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Poincaré Recurrence and Output Reversibility in Linear Dynamical Systems

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Abstract— **Reversibility of dynamical processes arises in many physical dynamical systems. For example, lossless Newtonian and Hamiltonian mechanical systems exhibit trajectories that can be obtained by time going forward and backward, providing an example of time symmetry that arises in natural sciences. Another example of such time symmetry is the phenomenon known as Poincaré recurrence wherein the dynamical system exhibits trajectories that return infinitely often to neighborhoods of their initial conditions. In this paper, we study output reversibility in linear dynamical systems, that is, the backward recoverability of the system output while time is going forward. Specifically, we provide necessary and sufficient conditions for output reversibility in terms of the spectrum of the system dynamics. In addition, we provide sufficient conditions for the absence of output reversibility. Furthermore, we establish that no system trajectory can retrace its time history backwards with time going forward which is also natural in light of the uniqueness of solutions to linear dynamical systems. Finally, we draw connections between output reversibility and Poincaré recurrence.**

I. INTRODUCTION

The notion of reversibility is one of the fundamental symmetries that arise in natural sciences. Specifically, time-reversal symmetry arises in many physical dynamical systems and, in particular, in classical and quantum mechanics. The governing dynamical system equations for such systems possess reversing symmetries, that is, the concept of time flow does not enter in these physical theories. In particular, Newtonian and Hamiltonian mechanics (including Einstein's relativistic and Schrödinger's quantum extensions) are invariant under time reversal, that is, they make no distinction of one direction of time and the other. Such theories possess a *time-reversal symmetry*, wherein, from any given moment of time, the governing dynamical laws treat past and future in exactly the same way [1], [2], [3].

In contrast, thermodynamics describes processes that give rise to *time-reversal non-invariance* [2], [3]. The term time-reversal is not meant literally here; that is, thermodynamic systems give rise to dynamical systems whose system trajectory reversal is or is not allowed and *not* a reversal of time itself. Nevertheless, many scientists have attributed this emergence of the direction of time flow to the second law of thermodynamics—the law that entropy always increases—due to its intimate connection to the irreversibility of dynamical processes.

Another key distinction between thermodynamics and mechanics is that thermodynamics is a theory of *open systems* [2], whereas mechanics is a theory of *closed systems*. In particular, thermodynamic systems exchange matter and energy with the environment, and hence, interact with the environment. Alternatively, in mechanics it is always possible to include interactions with the environment (via feedback interconnecting components) within the system description, to obtain an augmented closed system. In this case, the system can be described by an evolution law with, possibly, an output equation wherein past trajectories define the future trajectory uniquely and the system output depends on the instantaneous (present) value of the system state.

In this paper, we use system-theoretic notions to investigate system reversibility for closed linear dynamical systems. In particular, we consider the free response of a system on a given, finite interval. The system is *output reversible* if, for every initial condition and corresponding trajectory, there exists an alternative initial condition such that the corresponding trajectory is the time-reversed image of the original trajectory. Our goal is to characterize linear systems that are output reversible.

In [4], output reversibility was addressed for linear dynamical systems with single outputs. As special cases, it was shown that the class of output-reversible systems includes rigid body and Hamiltonian systems. This result suggests that stability and instability play a key role in the arrow of time, independently of dimensionality, nonlinearity, and initial-state sensitivity. In this paper, we extend the results of [4] to multi-output systems to obtain a spectral symmetry condition that characterizes output reversible systems. In particular, we show that a closed linear system is output reversible if and only if its non-imaginary spectrum is symmetric with respect to the imaginary axis.

Reversible dynamical systems tend to exhibit *Poincaré recurrence* [5], [2], that is, if the flow of a dynamical system preserves volume—namely, the volume of an arbitrary region of the state space is conserved by the time evolution of the system—and has only bounded orbits, then for each open set there exist orbits that intersect the set infinitely often. In this paper, we additionally provide connections between Poincaré recurrence and output reversibility.

Finally, time-reversibility in closed linear dynamical systems is also studied in [6], [7] using a class of behaviors which can be described through a set of linear constant coefficient differential equations. Specifically, the authors in [6], [7] consider linear differential equations defined by polynomial matrices. For these systems, the authors define the notion of J*-time-reversibility*. However, the notion of output reversibility of the present paper is distinct from the notion of J-time-reversibility defined in [6], [7]. Specifically, for linear systems defined by polynomial matrices that include state space systems as a special case, a system is J-time-reversible if the application of a linear transformation J to the trajectory yields a trajectory of the *time-reversed system*, that is, the modified system in which t is replaced by $-t$.

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For example, consider the unforced single-degree-of-freedom rigid body modeled by $\dot{x} = Ax$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then, applying $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to the trajectory provides a trajectory of the time-reversed system $\dot{\hat{x}} = \hat{A}\hat{x}$, where $\hat{x} \triangleq Jx$ and $\hat{A} \triangleq JAJ^{-1} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$. In contrast, the present paper considers only *forward* trajectories of $\dot{x} = Ax$ with the goal of determining initial conditions for which the output $y = Cx$ is the time-reversed image of a given output trajectory. Consequently, output reversibility and J-time-reversibility are distinct notions. In addition, the authors in [6], [7] do not give any connections between the spectral symmetry of the system dynamics and output reversibility, nor do they provide any connections between output reversibility and Poincaré recurrence.

