2016 IEEE 55th Conference on Decision and Control (CDC)
ARIA Resort & Casino
December 12-14, 2016, Las Vegas, USA

Combined State and Parameter Estimation and Identifiability of State Space Realizations

Ming Jui Yu and Dennis S. Bernstein

***Abstract*— The objective of combined state and parameter estimation is to estimate both unmeasured states and unknown entries of the dynamics matrix. Since the dynamics involve products of states and parameters, this is a nonlinear estimation problem. The classical approach to this problem is to use the extended Kalman filter, although more recent techniques, such as the unscented Kalman filter, can be used. The goal of this paper is to determine conditions under which the combined state and parameter estimation problem is feasible. To do this, we recast this problem as an identifiability problem and, for several special cases, we develop necessary and sufficient conditions for identifiability, which provides necessary and sufficient conditions for feasibility of the combined state and parameter estimation problem.**

I. Introduction

State estimation for linear time-invariant systems is one of the cornerstones of modern systems theory. The extension to nonlinear systems has immediate practical application, but is highly nontrivial and presents longstanding theoretical challenges. An important special case is state estimation for a linear time-invariant system with unknown entries in the dynamics matrix A. This combined state and parameter estimation (CSPE) problem is nonlinear due to products of the unknown parameters and unmeasured states.

The classical approach to CSPE is to apply the extended Kalman filter to an augmented model whose states include parameter estimates. This approach is considered for discrete-time systems in [1], where A is assumed to be in canonical form, and the goal is to estimate C and the entries in the first row of A.

The goal of the present paper is to explore the question of whether or not it is possible to estimate an arbitrary number of unknown entries in A. Using subspace identification methods [2, 3], it is indeed possible to estimate all of the entries of A along with the state. The resulting state space model is represented in an arbitrary basis, however, which obscures the meaning of the state components and does not distinguish between known and unknown entries in A. Consequently, the unknown entries of A are not uniquely specified and thus the parameter estimates are not physically meaningful. We say that the CSPE problem is *feasible* if the unmeasured states and unknown entries of A are uniquely determined by the measurements.

In the present paper we consider CSPE, where the output matrix is assumed to be known, and thus each state has a physical meaning. Furthermore, we assume that each entry of A is either known or unknown. To illustrate this problem, we consider a second-order system, where the first state is measured. Using the unscented Kalman filter (UKF), we show that it is possible to estimate both entries in the first row of A along with the unmeasured state. If, however, either both diagonal entries are unknown or both off-diagonal entries are unknown, then UKF fails to estimate these parameters and the unmeasured state.

In order to understand the source of the difficulty, we view the measurement as the impulse response of a state space model, and we recast the problem in terms of the identifiability of a state space realization. Identifiability of state space realizations is studied in [4–6], where sufficient conditions are given. It turns out, however, that these conditions are not satisfied for the cases described above, and thus no conclusion can be drawn.

The contribution of the present paper is the development of necessary and sufficient conditions for the identifiability of second- and third-order systems of special structure. These results are obtained by analyzing the equations that relate the parameters of the state space realization to the coefficients of the transfer function. In cases where uniqueness fails, the state space realization is not identifiable, and therefore the CSPE problem is not feasible.

II. Motivating Numerical Examples

Consider the asymptotically stable, linear time-invariant system

$$x(k+1) = Ax(k), \tag{1}$$

$$x(0) = x_0, \tag{2}$$

$$y(k) = Cx(k), \tag{3}$$

where

$$x(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix} \in \mathbb{R}^n, \tag{4}$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{5}$$

$y(k) \in \mathbb{R}$, and $C = [\ 1\ 0\ \cdots\ 0\ ] \in \mathbb{R}^{1\times n}$. We assume that C is known but some entries of A may be unknown. The objective is thus to use measurements of $y(k)$ to estimate the unknown entries of A and the unmeasured components $x_2(k), \ldots, x_n(k)$ of the state $x(k)$. Note that sensor noise is not included in (3) since the focus of this paper is on

Ming Jui Yu and Dennis S. Bernstein are with the Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI, USA. {mingray, dsbaero}@umich.edu

determining the feasibility of estimating these quantities rather than the statistical properties of the estimates.

To illustrate the problem, let $n = 2$, $C = [\ 1 \quad 0\ ]$, and

$$A = \begin{bmatrix} 0.4 & -0.6 \\ 0.98 & 0.99 \end{bmatrix}, \quad x_0 = \begin{bmatrix} -1 \\ 10 \end{bmatrix}. \tag{6}$$

We consider three special cases.

