of the forcing action (31), which is computed backwards and is supposed to be applied to the system forwards, starting from a time instant at $-\infty$. However, implementation requires that a finite, though arbitrary long, control sequence be considered. Let $-t_{pre}$ be the time when the actual control sequence starts being applied to (1), (2). Since the state of (1), (2) at $t = -t_{pre}$ is the zero state, the actual trajectory does not match the theoretic one on $\mathcal{V}_U^*$. Nonetheless, the error between the actual trajectory and the theoretical one can be made negligible, provided that (1), (2) be asymptotically stable and t_{pre} be sufficiently large. □

Remark 3: The control sequence (31) goes to zero asymptotically, as $t \to \infty$, at a ratio commensurate with the dynamics of the invariant zeros of (1), (2). Let ε be the tolerance, a positive real number sufficiently close to zero according to design specifications, and z_{dom} be the nonminimum-phase invariant zero with the smallest absolute value. If we choose

$$t_{pre} = \lceil -\log \varepsilon / \log |z_{dom}| \rceil, \tag{32}$$

we are guaranteed that the contribution of the mode of z_{dom} to (31) at $-t_{pre}$ is reduced in the ratio of 1 to ε with respect to the same contribution at $t=0$. Hence, (32) is a simple way to choose a value of t_{pre} consistent with the required accuracy and the invariant zero dynamics. □

On the other hand, the control sequence that drives the state from the origin to $x_{\mathcal{S}}$, while maintaining zero output until the last step but one, is given by (37) in Appendix. Corollary 2, that follows, shows how the control sequences must be combined in light of Remark 2.

Corollary 2: Consider (1), (2). Let $\mathcal{A}2'$, $\mathcal{A}3'$ hold. Let F be s.t. $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C+DF)$, and $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$. Let $\mathcal{V}_U^*$ be the subspace of Theorem 6. Let ρ be defined as in Appendix, Section B. Let $x_f = x_{\mathcal{V}_U^*} + x_{\mathcal{S}}$, with $x_{\mathcal{S}} \in \mathcal{S}^*$ and $x_{\mathcal{V}_U^*} \in \mathcal{V}_U^*$, be given. Let $t_{pre} > \rho + 1$. The control sequence u_t, with $t = -t_{pre} + \rho + 1, -t_{pre} + \rho + 2, \ldots, \rho$, driving the state from $x_{-t_{pre}+\rho+1} = 0$ to $x_{\rho+1} = x_f$, while maintaining zero output with arbitrary accuracy (i.e. with an error uniformly converging to 0 as $t_{pre} \to \infty$), until ρ is

$$u_t = \begin{cases} u_{\mathcal{V}_U^*,t-\rho-1} + u_{\mathcal{S},t}, & t = 0, 1, \ldots, \rho, \\ u_{\mathcal{V}_U^*,t-\rho-1}, & t = -t_{pre} + \rho + 1, \ldots, -1, \end{cases} \tag{33}$$

where $u_{\mathcal{V}_U^*,t}$, with $t = -t_{pre}, -t_{pre}+1, \ldots, -1$, is given by (31) and $u_{\mathcal{S},t}$, with $t = 0, 1, \ldots, \rho$, is given by (37).

Proof: By linearity and time invariance of (1), (2). ■

Let the input/output equation of the FIR system be $u_t = \sum_{\ell=0}^{N-1} \Phi_\ell h_{t-\ell}, \quad t \in \mathbb{Z}$, where $N \in \mathbb{Z}^+$ is the number of steps of the time window and Φ_ℓ, $\ell = 0, 1, \ldots, N-1$, is the gain matrix. Let $h_t = x_f \delta_t$, $t \in \mathbb{Z}$. Since any $x_f \in \mathbb{R}^n$ is a linear combination of the column vectors of the main basis I_n of $\mathbb{R}^n$, the gain matrix Φ_ℓ, $\ell = 0, 1, \ldots, N-1$, can be derived through a straightforward extension of (33). Let S^* and V_U^* be basis matrices of $\mathcal{S}^*$ and $\mathcal{V}_U^*$. Then, $I_n = S^* \alpha + V_U^* \beta$, where $\alpha \in \mathbb{R}^{n_R + n_S \times n}$ and $\beta \in \mathbb{R}^{n_V \times n}$ are given by $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = [S^* \; V_U^*]^{-1}$ (note that $[S^* \; V_U^*]$ is square and nonsingular due to proposition (ii) of Theorem 6). With the new definition of α and β, (37), (31) respectively provide the matrix control sequences, henceforth denoted by capital letters, $U_{\mathcal{S},t}$, with $t = 0, 1, \ldots, \rho$, $U_{\mathcal{V},-t}$, with $t = 1, 2, \ldots$, and U_t, with $t = -t_{pre} + \rho + 1, -t_{pre} + \rho + 2, \ldots, \rho$. Therefore, by linearity and time invariance of (1), (2), the gain matrix is $\Phi_\ell = U_{\ell - t_{pre} + \rho + 1}$, $\ell = 0, 1, \ldots, N-1$, with $N = t_{pre}$.