II. Mathematical Preliminaries

In this section, we introduce notation and definitions needed for developing the main results of this paper. Let $\mathbb{R}$ denote the set of real numbers, $\overline{\mathbb{Z}}_+$ denote the set of nonnegative integers, and $\mathbb{R}^n$ denote the set of $n \times 1$ column vectors. For $A \in \mathbb{R}^{n\times n}$ the multi-spectrum $\mathrm{mspec}\,(A) = \{\lambda_1, \ldots, \lambda_n\}_{\mathrm{m}}$ is the set of all eigenvalues of A including their multiplicity. We say that the multi-spectrum of A is *symmetric with respect to the imaginary axis* if $\{\lambda_1, \ldots, \lambda_n\}_{\mathrm{m}} = \{-\lambda_1, \ldots, -\lambda_n\}_{\mathrm{m}}$. We denote by $A^\dagger \in \mathbb{R}^{m\times n}$ the Moore-Penrose generalized inverse of $A \in \mathbb{R}^{n\times m}$ [8], by I or I_n the $n \times n$ identity matrix, and by $\mathcal{N}(A) \triangleq \{x \in \mathbb{R}^m : Ax = 0\}$ the null space of $A \in \mathbb{R}^{n\times m}$. We say that, for $A \in \mathbb{R}^{n\times n}$, $\lambda \in \mathrm{mspec(A)}$ is *semisimple* if the algebraic multiplicity of λ is equal to its geometric multiplicity, that is, the complex Jordan form of A is a diagonal matrix.

We begin by considering the nonlinear dynamical system given by

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{1}$$

with output

$$y(t) = g(x(t)), \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^l$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^l$ are continuous. We assume that solutions of (1) exist and are unique on all finite intervals $[0, T)$. For clarity we write the solution of (1) as $x(t, x_0)$ with the output given by $y(t) = y(t, x_0) = g(x(t, x_0))$.

Definition 2.1 ([4]): The system (1) and (2) is *output reversible* if, for all $x_0 \in \mathbb{R}^n$ and $t_1 > 0$, there exists $\hat{x}_0 \in \mathbb{R}^n$ such that

$$y(t, \hat{x}_0) = y(t_1 - t, x_0), \quad t \in [0, t_1]. \tag{3}$$

We wish to determine whether a given system (1) and (2) is output reversible. In the next section, we consider the special case of linear systems.

III. Linear Output Reversible Systems

In this section, we consider the linear dynamical system

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \geq 0, \tag{4}$$

with output

$$y(t) = Cx(t), \tag{5}$$

where $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$. For the remainder of the paper we assume that (A, C) is observable. It follows from Definition 2.1 that (4) and (5) is output reversible if and only if, for all $x_0 \in \mathbb{R}^n$ and $t_1 > 0$, there exists $\hat{x}_0 \in \mathbb{R}^n$ such that

$$Ce^{At}\hat{x}_0 = Ce^{A(t_1-t)}x_0, \quad t \in [0, t_1]. \tag{6}$$

Note that output reversibility is a basis-independent property.

The following result shows that if (4) and (5) is output reversible, then $\hat{x}_0$ satisfying (6) is unique.

Proposition 3.1: Let $x_0 \in \mathbb{R}^n$ and $t_1 > 0$, assume that (4) and (5) is output reversible, and let $\hat{x}_0 \in \mathbb{R}^n$ satisfy (6). Then $\hat{x}_0$ satisfies

$$\mathcal{O}\hat{x}_0 = \mathcal{S}\mathcal{O}e^{At_1}x_0 \tag{7}$$

and is given uniquely by

$$\hat{x}_0 = \mathcal{O}^\dagger\mathcal{S}\mathcal{O}e^{At_1}x_0, \tag{8}$$

where $\mathcal{O} \in \mathbb{R}^{nl\times n}$ and $\mathcal{S} \in \mathbb{R}^{nl\times nl}$ are defined by

$$\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \tag{9}$$

$$\mathcal{S} \triangleq \begin{bmatrix} I_l & 0_{l\times l} & & \cdots & 0_{l\times l} \\ 0_{l\times l} & -I_l & & & \\ & & I_l & & \vdots \\ \vdots & & & \ddots & 0_{l\times l} \\ 0_{l\times l} & \cdots & & 0_{l\times l} & (-1)^{n-1}I_l \end{bmatrix}. \tag{10}$$

Proof. Since (4) and (5) is output reversible, there exists $\hat{x}_0$ satisfying (6). Differentiating (6) $n-1$ times and setting $t = 0$ yields (7). Since (A, C) is observable, $\mathcal{O}^\dagger$ is a left inverse of $\mathcal{O}$. Hence, (7) implies (8). □

Note that since (A, C) is observable, $\mathrm{rank}\,\mathcal{O} = n$. In addition, if, for some $x_0 \notin \mathcal{N}(A)$ and $t_1 > 0$, there does not exist $\hat{x}_0 \in \mathbb{R}^n$ satisfying (7), then (6) does not have a solution. In this case, (4) and (5) is not output reversible.

Corollary 3.1: Let $x_0 \in \mathbb{R}^n$, $t_1 > 0$, and assume that $\mathrm{rank}\,[\mathcal{O} \quad \mathcal{S}\mathcal{O}e^{At_1}x_0] > n$. Then (4) and (5) is not output reversible.

Proposition 3.2: Assume that $l = n$ and $C \in \mathbb{R}^{n\times n}$ is invertible. Then (4) and (5) is not output reversible.

Proof. Since $C \in \mathbb{R}^{n\times n}$ is invertible, then (A, C) is observable. Let $x_0 \in \mathbb{R}^n$ and $t_1 > 0$, and note that (7) is equivalent to

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \hat{x}_0 = \begin{bmatrix} C \\ -CA \\ \vdots \\ (-1)^{n-1}CA^{n-1} \end{bmatrix} e^{At_1}x_0. \tag{11}$$

Since $C \in \mathbb{R}^{n\times n}$ is invertible, it follows from (11) that

$$\begin{aligned} \hat{x}_0 &= e^{At_1}x_0, \\ A\hat{x}_0 &= -Ae^{At_1}x_0, \\ &\vdots \\ A^{n-1}\hat{x}_0 &= (-1)^{n-1}A^{n-1}e^{At_1}x_0, \end{aligned}$$

which implies that $A\hat{x}_0 = Ae^{At_1}x_0 = -Ae^{At_1}x_0 = 0$. Thus, $\hat{x}_0 \in \mathcal{N}(A)$ is an equilibrium of (4). Furthermore, $x_0 =$

$e^{-At_1}\hat{x}_0$ and, hence, $Ax_0 = Ae^{-At_1}\hat{x}_0 = 0$, which implies that $x_0 \in \mathcal{N}(A)$ is an equilibrium point of (4) and, hence, $\hat{x}_0 = x_0$. Thus, (7) has only solutions $\hat{x}_0 = x_0$ that are equilibrium points of (4). Therefore, (4) and (5) is not output reversible. □

Proposition 3.2 implies that full state reversibility, i.e., $C = I_n$, in linear dynamical systems is impossible. That is, if $C = I_n$, then there does *not* exist an initial condition to generate the solution that will retrace backwards the original solution with time going forward. This conclusion is natural in light of uniqueness of solutions of linear dynamical systems.