Example II.1 Consider the case where the entries a_{11} and a_{12} in the first row of A are unknown. To apply the unscented Kalman filter (UKF) [7] to combined state and parameter estimation, we define the augmented system

$$X(k+1) = \tilde{A}(k)X(k), \tag{7}$$
$$X(0) = X_0, \tag{8}$$
$$Y(k) = \tilde{C}X(k), \tag{9}$$

where

$$\tilde{A}(k) = \begin{bmatrix} \frac{1}{2}\hat{a}_{11}(k) & \frac{1}{2}\hat{a}_{12}(k) & \frac{1}{2}\hat{x}_1(k) & \frac{1}{2}\hat{x}_2(k) \\ 0.98 & 0.99 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{10}$$

$$\tilde{C} = [\ C \quad 0_{1\times 2}\ ], \quad X(k) = \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \hat{a}_{11}(k) \\ \hat{a}_{12}(k) \end{bmatrix}, \tag{11}$$

$\hat{x}_1(k)$, $\hat{x}_2(k)$ are estimates of $x_1(k)$, $x_2(k)$ and $\hat{a}_{11}(k)$, $\hat{a}_{12}(k)$ are estimates of a_{11}, a_{12}. At each step, UKF uses $\tilde{A}(k)$, $\tilde{C}$, and $X(k)$ to update $X(k)$, beginning with $\hat{x}_1(0) = x_1(0) = -1$, $\hat{x}_2(0) = 0$, $\hat{a}_{11}(0) = -0.1$, and $\hat{a}_{12}(0) = -0.1$. For UKF, we set the initial covariance matrix $P = 100I_4$ and tuning parameters $\alpha = 1$, $\kappa = 0$, $\beta = 2$ [7]. Figure 1

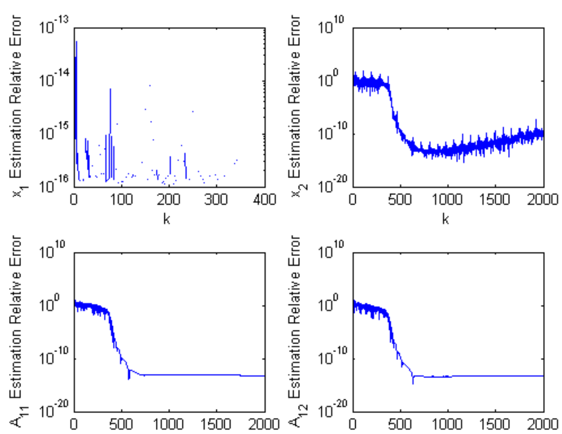

Fig. 1. State and parameter estimation of (1) – (3), where the first row of A is unknown and the second component of x is not measured. Since the first state is measured, the error is approximately equal to machine precision.

shows the resulting state and parameter estimation relative error. Note that UKF estimates a_{11} and a_{12} and significantly decreases the relative error in the unmeasured state. The relative error in the unmeasured state eventually increases as $y(k)$ approaches zero due to numerical resolution. ■

Example II.2 Consider the case where the diagonal entries a_{11} and a_{22} of A are unknown. We thus use (7) – (9) with

$$\tilde{A}(k) = \begin{bmatrix} \frac{1}{2}\hat{a}_{11}(k) & -0.6 & \frac{1}{2}\hat{x}_1(k) & 0 \\ 0.98 & \frac{1}{2}\hat{a}_{22}(k) & 0 & \frac{1}{2}\hat{x}_2(k) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{12}$$

$$\tilde{C} = [\ C \quad 0_{1\times 2}\ ], \quad X(k) = \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \hat{a}_{11}(k) \\ \hat{a}_{22}(k) \end{bmatrix}, \tag{13}$$

where $\hat{x}_1(k)$, $\hat{x}_2(k)$ are estimates of $x_1(k)$, $x_2(k)$ and $\hat{a}_{11}(k)$, $\hat{a}_{22}(k)$ are estimates of a_{11}, a_{22}. We apply UKF as in Example II.1 using the initial estimates $\hat{x}_1(0) = x_1(0) = -1$, $\hat{x}_2(0) = 0$, $\hat{a}_{11}(0) = -0.1$, and $\hat{a}_{22}(0) = -0.1$. Figure