The discussion developed so far has shown how to design a FIR system that reconstructs, with a possible delay, the unknown initial state and the subsequent state trajectory by processing the available measurements. Then, the sequence $\tilde{u}_t$, $t \in \mathbb{Z}_0^+$, of the reconstructed input can be derived from the output sequence and the reconstructed state trajectory with a further delay of one step. In particular $\tilde{u}_t$, $t \in \mathbb{Z}_0^+$, is significant from time $t = N$ on and is given by

$$\tilde{u}_t = \begin{bmatrix} B \\ D \end{bmatrix}^\dagger \left(\begin{bmatrix} \tilde{x}_{t+1} \\ y_{t-N} \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} \tilde{x}_t \right), \; t = N, N+1, \ldots, \tag{34}$$

VIII. An Illustrative Example

Consider (1), (2), with $A \in \mathbb{R}^{6\times 6}$, $B \in \mathbb{R}^{6\times 3}$, $C \in \mathbb{R}^{4\times 6}$, $D \in \mathbb{R}^{4\times 3}$ matrices whose nonzero entries are $A_{11} = A_{22} = 0.16$, $A_{12} = -0.58$, $A_{13} = 0.44$, $A_{14} = 1.01$, $A_{15} = -3.69$, $A_{21} = 0.49$, $A_{23} = 0.39$, $A_{24} = 0.68$, $A_{25} = -3.47$, $A_{44} = 0.5$, $A_{66} = 0.95$, $B_{11} = -2.91$, $B_{12} = 1.23$, $B_{13} = -2.58$, $B_{21} = -0.49$, $B_{22} = 0.48$, $B_{23} = 2.55$, $B_{51} = 0.05$, $B_{53} = C_{31} = 0.03$, $C_{11} = -C_{43} = -0.01$, $C_{12} = C_{32} = 0.09$, $C_{13} = -C_{42} = 0.07$, $C_{21} = -0.48$, $C_{22} = -0.59$, $C_{25} = -49.51$, $C_{33} = -0.06$, $C_{41} = 0.26$, $C_{46} = D_{21} = D_{22} = 1$, $D_{23} = 10$. The sampling time is

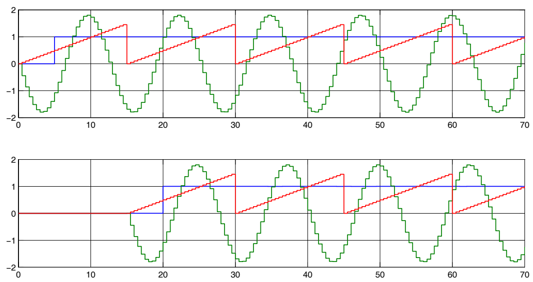

Fig. 1. True input (upper plot) and reconstructed input (lower plot) – Amplitude versus time.

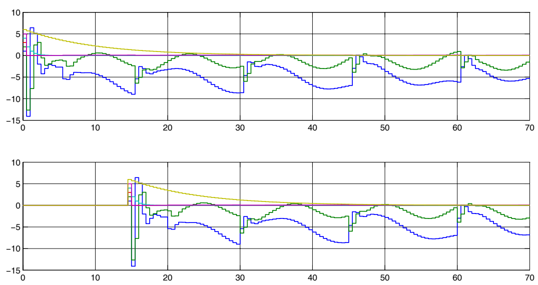

Fig. 2. True state (upper plot) and reconstructed state (lower plot) – Amplitude versus time.