Next, we use the fact [8, p. 422] that the matrix exponential e^{At} of $A \in \mathbb{R}^{n\times n}$ can be written as a polynomial in A of the form

$$e^{At} = \sum_{i=0}^{n-1} \phi_i(t)A^i. \tag{12}$$

The coefficients $\phi_0(t), \ldots, \phi_{n-1}(t)$ are real linear combinations of terms of the form $t^r \mathrm{Re}\, e^{\lambda t}$ and $t^r \mathrm{Im}\, e^{\lambda t}$, where λ is an eigenvalue of A and r is a nonnegative integer. Explicitly, $\phi_i(t)$ is given by the contour integration [8, p. 423]

$$\phi_i(t) = \frac{1}{2\pi \jmath} \oint_{\mathcal{C}} \frac{p^{[i+1]}(z)}{p(z)} e^{tz}\, \mathrm{d}z, \quad i = 0, \ldots, n-1, \tag{13}$$

where $\mathcal{C}$ is a clockwise contour enclosing the spectrum of A,

$$p(s) = s^n + \beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0 \tag{14}$$

is the characteristic polynomial of A, that is, $p(s) = \det(sI - A)$, and, for all $i = 0, \ldots, n-1$,

$$p^{[i+1]}(s) = s^{n-i-1} + \beta_{n-1}s^{n-i-2} + \beta_{n-2}s^{n-i-3} + \cdots + \beta_{i+1}. \tag{15}$$

Note that $p^{[n]}(s) = 1$. The polynomials $p^{[i+1]}(s)$ satisfy the recursion [8]

$$sp^{[i+1]}(s) = p^{[i]}(s) - \beta_i, \quad i = 0, \ldots, n-1, \tag{16}$$

where $p^{[0]}(s) \triangleq p(s)$.

Since (A, C) is observable it follows from [8, p. 552] that, if $l = 1$, then A is cyclic (nonderogatory), and thus, its minimal polynomial coincides with its characteristic polynomial [8, p. 179]. Recall that A is cyclic if and only if A has exactly one Jordan block associated with each distinct eigenvalue. The next proposition shows that if A is cyclic, then the coefficients satisfying (12) are unique.

Proposition 3.3: If $A \in \mathbb{R}^{n\times n}$ is cyclic, then the functions $\phi_0(t), \ldots, \phi_{n-1}(t)$, $t \geq 0$, satisfying (12) are unique.

Proof. Let $\hat{\phi}_0(t), \ldots, \hat{\phi}_{n-1}(t)$, $t \geq 0$, satisfy

$$e^{At} = \sum_{i=0}^{n-1} \hat{\phi}_i(t)A^i, \quad t \geq 0. \tag{17}$$

Subtracting (17) from (12) yields

$$\sum_{i=0}^{n-1} [\phi_i(t) - \hat{\phi}_i(t)]A^i = 0, \quad t \geq 0. \tag{18}$$

For each $t \geq 0$, the left-hand side of (18) represents a polynomial of degree $n-1$ with root A. However, since A is cyclic, its minimal polynomial is equal to its characteristic polynomial; see Proposition 5.5.20 of [8]. Thus, $\phi_i(t) - \hat{\phi}_i(t) = 0$, $t \geq 0$, $i = 0, \ldots, n-1$. □

Define $r \triangleq \mathrm{rank}\, A$ and note that the dimension of $\mathcal{N}(A)$ is $n - r$ [8, Corollary 2.5.1], that is, $\mathcal{N}(A)$ contains $n - r$ linearly independent vectors.

Lemma 3.1: Let $t^* > 0$ and assume there exist r linearly independent vectors $x_1, \ldots, x_r \in \mathbb{R}^n$ such that $x_i \notin \mathcal{N}(A)$, $i = 1, \ldots, r$, and

$$\mathrm{rank}\,[\mathcal{O} \quad \mathcal{S}\mathcal{O}e^{At^*}x_i] = n, \quad i = 1, \ldots, r. \tag{19}$$

Then, for all $x_0 \in \mathbb{R}^n$ and $t_1 > 0$,

$$\mathrm{rank}\,[\mathcal{O} \quad \mathcal{S}\mathcal{O}e^{At_1}x_0] = n. \tag{20}$$

Proof. Let $x_{r+1}, \ldots, x_n \in \mathcal{N}(A)$ be linearly independent. Next, let $\hat{x}_0 \in \mathbb{R}^n$ satisfy

$$\mathcal{O}\hat{x}_0 = \mathcal{S}\mathcal{O}e^{At^*}x_i, \quad i = r+1, \ldots, n. \tag{21}$$

Since $Ax_i = 0$, $i = r+1, \ldots, n$, it follows that (21) is equivalent to

$$\begin{bmatrix} C\hat{x}_0 \\ CA\hat{x}_0 \\ \vdots \\ CA^{n-1}\hat{x}_0 \end{bmatrix} = \begin{bmatrix} Cx_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad i = r+1, \ldots, n. \tag{22}$$

Note that (21) holds with $\hat{x}_0 = x_i$ for each $i = r+1, \ldots, n$. It follows from Theorem 2.6.3 of [8] and the fact that (A, C) is observable, that, for all $i = r+1, \ldots, n$, $\mathrm{rank}\,[\mathcal{O} \quad \mathcal{S}\mathcal{O}e^{At^*}x_i] = n$, and thus, $\hat{x}_0 = x_i$ is the unique solution to (21) for each $i = r+1, \ldots, n$. Thus, it follows from the above arguments and (19) that each vector $\mathcal{S}\mathcal{O}e^{At^*}x_i$, $i = 1, \ldots, n$, is a linear combination of the columns of $\mathcal{O} \in \mathbb{R}^{nl\times n}$.