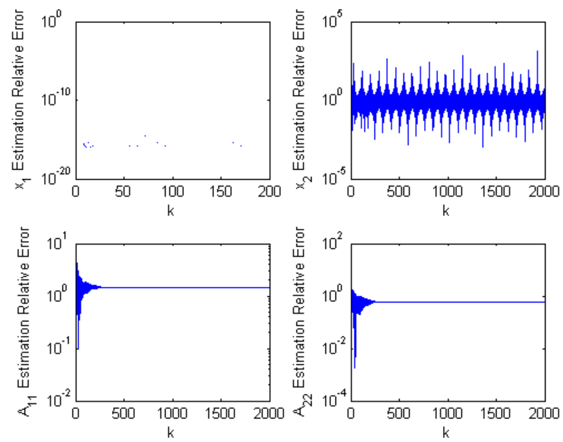

Fig. 2. State and parameter estimation of (1) – (3), where the diagonal entries of A are unknown and the second component of x is not measured.

2 shows the resulting state and parameter estimation relative error. Note that the parameter estimation errors for a_{11} and a_{22} remain on the same order of magnitude as the initial errors. ■

Example II.3 Consider the case where the off-diagonal entries a_{12} and a_{21} of A are unknown. We thus use (7) – (9) with

$$\tilde{A}(k) = \begin{bmatrix} 0.4 & \frac{1}{2}\hat{a}_{12}(k) & \frac{1}{2}\hat{x}_2(k) & 0 \\ \frac{1}{2}\hat{a}_{21}(k) & 0.99 & 0 & \frac{1}{2}\hat{x}_1(k) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{14}$$

$$\tilde{C} = [\ C \quad 0_{1\times 2}\ ], \quad X(k) = \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \hat{a}_{12}(k) \\ \hat{a}_{21}(k) \end{bmatrix}, \tag{15}$$

where $\hat{x}_1(k)$, $\hat{x}_2(k)$ are estimates of $x_1(k)$, $x_2(k)$ and $\hat{a}_{12}(k)$, $\hat{a}_{21}(k)$ are estimates of a_{12}, a_{21}. We apply UKF as in Example II.1 using the initial estimates $\hat{x}_1(0) = x_1(0) = -1$, $\hat{x}_2(0) = 0$, $\hat{a}_{12}(0) = -0.1$, and $\hat{a}_{21}(0) = -0.1$. Figure 3 shows the resulting state and parameter estimation relative error. As in Example II.2, the parameter estimation errors for a_{12} and a_{21} remain on the same order of magnitude as the initial errors. ■

Note that UKF successfully estimates the two unknown

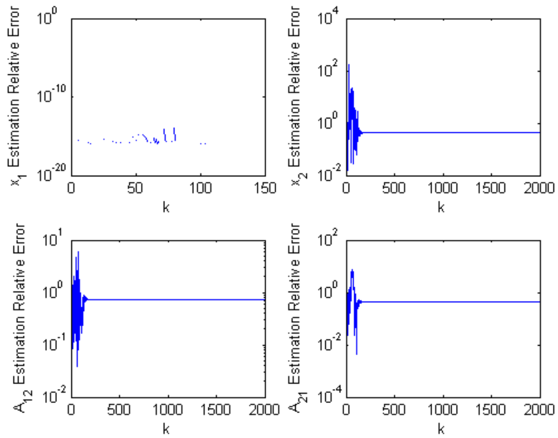

Fig. 3. State and parameter estimation of (1) – (3), where the off-diagonal entries of A are unknown and the second component of x is not measured.

parameters and the unmeasured state for Example II.1, but fails for Example II.2 and Example II.3. In the following sections, we explain this behavior in terms of the identifiability of a state space realization.