$T_s = 0.5\,$s. (A, B, C, D) satisfies $\mathcal{A}1$, $\mathcal{A}2$, and $\mathcal{A}3$ since $\mathcal{Z}(A, B, C, D) = \{0.5,\ 1.5253\}$. Moreover, $\sigma(A) = \{0.16 \pm 0.5331j,\ 0, 0, 0.5, 0.95\}$. The synthesis procedure described in Sections V–VII with $N = 30$, leads to a device which guarantees the behavior illustrated in Figs, 1, 2. The initial state, regarded as unknown, is $x_0 = [1\ 2\ 3\ 4\ 5\ 6]^\top$. The unknown input consists of the three signals shown in the upper plot of Fig. 1: a step, a sinusoid, and a sawtooth.

IX. Conclusion

This work has dealt with unknown-state, unknown-input reconstruction in discrete-time, nonminimum-phase systems with feedthrough terms. The synthesis of the reconstructor has been developed through the methods of the geometric approach. A numerical example has illustrated the effectiveness of the procedure described.

Appendix

This section reviews the geometric algorithms used for computing the maximal output-nulling controlled invariant subspace $\mathcal{V}^* = \max \mathcal{V}(A, B, C, D)$ and the minimal input-containing conditioned invariant subspace $\mathcal{S}^* = \min \mathcal{S}\,(A, B, C, D)$, respectively.

Consider (1), (2). Let the triple $(\tilde{A}, \tilde{B}, \tilde{C})$ be defined by

$$\tilde{A} = \begin{bmatrix} A & O \\ C & O \end{bmatrix},\ \tilde{B} = \begin{bmatrix} B \\ D \end{bmatrix},\ \tilde{C} = \begin{bmatrix} O & I_q \end{bmatrix}. \quad (35)$$

Let $\tilde{\mathcal{V}}^* = \max \mathcal{V}(\tilde{A}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})$ be the maximal controlled invariant subspace of $(\tilde{A}, \tilde{B}, \tilde{C})$: i.e., the maximal $(\tilde{A}, \tilde{\mathcal{B}})$-controlled invariant subspace contained in $\tilde{\mathcal{C}}$, where $\tilde{\mathcal{B}} = \operatorname{im} \tilde{B}$ and $\tilde{\mathcal{C}} = \ker \tilde{C}$. An algorithm for computing $\tilde{\mathcal{V}}^*$ is given in [1] (Algorithm 4.1-2). As is shown in [1] (Section 4.5), a basis matrix $\tilde{V}^*$ of $\tilde{\mathcal{V}}^*$ has the structure $\tilde{V}^* = \left[\begin{smallmatrix} V^* \\ O \end{smallmatrix}\right]$ where V^* is a basis matrix of $\mathcal{V}^* = \max \mathcal{V}(A, B, C, D)$. Hence, the computation of the maximal output-nulling controlled invariant subspace $\mathcal{V}^*$ reduces to the application of the algorithm for the controlled invariant subspace to the extended triple (35).

Consider (1), (2). Let the triple $(\tilde{A}, \tilde{B}, \tilde{C})$ be defined by

$$\tilde{A} = \begin{bmatrix} A & B \\ O & O \end{bmatrix},\ \tilde{B} = \begin{bmatrix} O \\ I_p \end{bmatrix},\ \tilde{C} = \begin{bmatrix} C & D \end{bmatrix}. \quad (36)$$