Note that $x_1, \ldots, x_n$ are linearly independent vectors and, thus, form a basis in $\mathbb{R}^n$. To see this, note that every vector $x \in \mathbb{R}^n$ can be represented as $x = y + z$, where $y \in \mathcal{N}(A)$ and $z \in \mathbb{R}^n \backslash \mathcal{N}(A)$. Since $x_1, \ldots, x_r$ form a basis in $\mathbb{R}^n \backslash \mathcal{N}(A)$ and $x_{r+1}, \ldots, x_n$ form a basis in $\mathcal{N}(A)$ it follows that y and z are linear combinations of $x_{r+1}, \ldots, x_n$ and $x_1, \ldots, x_r$, respectively. Hence, $x \in \mathbb{R}^n$ is a linear combination of x_i, $i = 1, \ldots, n$, and hence, $x_1, \ldots, x_n$ form a basis in $\mathbb{R}^n$.

Now, let $x_0 \in \mathbb{R}^n$ and $t_1 > 0$. Since $x_1, \ldots, x_n$ form a basis in $\mathbb{R}^n$, there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $e^{A(t_1 - t^*)}x_0 = \sum_{i=1}^n \alpha_i x_i$. Furthermore,

$$\mathcal{S}\mathcal{O}e^{At_1}x_0 = \mathcal{S}\mathcal{O}e^{At^*}e^{A(t_1-t^*)}x_0 = \sum_{i=1}^{n} \alpha_i \mathcal{S}\mathcal{O}e^{At^*}x_i. \tag{23}$$

Since, for all $i = 1, \ldots, n$, $\mathcal{S}\mathcal{O}e^{At^*}x_i$ is a linear combination of the columns of $\mathcal{O}$, it follows from (23) that $\mathcal{S}\mathcal{O}e^{At_1}x_0$ is also a linear combination of the columns of $\mathcal{O}$. Thus, $\mathrm{rank}\,[\mathcal{O} \quad \mathcal{S}\mathcal{O}e^{At_1}x_0] = n$ for every $x_0 \in \mathbb{R}^n$ and $t_1 > 0$, which proves the result. □

For the following result, define $\Phi(\cdot) \in \mathbb{R}^{nl\times l}$ and $\phi(\cdot) \in \mathbb{R}^n$ given by

$$\Phi(t) \triangleq \begin{bmatrix} \phi_0(t)I_l \\ \vdots \\ \phi_{n-1}(t)I_l \end{bmatrix}, \quad \phi(t) \triangleq \begin{bmatrix} \phi_0(t) \\ \vdots \\ \phi_{n-1}(t) \end{bmatrix}, \tag{24}$$

for all $t \geq 0$. Substituting (12) into (6) yields

$$\Phi^{\mathrm{T}}(t)\mathcal{O}\hat{x}_0 = \Phi^{\mathrm{T}}(-t)\mathcal{O}e^{At_1}x_0, \quad t \geq 0. \tag{25}$$

Note that (6) and (25) are equivalent.

Proposition 3.4: Let $t^* > 0$ and $r = \mathrm{rank}\,(A)$, and assume there exist r linearly independent vectors $x_1, \ldots, x_r \in \mathbb{R}^n$ such that $x_i \notin \mathcal{N}(A)$, $i = 1, \ldots, r$, and

$$\mathrm{rank}\,[\mathcal{O} \quad \mathcal{S}\mathcal{O}e^{At^*}x_i] = n, \quad i = 1, \ldots, r. \tag{26}$$

The linear system (4), (5) is output reversible if and only if

$$\phi(-t) = S\phi(t), \quad t \geq 0, \tag{27}$$

where $S \triangleq \mathrm{diag}\,[1, -1, 1, \cdots, (-1)^{n-1}]$.

Proof. First, note that it follows from Lemma 3.1 that for all $x_0 \in \mathbb{R}^n$ and $t_1 > 0$, $\mathrm{rank}\,[\mathcal{O} \quad \mathcal{S}\mathcal{O}e^{At_1}x_0] = n$. In addition, note that (27) is equivalent to $\Phi(-t) = \mathcal{S}\Phi(t)$, $t \geq 0$, where $\mathcal{S}$ is given by (10). To prove necessity, assume that (4) and (5) is output reversible so that, by Proposition 3.1, $\hat{x}_0$ satisfies (7). Substituting (7) into (25) implies that, for all $x_0 \in \mathbb{R}^n$ and $t_1 > 0$, equality $\Phi^{\mathrm{T}}(t)\mathcal{S}\mathcal{O}e^{At_1}x_0 = \Phi^{\mathrm{T}}(-t)\mathcal{O}e^{At_1}x_0$, $t \geq 0$, holds.

Consequently, it follows that, for all $q \in \mathbb{R}^n$, $\Phi^{\mathrm{T}}(t)\mathcal{S}\mathcal{O}q = \Phi^{\mathrm{T}}(-t)\mathcal{O}q$, $t \geq 0$. Furthermore, since (A, C) is observable it follows that $\mathcal{O} \in \mathbb{R}^{nl \times n}$ is full rank, and hence, for all $z \in \mathbb{R}^{nl}$, $\Phi^{\mathrm{T}}(t)\mathcal{S}z = \Phi^{\mathrm{T}}(-t)z$, $t \geq 0$, which implies that $\Phi^{\mathrm{T}}(t)\mathcal{S} = \Phi^{\mathrm{T}}(-t)$, $t \geq 0$, which is equivalent to (27).