III. Identifiability

Let $\delta(k)$ be the unit impulse signal, let

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n, \tag{16}$$

and consider the single-input, single-output system

$$x(k+1) = Ax(k) + B\delta(k), \tag{17}$$
$$x(0) = 0, \tag{18}$$
$$y(k) = Cx(k), \tag{19}$$

where $C = [\ 1\ 0\ \cdots\ 0\ ] \in \mathbb{R}^{1\times n}$. We assume that $B = x_0$, where x_0 is the initial condition in (2). It thus follows that (1) – (3) and (17) – (19) are equivalent in the sense that (1) – (3) and (17) – (19) have the same output $y(k)$ for all $k \geq 0$. This output is the impulse response of the transfer function corresponding to (A, B, C). Furthermore, since $B = x_0$ and $y(0) = x_1(0)$, it follows that b_1 is known but the remaining components of B are unknown. Therefore, for (1) – (3), the feasibility of estimating the unmeasured states and unknown parameters in A is equivalent to determining whether or not the unknown entries in A and B are uniquely specified given the transfer function corresponding to (A, B, C). The equivalence of (1) – (3) and (17) – (19) thus allows us to determine the feasibility of the combined state and parameter estimation problem in terms of the identifiability of (A, B).

Now consider the transfer function

$$G(z) = \frac{\beta_{n-1}z^{n-1} + \cdots + \beta_1 z + \beta_0}{z^n + \cdots + \alpha_1 z + \alpha_0}, \tag{20}$$

whose numerator and denominator are coprime. Let $q \leq n^2 + n - 1$ be the number of unknown entries in (A, B). We define $\mathcal{S} \subset \mathbb{R}^q$ to be the set of q-tuples of unknown entries in A and B such that (A, B, C) is a minimal realization of (20). If $\mathcal{S}$ contains exactly one element, then (17) is *identifiable*. The goal of the present paper is to determine necessary and sufficient conditions under which (A, B) is identifiable.

IV. Identifiability Analysis for $n = 2$

We now state necessary and sufficient conditions under which (A, B) is not identifiable in the case $n = 2$. Note that b_1 is known and b_2 is unknown.

Theorem IV.1: Let $n = 2$,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{21}$$

and assume that (A, B, C) is a minimal realization of

$$G(z) = \frac{\beta_1 z + \beta_0}{z^2 + \alpha_1 z + \alpha_0}, \tag{22}$$

whose numerator and denominator are coprime. Then (A, B) is not identifiable if and only if at least one of the following statements holds:

1) $a_{12} = 0$.
2) $a_{12} \neq 0$, $a_{21} = 0$, and a_{12} is unknown.
3) $a_{12} \neq 0$, $\alpha_1^2 - 4(\alpha_0 + \bar{a}_{12}\bar{a}_{21}) \neq 0$, and a_{11} and a_{22} are unknown.
4) $a_{12} \neq 0$, $a_{21} \neq 0$, and a_{12} and a_{21} are unknown.
5) $q \geq 4$.

Furthermore, if 4) holds, then $\mathcal{S}$ has exactly two elements, and, if 1), 2), 3), or 5) holds, then $\mathcal{S}$ has infinitely many elements.

Proof: Set

$$G(z) = [\ 1\quad 0\ ]\,(zI_2 - A)^{-1}B. \tag{23}$$

It thus follows that

$$b_1 = \beta_1, \tag{24}$$
$$b_2 a_{12} - b_1 a_{22} = \beta_0, \tag{25}$$
$$a_{11} + a_{22} = -\alpha_1, \tag{26}$$
$$a_{11}a_{22} - a_{12}a_{21} = \alpha_0. \tag{27}$$

Since (24) constrains only known parameters, it can be disregarded. Henceforth the notation $\bar{a}$ indicates that a is a known parameter.

A. Sufficiency

1) Note that b_2 does not appear in (25) – (27), and thus is arbitrary. Hence, $\mathcal{S}$ has infinitely many elements.

2) Note that (25) implies

$$b_2 a_{12} = b_1 a_{22} + \beta_0, \tag{28}$$

and, since $a_{21} = 0$ in (27), there are no additional constraints on a_{12} and b_2. Hence, $\mathcal{S}$ has infinitely many elements.

3) Since a_{11}, a_{22}, and b_2 are unknown, (25) – (27) have the form

$$b_2\bar{a}_{12} - \bar{b}_1 a_{22} = \beta_0, \tag{29}$$
$$a_{11} + a_{22} = -\alpha_1, \tag{30}$$
$$a_{11}a_{22} - \bar{a}_{12}\bar{a}_{21} = \alpha_0. \tag{31}$$

Since $\bar{a}_{12} \neq 0$, (29)–(31) imply

$$b_2 = \frac{\beta_0 + \bar{b}_1 a_{22}}{\bar{a}_{12}}, \tag{32}$$

$$a_{22} = \frac{1}{2}\left(-\alpha_1 \pm \sqrt{\alpha_1^2 - 4(\alpha_0 + \bar{a}_{12}\bar{a}_{21})}\right). \tag{33}$$

Since $\alpha_1^2 - 4(\alpha_0 + \bar{a}_{12}\bar{a}_{21}) \neq 0$, $\mathcal{S}$ has exactly two elements. Note that both alternatives in (33) are real since A has real entries.