Let $\tilde{\mathcal{S}}^* = \min \mathcal{S}(\tilde{A}, \tilde{\mathcal{C}}, \tilde{\mathcal{B}})$ be the minimal conditioned invariant subspace of $(\tilde{A}, \tilde{B}, \tilde{C})$: i.e., the minimal $(\tilde{A}, \tilde{\mathcal{C}})$-conditioned invariant subspace containing $\tilde{\mathcal{B}}$, where $\tilde{\mathcal{B}} = \operatorname{im} \tilde{B}$ and $\tilde{\mathcal{C}} = \ker \tilde{C}$. An algorithm for computing $\tilde{\mathcal{S}}^*$ is given in [1] (Algorithm 4.1-1): namely, $\tilde{\mathcal{S}}^*$ is the last term of the sequence $\tilde{\mathcal{S}}_0 = \tilde{\mathcal{B}}$, $\tilde{\mathcal{S}}_i = \tilde{A}(\tilde{\mathcal{S}}_{i-1} \cap \tilde{\mathcal{C}}) + \tilde{\mathcal{B}}$, where $i = 1, 2, \ldots, \tilde{\rho}$, with $\tilde{\rho}$ denoting the least integer s.t. $\tilde{\mathcal{S}}_i = \tilde{\mathcal{S}}_{i+1}$. It can be shown that a basis matrix $\tilde{S}^*$ of $\tilde{\mathcal{S}}^*$ has the structure $\tilde{S}^* = \left[\begin{smallmatrix} S^* & O \\ O & I_p \end{smallmatrix}\right]$ where S^* is a basis matrix of $\mathcal{S}^* = \min \mathcal{S}(A, B, C, D)$. Hence, the computation of the minimal input-containing conditioned invariant subspace $\mathcal{S}^*$ reduces to the application of the algorithm for the conditioned invariant subspace to the extended triple (36).

The subspace $\tilde{\mathcal{S}}^*$ is the subspace of the states of $(\tilde{A}, \tilde{B}, \tilde{C})$ reachable from the origin in $\tilde{\rho} + 1$ steps, along trajectories that do not affect the output until the last step but one (this is an immediate consequence of the algorithm). Let $\tilde{\mathcal{S}}_0$, $\tilde{\mathcal{S}}_1$, ..., $\tilde{\mathcal{S}}_{\tilde{\rho}}$ denote the subspaces generated by the algorithm for $\tilde{\mathcal{S}}^*$. Let $\tilde{M}_1$, $\tilde{M}_2$, ..., $\tilde{M}_{\tilde{\rho}}$ denote basis matrices of the subspaces $\tilde{\mathcal{S}}_0 \cap \tilde{\mathcal{C}}$, $\tilde{\mathcal{S}}_1 \cap \tilde{\mathcal{C}}$, ..., $\tilde{\mathcal{S}}_{\tilde{\rho}-1} \cap \tilde{\mathcal{C}}$, respectively. Let $\tilde{M}_{\tilde{\rho}+1}$ denote a basis matrix of $\tilde{\mathcal{S}}_\rho = \tilde{\mathcal{S}}^*$. Let $\tilde{x}_{\tilde{\mathcal{S}}} = [x_{\mathcal{S}}^\top O]^\top \in \tilde{\mathcal{S}}^*$, so that $x_{\mathcal{S}} \in \mathcal{S}^*$. Let $\tilde{\alpha}_{\tilde{\rho}+1}$ be s.t. $\tilde{x}_{\tilde{\mathcal{S}}} = \tilde{M}_{\tilde{\rho}+1} \tilde{\alpha}_{\tilde{\rho}+1}$. Then, the control sequence $\tilde{u}_{\tilde{\mathcal{S}},0}$, $\tilde{u}_{\tilde{\mathcal{S}},1}$, ..., $\tilde{u}_{\tilde{\mathcal{S}},\tilde{\rho}}$ that drives the state from the origin to $\tilde{x}_{\tilde{\mathcal{S}},\tilde{\rho}+1} = \tilde{x}_{\mathcal{S}}$ while maintaining the output at zero until the last step but one is $\left[\begin{smallmatrix} \tilde{u}_{\tilde{\mathcal{S}},t} \\ \tilde{\alpha}_t \end{smallmatrix}\right] = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{M}_t \end{bmatrix}^\dagger \tilde{M}_{t+1} \tilde{\alpha}_{t+1}$, $t = \tilde{\rho}, \tilde{\rho} - 1, \ldots, 1$, $\tilde{u}_{\tilde{\mathcal{S}},0} = \tilde{B}^\dagger \tilde{M}_1 \tilde{\alpha}_1$. In light of (36), the control sequence that drives the state of (1), (2) from the origin to $x_{\mathcal{S}} \in \mathcal{S}^*$ along a trajectory not visible at the output until the last step but one is

$$u_{\mathcal{S},t} = \tilde{u}_{\tilde{\mathcal{S}},t}, \quad t = 0, 1, \ldots \rho, \quad \text{with } \rho = \tilde{\rho} - 1. \quad (37)$$

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