Conversely, it follows from (27) that $\Phi^{\mathrm{T}}(-t)\mathcal{O}e^{At_1}x_0 = \Phi^{\mathrm{T}}(t)\mathcal{S}\mathcal{O}e^{At_1}x_0$. Since (A, C) is observable and $\mathrm{rank}\,[\mathcal{O} \quad \mathcal{S}\mathcal{O}e^{At_1}x_0] = n$, there exists a unique solution $\hat{x}_0 \in \mathbb{R}^n$ satisfying (7) and, hence,

$$\begin{aligned}\Phi^{\mathrm{T}}(-t)\mathcal{O}e^{At_1}x_0 &= \Phi^{\mathrm{T}}(t)\mathcal{S}\mathcal{O}e^{At_1}x_0 \\ &= \Phi^{\mathrm{T}}(t)\mathcal{O}\hat{x}_0, \quad t \geq 0,\end{aligned}$$

which implies (25). Hence, (4) and (5) is output reversible. □

Remark 3.1: In case of a single output, that is $C \in \mathbb{R}^{1 \times n}$, with (A, C) observable, condition $\mathrm{rank}\,[\mathcal{O} \quad \mathcal{S}\mathcal{O}e^{At_1}x_0] = n$ is satisfied since $\mathcal{O} \in \mathbb{R}^{n \times n}$ is invertible. Thus, in the single output case the output reversibility of (4) and (5) is independent of C so long as (A, C) is observable.

The following lemma is needed for the main result of this section.

Lemma 3.2: The multi-spectrum of $A \in \mathbb{R}^{n \times n}$ is symmetric with respect to the imaginary axis if and only if $p(-s) = (-1)^n p(s)$ for all $s \in \mathbb{C}$.

Proof. Sufficiency is immediate. To show necessity, assume that the spectrum of A is symmetric with respect to the imaginary axis. In this case, the multi-spectrum of A is given by

$$\mathrm{mspec}\,(A) = \{0, \ldots, 0, \lambda_1, \ldots, \lambda_k, -\lambda_1, \ldots, -\lambda_k\}_{\mathrm{m}},$$

where $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ are nonzero and the multiplicity of the zero eigenvalue is $r = n - 2k$. Thus,

$$p(s) = s^r \prod_{i=1}^{k}(s - \lambda_i) \prod_{i=1}^{k}(s + \lambda_i).$$

Hence,

$$\begin{aligned}p(-s) &= (-1)^r(-1)^{2k}s^r \prod_{i=1}^{k}(s - \lambda_i) \prod_{i=1}^{k}(s + \lambda_i) \\ &= (-1)^{r+2k}p(s) = (-1)^n p(s).\end{aligned}$$ □

The following theorem is the main result of the section.

Theorem 3.1: Let $t^* > 0$ and $r = \mathrm{rank}\,(A)$, and assume there exist r linearly independent vectors $x_1, \ldots, x_r \in \mathbb{R}^n$ such that $x_i \notin \mathcal{N}(A)$, $i = 1, \ldots, r$, and

$$\mathrm{rank}\,[\mathcal{O} \quad \mathcal{S}\mathcal{O}e^{At^*}x_i] = n, \quad i = 1, \ldots, r. \tag{28}$$

Then (4) and (5) is output reversible if and only if the multi-spectrum of A is symmetric with respect to the imaginary axis.

Proof. To prove sufficiency, assume that the spectrum of A is symmetric with respect to the imaginary axis. In this case, it follows from Lemma 3.2 that $p(-s) = (-1)^n p(s)$ for all $s \in \mathbb{C}$. Let $i \in \{0, \ldots, n-1\}$ and $t \geq 0$. Then it follows from (13) that

$$\begin{aligned}\phi_i(-t) &= \frac{1}{2\pi \jmath}\oint_{\mathcal{C}} \frac{p^{[i+1]}(z)}{p(z)} e^{-tz}\,\mathrm{d}z \\ &= \frac{(-1)^{n-1}}{2\pi \jmath}\oint_{\mathcal{C}} \frac{p^{[i+1]}(-z)}{p(z)} e^{tz}\,\mathrm{d}z,\end{aligned}$$

where $p(s)$ and $p^{[i+1]}(s)$ are given by (14) and (15), respectively.

Next, equating coefficients of equal powers in $p(-s) = (-1)^n p(s)$ yields $\beta_{n-1} = \beta_{n-3} = \cdots = \beta_1 = 0$ if n is even, and $\beta_{n-1} = \beta_{n-3} = \cdots = \beta_0 = 0$ if n is odd. Now, assume that $n - i$ is even. Then

$$p^{[i+1]}(s) = s^{n-i-1} + \beta_{n-2}s^{n-i-3} + \cdots + \beta_{i+2}s. \tag{29}$$

Hence,

$$\begin{aligned}p^{[i+1]}(-s) &= -(s^{n-i-1} + \beta_{n-2}s^{n-i-3} + \cdots + \beta_{i+2}s) \\ &= -p^{[i+1]}(s).\end{aligned}$$

Alternatively, assume that $n - i$ is odd. Then

$$\begin{aligned}p^{[i+1]}(s) &= s^{n-i-1} + \beta_{n-2}s^{n-i-3} + \cdots + \beta_{i+3}s^2 \\ &\quad + \beta_{i+1}.\end{aligned} \tag{30}$$

Thus,

$$p^{[i+1]}(-s) = p^{[i+1]}(s). \tag{31}$$

Hence, in both cases,

$$p^{[i+1]}(-s) = (-1)^{n-i-1}p^{[i+1]}(s). \tag{32}$$

Thus,

$$\begin{aligned}\phi_i(-t) &= \frac{(-1)^{n-1}(-1)^{n-i-1}}{2\pi \jmath}\oint_{\mathcal{C}} \frac{p^{[i+1]}(z)}{p(z)} e^{tz}\,\mathrm{d}z \\ &= (-1)^i \phi_i(t), \quad t \geq 0.\end{aligned}$$

Consequently, $\phi(-t) = S\phi(t)$ for all $t \geq 0$. Hence, it follows from Proposition 3.4 that (4) and (5) is output reversible.