4) Note that (26) constrains only known parameters and thus can be disregarded. Hence, (25) and (27) have the form

$$b_2 a_{12} - \bar{b}_1 \bar{a}_{22} = \beta_0, \tag{34}$$

$$\bar{a}_{11}\bar{a}_{22} - a_{12}a_{21} = \alpha_0. \tag{35}$$

Since three unknown parameters satisfy two equations, $\mathcal{S}$ has infinitely many elements.

5) Since either four or five unknown parameters satisfy (25)–(27), $\mathcal{S}$ has infinitely many elements.

B. Necessity

To establish necessity, we show that $\mathcal{S}$ has exactly one element in the case where one of the following mutually exclusive statements holds:

i) $a_{12} \neq 0$, a_{12} is known, and $q = 2$.
ii) $a_{12} \neq 0$, $a_{21} \neq 0$, a_{12} is unknown, and $q = 2$.
iii) $a_{12} \neq 0$, a_{12} is known, $q = 3$, and either a_{11} or a_{22} is known.
iv) $a_{12} \neq 0$, $a_{21} \neq 0$, a_{12} is unknown, a_{21} is known, and $q = 3$.
v) $a_{12} \neq 0$, $\alpha_1^2 - 4(\alpha_0 + \bar{a}_{12}\bar{a}_{21}) = 0$, and a_{11} and a_{22} are unknown.

In Figure 4, C1 denotes the case where either one row or one column of A is unknown. C2 denotes the case where the diagonal entries of A are unknown and $\alpha_1^2 - 4(\alpha_0 + \bar{a}_{12}\bar{a}_{21}) = 0$. C3 denotes the case where the diagonal entries of A are unknown and $\alpha_1^2 - 4(\alpha_0 + \bar{a}_{12}\bar{a}_{21}) \neq 0$. C4 denotes the case where the off-diagonal entries of A are unknown. Note that 1)–5) together correspond to the red region in Figure 4, and $i)$–$v)$ together correspond to the green region in Figure 4. Therefore, 1)–5) and $i)$–$v)$ together cover all possible cases.

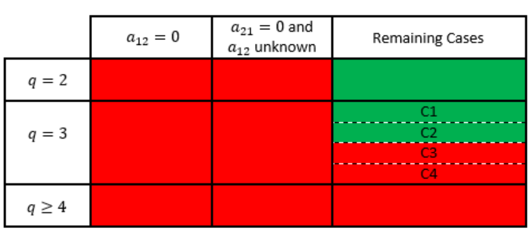

Fig. 4. Cases in the green region are identifiable, whereas cases in the red region are not identifiable.

i) Consider the case where a_{21} is unknown. (26) constrains only known parameters and thus can be disregarded. Hence, (25) and (27) have the form

$$b_2 \bar{a}_{12} - \bar{b}_1 \bar{a}_{22} = \beta_0, \tag{36}$$

$$\bar{a}_{11}\bar{a}_{22} - \bar{a}_{12}a_{21} = \alpha_0. \tag{37}$$

Since $\bar{a}_{21} \neq 0$, (36) and (37) imply

$$a_{21} = \frac{\bar{a}_{11}\bar{a}_{22} - \alpha_0}{\bar{a}_{12}}, \tag{38}$$

$$b_2 = \frac{\beta_0 + \bar{b}_1 \bar{a}_{22}}{\bar{a}_{12}}, \tag{39}$$

and thus $\mathcal{S}$ has exactly one element.

Consider the case where a_{11} is unknown. We write (25)–(27) as

$$M \begin{bmatrix} a_{11} \\ b_2 \end{bmatrix} = \begin{bmatrix} \beta_0 + \bar{b}_1 \bar{a}_{22} \\ -\alpha_1 - \bar{a}_{22} \\ \alpha_0 + \bar{a}_{12}\bar{a}_{21} \end{bmatrix}, \tag{40}$$

where

$$M \triangleq \begin{bmatrix} 0 & \bar{a}_{12} \\ 1 & 0 \\ \bar{a}_{22} & 0 \end{bmatrix}. \tag{41}$$

Since $\bar{a}_{12} \neq 0$, it follows that M has full column rank, and thus (40) has a unique solution. Hence, $\mathcal{S}$ has exactly one element. In the case where a_{22} is unknown, a similar argument shows that $\mathcal{S}$ has exactly one element.

ii) Consider the case where a_{12} is unknown. (26) constrains only known parameters and thus can be disregarded. Hence, (25) and (27) have the form

$$b_2 a_{12} - \bar{b}_1 \bar{a}_{22} = \beta_0, \tag{42}$$

$$\bar{a}_{11}\bar{a}_{22} - a_{12}\bar{a}_{21} = \alpha_0. \tag{43}$$

Since $\bar{a}_{21}, a_{12} \neq 0$, (42) and (43) imply

$$a_{12} = \frac{\bar{a}_{11}\bar{a}_{22} - \alpha_0}{\bar{a}_{21}}, \tag{44}$$

$$b_2 = \frac{\beta_0 + \bar{b}_1 \bar{a}_{22}}{a_{12}}, \tag{45}$$

and thus $\mathcal{S}$ has exactly one element.

iii) Consider the case where a_{11} and a_{21} are unknown. Then (25)–(27) have the form

$$b_2 \bar{a}_{12} - \bar{b}_1 \bar{a}_{22} = \beta_0, \tag{46}$$

$$a_{11} + \bar{a}_{22} = -\alpha_1, \tag{47}$$

$$a_{11}\bar{a}_{22} - \bar{a}_{12}a_{21} = \alpha_0. \tag{48}$$

Since $\bar{a}_{12} \neq 0$, (52)–(54) imply

$$a_{11} = -\alpha_1 - \bar{a}_{22}, \tag{49}$$

$$a_{21} = \frac{a_{11}\bar{a}_{22} - \alpha_0}{\bar{a}_{12}}, \tag{50}$$

$$b_2 = \frac{\beta_0 + \bar{b}_1 \bar{a}_{22}}{\bar{a}_{12}}, \tag{51}$$

and thus $\mathcal{S}$ has exactly one element. In the case where a_{21} and a_{22} are unknown, a similar argument shows that $\mathcal{S}$ has exactly one element.

iv) Consider the case where a_{11} and a_{12} are unknown. Thus (25)–(27) have the form

$$b_2 a_{12} - \bar{b}_1 \bar{a}_{22} = \beta_0, \tag{52}$$

$$a_{11} + \bar{a}_{22} = -\alpha_1, \tag{53}$$

$$a_{11}\bar{a}_{22} - a_{12}\bar{a}_{21} = \alpha_0. \tag{54}$$

Since $\bar{a}_{21}, a_{12} \neq 0$, (52)–(54) imply

$$a_{11} = -\alpha_1 - \bar{a}_{22}, \tag{55}$$

$$a_{12} = \frac{a_{11}\bar{a}_{22} - \alpha_0}{\bar{a}_{21}}, \tag{56}$$

$$b_2 = \frac{\beta_0 + \bar{b}_1 \bar{a}_{22}}{a_{12}}, \tag{57}$$

and thus $\mathcal{S}$ has exactly one element. In the case where a_{12} and a_{22} are unknown, a similar argument shows that $\mathcal{S}$ has exactly one element.

v) Since $\alpha_1^2 - 4(\alpha_0 + \bar{a}_{12}\bar{a}_{21}) = 0$ and $\bar{a}_{12} \neq 0$, (29)–(31) imply

$$a_{22} = a_{11} = \frac{-\alpha_0}{2}, \tag{58}$$

$$b_2 = \frac{\beta_0 + \bar{b}_1 a_{22}}{\bar{a}_{12}}. \tag{59}$$

Thus $\mathcal{S}$ has exactly one element.

□

Theorem IV.1 shows that (A, B) is not identifiable in the case where either the diagonal or off-diagonal entries of A are unknown, which is consistent with the results obtained in Section II using UKF.

V. Identifiability Analysis for $n = 3$

We now state necessary and sufficient conditions under which (A, B) is generically not identifiable in the case $n = 3$ and $C = [\ 1\ 0\ 0\ ]$. Note that b_1 is known and b_2 and b_3 are unknown.