To prove necessity, assume that (4) and (5) is output reversible. Hence, it follows from Proposition 3.4 that $\phi(-t) = S\phi(t)$, $t \geq 0$, or, equivalently, for every $i = 0, \ldots, n-1$,

$$\begin{aligned}\phi_i(-t) &= \frac{1}{2\pi \jmath}\oint_{\mathcal{C}} \frac{p^{[i+1]}(z)}{p(z)} e^{-tz}\,\mathrm{d}z \\ &= \frac{(-1)^i}{2\pi \jmath}\oint_{\mathcal{C}} \frac{p^{[i+1]}(z)}{p(z)} e^{tz}\,\mathrm{d}z \\ &= \frac{(-1)^{i+1}}{2\pi \jmath}\oint_{\mathcal{C}^-} \frac{p^{[i+1]}(-z)}{p(-z)} e^{-tz}\,\mathrm{d}z,\end{aligned} \tag{33}$$

for all $t \geq 0$, where $\mathcal{C}$ and $\mathcal{C}^-$ are contours in $\mathbb{C}$ enclosing the roots of $p(s) = 0$ and $p(-s) = 0$, respectively. Since $p(s) = \det(sI - A)$ is a characteristic polynomial of A, the roots of $p(s) = 0$ are given by $\operatorname{mspec}(A) = \{\lambda_1, \ldots, \lambda_n\}_{\mathrm{m}}$, while the roots of $p(-s) = 0$ are given by $\{-\lambda_1, \ldots, -\lambda_n\}_{\mathrm{m}}$. Therefore, for $i = n - 1$, (33) is equivalent to

$$\frac{1}{2\pi \jmath} \oint_{\mathcal{C}} \frac{1}{p(z)} e^{-tz} \, \mathrm{d}z = \frac{(-1)^n}{2\pi \jmath} \oint_{\mathcal{C}^-} \frac{1}{p(-z)} e^{-tz} \, \mathrm{d}z, \quad (34)$$

where $t \geq 0$, which implies that

$$\sum_{k=1}^{n} \operatorname*{Res}_{z=\lambda_k} \left[\frac{1}{p(z)}\right] e^{-t\lambda_k} = (-1)^n \sum_{k=1}^{n} \operatorname*{Res}_{z=-\lambda_k} \left[\frac{1}{p(-z)}\right] e^{t\lambda_k}, \quad t \geq 0, \quad (35)$$

where $\operatorname{Res}[\cdot]$ denotes residue. If $\operatorname{Re} \lambda_k = 0$, $k = 1, \ldots, n$, then $\operatorname{mspec}(A)$ is symmetric with respect to the imaginary axis. Alternatively, assume there exist $\lambda_l \in \operatorname{mspec}(A)$ such that $\operatorname{Re} \lambda_l \neq 0$ for all $l \in \mathcal{N} \subseteq \{1, \ldots, n\}$. Let λ_m, $m \in \mathcal{N}$, be such that $|\operatorname{Re} \lambda_m| = \max_{l \in \mathcal{N}} |\operatorname{Re} \lambda_l|$. Thus, in order for (35) to hold for large $t > 0$, there must exist $\lambda_p \in \operatorname{mspec}(A)$ such that $-\lambda_p = \lambda_m$ and

$$\operatorname*{Res}_{z=\lambda_p} \left[\frac{1}{p(z)}\right] = (-1)^n \operatorname*{Res}_{z=-\lambda_m} \left[\frac{1}{p(-z)}\right], \quad (36)$$

which implies that

$$\operatorname*{Res}_{z=\lambda_p} \left[\frac{1}{p(z)}\right] e^{-\lambda_p t} - (-1)^n \operatorname*{Res}_{z=-\lambda_m} \left[\frac{1}{p(-z)}\right] e^{\lambda_m t} = 0, \quad t \geq 0. \quad (37)$$

Next, let λ_q, $q \in \mathcal{N}$, be such that the absolute value of its real part is closest to $|\mathrm{Re}\lambda_m|$ to establish the existence of $\lambda_s \in \operatorname{mspec}(A)$ such that $-\lambda_s = \lambda_q$. Recursively repeating this procedure for all λ_l, $l \in \mathcal{N}$, yields that

$$\{\lambda_1, \ldots, \lambda_n\} = \{-\lambda_1, \ldots, -\lambda_n\}, \quad (38)$$

which implies that the eigenvalues of A are symmetric with respect to the imaginary axis. □

The following corollary specializes Theorem 3.1 to the case where $l = 1$ and recovers Theorem 2.8 of [4].

Corollary 3.2: Assume $C \in \mathbb{R}^{1 \times n}$. Then (4) and (5) is output reversible if and only if the spectrum of A is symmetric with respect to the imaginary axis.

Proof. Since $C \in \mathbb{R}^{1 \times n}$ and (A, C) is observable, then $\mathcal{O} \in \mathbb{R}^{n \times n}$ is invertible, and hence, $\operatorname{rank}[\mathcal{O} \quad \mathcal{S}\mathcal{O}e^{At_1}x_0] = n$ for all $x_0 \in \mathbb{R}^n$ and $t_1 > 0$. The result now follows from Theorem 3.1. □

IV. Poincaré Recurrence and Output Reversibility

Reversible dynamical systems [2] tend to exhibit a phenomenon known as *Poincaré recurrence* [5]. Specifically, if the flow of a dynamical system preserves volume and has only bounded orbits, then for each open bounded set there exist orbits that intersect this set infinitely often [2]. In this section, we connect Poincaré recurrence with output reversibility.