Theorem V.1: Let $n = 3$,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \tag{60}$$

and assume that (A, B, C) is a minimal realization of

$$G(z) = \frac{\beta_2 z^2 + \beta_1 z + \beta_0}{z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0}, \tag{61}$$

whose numerator and denominator are coprime. Then (A, B) is not identifiable if and only if at least one of the following statements holds:

1) $\begin{bmatrix} a_{12} & a_{13} \\ a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{33} - a_{13}a_{22} \end{bmatrix}$ is singular.
2) $q = 3$, a_{ij} is unknown, $a_{ji} = 0$, and $a_{kk}a_{ji} - a_{jk}a_{ki} = 0$, where $i \neq j \neq k \leq n$.
3) $q = 4$, a_{ii} and a_{jj} are unknown, and $a_{jk}a_{kj} = a_{ik}a_{ki}$, where $i \neq j \neq k \leq n$.
4) Generically, $q = 4$, a_{ij} and a_{kl} are unknown, where $i \neq k$, $j \neq l$, $i \neq j$, and $k \neq l$.
5) $q = 4$, a_{ii} and a_{jk} are unknown, $a_{kj} = 0$, and $a_{kj}a_{ll} - a_{kl}a_{lj} = 0$, where either $i = j \neq k \neq l \leq n$ or $i = k \neq l \neq l \leq n$.
6) $q = 4$, a_{ii} and a_{jk} are unknown, $a_{kj} = 0$, and $a_{kj}(\alpha_2 - a_{jj} - a_{kk}) + a_{ij}a_{ki} = 0$, where $i \neq j \neq k \leq n$.
7) $q = 4$, a_{ij} and a_{lm} are unknown, and $\begin{bmatrix} -a_{ji} & -a_{ml} \\ a_{ji}a_{kk} - a_{jk}a_{ki} & a_{ml}a_{oo} - a_{mo}a_{ol} \end{bmatrix}$ is singular, where $i = l \neq j \neq m \leq n$ or $j = m \neq i \neq l \leq n$, $i \neq j \neq k \leq n$, and $l \neq m \neq o \leq n$.
8) $q = 5$, a_{ii}, a_{jj}, and a_{kk} are known, where $i \neq j \neq k \leq n$.
9) Generically, $q = 5$ and a_{ii} and a_{jj} are unknown, where $i \neq j \leq n$.
10) Generically, $q = 5$, a_{ii}, a_{jk}, and a_{lm} are unknown, where $i \leq n$, $j \neq l \leq n$, and $k \neq m \leq n$.
11) $q = 5$, a_{ii}, a_{jk}, and a_{mo} are unknown, and $\begin{bmatrix} -a_{kj} & -a_{om} \\ a_{kj}a_{ll} - a_{kl}a_{lj} & a_{om}a_{pp} - a_{op}a_{pm} \end{bmatrix}$ is singular, where $i \leq n$, $j = m \neq k \neq o \leq n$ or $k = o \neq j \neq m \leq n$, $j \neq k \neq l \leq n$, and $m \neq o \neq p \leq n$.
12) $q \geq 6$.

Proof: The proof is similar to the proof for Theorem IV.1 and thus is omitted.

For $q = 4$, Theorem V.1 shows that (A, B) is identifiable only in the cases where either at least one entry on the diagonal of A is unknown, or both unknown off-diagonal entries of A are in the same row or column. For $q = 5$, Theorem V.1 shows that (A, B) is identifiable only in the cases where exactly one entry on the diagonal of A is unknown, and the remaining unknown entries are in the same row or column.

VI. UKF Examples for $n = 3$

Example VI.1 Let $n = 3$, $C = [\ 1\ 0\ 0\ ]$, and

$$A = \begin{bmatrix} 0.51 & -0.29 & 0.4 \\ -1.12 & 0.34 & 1 \\ 0.03 & 0 & 0.34 \end{bmatrix}, \ x_0 = \begin{bmatrix} -1 \\ 10 \\ -20 \end{bmatrix}. \tag{62}$$

Consider the case where the entry a_{21} of A is unknown. It follows from 2) of Theorem V.1 that CSPE for this example is feasible. We thus use (7)–(9) with

$$\tilde{A}(k) = \begin{bmatrix} 0.51 & -0.29 & 0.4 & 0 \\ \frac{1}{2}\hat{a}_{21}(k) & 0.34 & 1 & \frac{1}{2}\hat{x}_1(k) \\ 0.03 & 0 & 0.34 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{63}$$

$$\tilde{C} = [\ C \quad 0\ ], \quad X(k) = \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \\ \hat{x}_3(k) \\ \hat{a}_{21}(k) \end{bmatrix}, \tag{64}$$

$\hat{x}_1(k), \hat{x}_2(k), \hat{x}_3(k)$ are estimates of $x_1(k)$, $x_2(k)$, and $x_3(k)$ and $\hat{a}_{21}(k)$ are the estimates of a_{21}. At each step, UKF uses $\tilde{A}(k)$, $\tilde{C}$, and $X(k)$ to update $X(k)$, beginning with $\hat{x}_1(0) = x_1(0) = -1$, $\hat{x}_2(0) = 0$, $\hat{x}_3(0) = 0$, and $\hat{a}_{21}(0) = 0.1$. For UKF, we set the initial covariance matrix $P = 10^8 I_4$ and tuning parameters $\alpha = 1$, $\kappa = 0$, $\beta = 2$ [7]. Figure 5 shows the resulting state and parameter estimation relative error. Note that UKF estimates a_{21} and significantly decreases the relative error in the unmeasured state. The relative error in

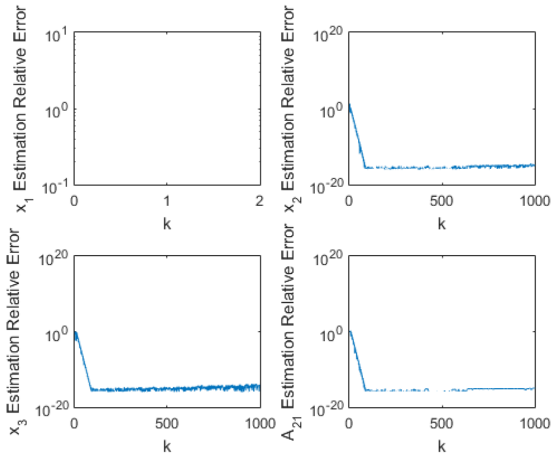

Fig. 5. State and parameter estimation of (1) – (3), where a_{21} is unknown and the second and third components of x is not measured. Since the first state is measured, the error is approximately equal to machine precision.

the unmeasured state eventually increases as $y(k)$ approaches zero due to numerical resolution. ■

Example VI.2 This example is the same as *Example VI.1* except with A given by

$$A = \begin{bmatrix} 0.51 & 0 & 10.4 \\ -1.12 & 0.34 & 1 \\ 0.03 & 0 & 0.34 \end{bmatrix}. \tag{65}$$

It follows from 2) of Theorem V.1 that CSPE for this example is not feasible. Figure 6 shows the resulting state and parameter estimation relative error. Note that the parameter estimation errors for a_{21} remain on the same order of magnitude as the initial errors. ■

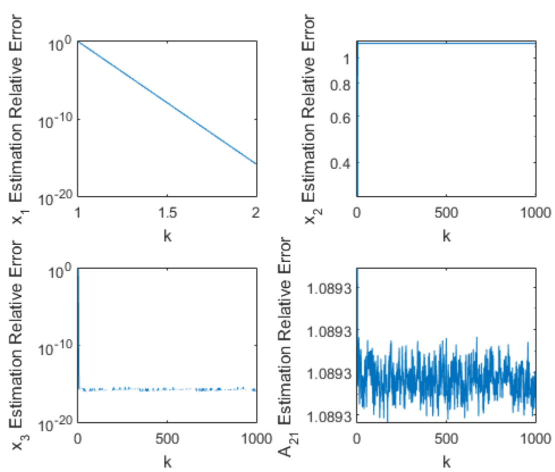

Fig. 6. State and parameter estimation of (1) – (3), where a_{21} is unknown and the second and third components of x is not measured. Since the first state is measured, the error is approximately equal to machine precision.

VII. Conclusion

The purpose of this paper is to establish the limitations of combined state and parameter estimation (CSPE) for linear systems with unknown entries in the dynamics matrix A. The analysis was confined to the cases $n = 2$ and $n = 3$, assuming that $C = [\,1 \quad 0\,]$ and $C = [\,1 \quad 0 \quad 0\,]$. This analysis is a first step toward developing necessary and sufficient conditions for CSPE assuming an affine but arbitrary uncertainty structure in A and C. Once these conditions are established, nonlinear estimation techniques, such as UKF, can be compared in terms of the speed and accuracy of their estimates.

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