Consider the nonlinear dynamical system given by (1). Given $t \in \mathbb{R}$, we denote the *flow* $s(t, \cdot) : \mathcal{D} \to \mathcal{D}$ of (1) by $s_t(x_0)$ for $x_0 \in \mathcal{D}$, and given $x \in \mathcal{D}$, we denote the *trajectory* $s(\cdot, x) : \mathbb{R} \to \mathcal{D}$ of (1) by $s^x(t)$. The following definition provides several equivalent statements for Poincaré recurrence. The equivalence of these statements is established in [2]. For this definition, $\omega(x_0)$, $x_0 \in \mathcal{D}$, denotes the positive limit set of (1) and $s_t(\mathcal{N}) \triangleq \{y \in \mathcal{D} : y = s_t(x_0) \text{ for all } x_0 \in \mathcal{N}\}$ denotes the image of $\mathcal{N} \subseteq \mathcal{D}$ under the flow $s_t(\cdot)$.

Definition 4.1 ([2]): The nonlinear dynamical system (1) exhibits *Poincaré recurrence* in $\mathcal{D}_{\mathrm{c}} \subseteq \mathcal{D}$ if either of the following statements hold:

i) For every open bounded set $\mathcal{N} \subset \mathcal{D}_{\mathrm{c}}$, there exists $t > t_0$ such that $s_t(\mathcal{N}) \cap \mathcal{N} \neq \emptyset$.

ii) For every open bounded set $\mathcal{N} \subset \mathcal{D}_{\mathrm{c}}$, there exists a point $x_0 \in \mathcal{N}$ which returns to $\mathcal{N}$ under the flow of (1), that is, $s(t, x_0) \in \mathcal{N}$ for some $t > t_0$.

iii) For every open bounded set $\mathcal{N} \subset \mathcal{D}_{\mathrm{c}}$, there exists a point $x_0 \in \mathcal{N}$ which returns to $\mathcal{N}$ infinitely often under the flow of (1), that is, $s(t_k, x_0) \in \mathcal{N}$ for some sequence $\{t_k\}_{k=1}^{\infty}$, with $t_k \to \infty$ as $k \to \infty$.

iv) For every open bounded set $\mathcal{N} \subset \mathcal{D}_{\mathrm{c}}$, there exists a point $x_0 \in \mathcal{N}$ such that $\lim_{k\to\infty} s(t_k, x_0) = x_0$ for some sequence $\{t_k\}_{k=1}^{\infty}$, with $t_k \to \infty$ as $k \to \infty$, or, equivalently, $x_0 \in \omega(x_0)$.

v) For every open bounded set $\mathcal{N} \subset \mathcal{D}_{\mathrm{c}}$, there exists a dense subset $\mathcal{V} \subset \mathcal{N}$ such that for every point $x_0 \in \mathcal{V}$, $\lim_{k\to\infty} s(t_k, x_0) = x_0$ for some sequence $\{t_k\}_{k=1}^{\infty}$, with $t_k \to \infty$ as $k \to \infty$, or, equivalently, $x_0 \in \omega(x_0)$.

The following theorem proven in [9] provides necessary and sufficient conditions for Poincaré recurrence in linear dynamical systems.

Theorem 4.1 ([9]): The linear dynamical system given by (4) exhibits Poincaré recurrence in $\mathbb{R}^n$ if and only if $\operatorname{Re} \lambda = 0$ and λ is semisimple, where $\lambda \in \operatorname{spec}(A)$.

It follows from Theorem 4.1 that if (4) exhibits Poincaré recurrence, then the multi-spectrum of A is symmetric with respect to the imaginary axis. The next result shows that Poincaré recurrence is a sufficient condition for output reversibility in linear dynamical systems.

Theorem 4.2: Let $t^* > 0$ and $r = \operatorname{rank}(A)$, and assume there exist r linearly independent vectors $x_1, \ldots, x_r \in \mathbb{R}^n$ such that $x_i \notin \mathcal{N}(A)$, $i = 1, \ldots, r$, and

$$\operatorname{rank}[\mathcal{O} \quad \mathcal{S}\mathcal{O}e^{At^*}x_i] = n, \quad i = 1, \ldots, r. \quad (39)$$

If (4) exhibits Poincaré recurrence in $\mathbb{R}^n$, then (4) and (5) is output reversible.

Proof. The proof follows immediately from Theorems 3.1 and 4.1. □

The converse to Theorem 4.2 is not true. In particular, if (4) and (5) is output reversible, then the system matrix $A \in \mathbb{R}^{n \times n}$ can have both stable and unstable eigenvalues which precludes Poincaré recurrence [9].

Example 4.1: For this example we consider two coupled oscillators with masses m_1 and m_2, and spring stiffness coefficients k_1 and k_2 so that the system inertia and stiffness matrices are given by

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \quad (40)$$

with the system state matrix $A \in \mathbb{R}^{4 \times 4}$ given by

$$A \triangleq \begin{bmatrix} 0 & I_2 \\ -M^{-1}K & 0 \end{bmatrix}, \quad (41)$$

where $\operatorname{mspec}(A) = \{\pm \jmath\omega_1, \pm \jmath\omega_2\}$, $\omega_1 > 0$, and $\omega_2 > 0$. Figures 1 and 2 show the position and velocity

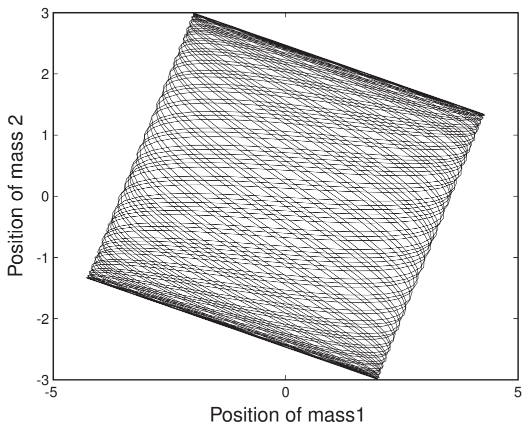

Fig. 1. Position phase portrait

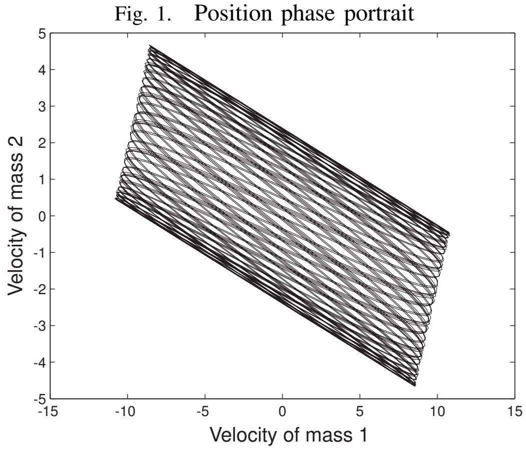

Fig. 2. Velocity phase portrait

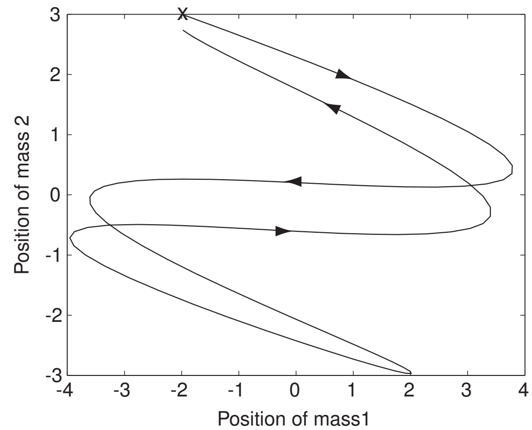

Fig. 3. Original position phase portrait

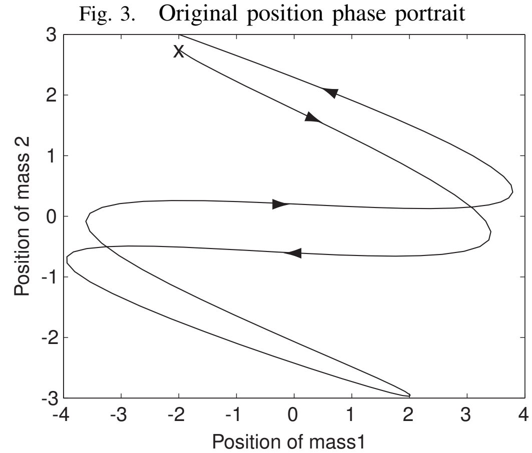

Fig. 4. Reversed position phase portrait

phase portraits, respectively, for the initial condition $x_0 = [-2, 3, 0, 0]^{\mathrm{T}}$ and system parameters $m_1 = 2$, $m_2 = 4$, $k_1 = 9$, and $k_2 = 8$. Note that in this case the ratio $\frac{\omega_1}{\omega_2}$ is an irrational number, and hence, the solutions to (4) with A given by (41) are not periodic. Nevertheless, it follows from Theorem 4.1 that (4) exhibits Poincaré recurrence and the state trajectories of (4) return to any neighborhood of their initial conditions infinitely often.

Next, we consider reversibility of mass positions, that is, output reversibility with

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{42}$$

Note that for the data chosen $\operatorname{rank} A = 4$. Let $t^* = 1$ and $x_i \in \mathbb{R}^4$, $i = 1, \ldots, 4$, be the ith column of I_4. It can be shown that $\operatorname{rank}[\mathcal{O} \quad \mathcal{S}\mathcal{O}e^{At^*}x_i] = 4$, $i = 1, \ldots, 4$. Thus, it follows from Theorem 4.2 that (4) and (5) with A and C given by (41) and (42), respectively, is output reversible. For the initial condition $x_0 = [-2, 3, 0, 0]^{\mathrm{T}}$ and $t_1 = 6$, it follows from Proposition 3.1 that $\hat{x}_0 = [-1.9796, 2.7365, 2.3757, -1.6953]^{\mathrm{T}}$ will generate the solution to (4) that will retrace the original time history of mass positions backwards with time going forward. Figure 3 shows the original phase portrait of mass positions with the initial condition $x_0 = [-2, 3, 0, 0]^{\mathrm{T}}$ up to $t = 6$ and Figure 4 shows the phase portrait of mass positions with the initial condition $\hat{x}_0 = [-1.9796, 2.7365, 2.3757, -1.6953]^{\mathrm{T}}$ up to $t = 6$.

References

[1] J. S. W. Lamb and J. A. G. Roberts, "Time reversal symmetry in dynamical systems: A survey," *Phys. D*, vol. 112, pp. 1–39, 1998.

[2] W. M. Haddad, V. Chellaboina, and S. G. Nersesov, *Thermodynamics. A Dynamical Systems Approach*. Princeton, NJ: Princeton University Press, 2005.

[3] W. M. Haddad, "Temporal asymmetry, entropic irreversibility, and finite-time thermodynamics: From Parmenides-Einstein time-reversal symmetry to the Heraclitan entropic arrow of time," *Entropy*, vol. 14, pp. 407–455, 2012.

[4] D. S. Bernstein and S. P. Bhat, "Linear output-reversible systems," in *Proc. Amer. Contr. Conf.*, Denver, CO, 2003, pp. 3240–3241.

[5] V. I. Arnold, *Mathematical Models of Classical Mechanics*. New York, NY: Springer-Verlag, 1989.

[6] F. Fagnani and J. C. Willems, "Representations of time-reversible systems," *Journal of Mathematical Systems, Estimation, and Control*, vol. 1, no. 1, pp. 5–28, 1991.

[7] ——, "Representations of symmetric linear dynamical systems," *SIAM J. Control and Optimization*, vol. 31, no. 5, pp. 1267–1293, 1993.

[8] D. S. Bernstein, *Matrix Mathematics*. Princeton, NJ: Princeton University Press, 2005.

[9] S. G. Nersesov and W. M. Haddad, "Reversibility and Poincaré recurrence in linear dynamical systems," *IEEE Trans. Autom. Contr.*, vol. 53, no. 9, pp. 2160–2165, 2008.