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Guaranteed cost inequalities for robust stability and performance analysis

Dennis S. Bernstein[1,*,†] and Scot L. Osburn[2]

[1] *Department of Aerospace Engineering, The University of Michigan, Ann Arbor MI, 48109-2140, USA*
[2] *The Aerospace Corporation, P.O. Box 92957-M4/972, El Segundo CA, 90009, USA*

SUMMARY

In this paper, we formulate robust stability and performance bounds in terms of guaranteed cost inequalities. We derive new guaranteed cost bounds for plants with real structured uncertainty, and we reformulate them as linear matrix inequalities (LMIs). In particular, we obtain a shifted linear bound and a shifted inverse bound, and give LMI forms for a shifted bounded real bound, a shifted Popov bound, a shifted linear bound and a shifted inverse bound. Several examples are used to compare the shifted bounds with their unshifted counterparts and to make comparisons among these new bounds and vertex LMI bounds. Copyright © 2002 John Wiley & Sons, Ltd.

1. INTRODUCTION

For unstructured time-varying or complex uncertainty, the small gain theorem provides a non-conservative test for robust stability [1–6]. For structured and possibly real uncertainty, however, the small gain theorem is known to be conservative, and structured singular value bounds, which involve multipliers and complex scalings, can be used [7,8]. Linear matrix inequalities (LMIs) are also used to guarantee robust stability [9–15].

Within the context of robust H_2 performance, the small gain theorem is equivalent to the bounded real bound [1–3], which plays the role of a guaranteed cost bound [4]. Various guaranteed cost bounds have been developed including quadratic and non-quadratic bounds. Quadratic bounds include the bounded real [1–3], positive real [6,16], and Popov bounds [6,17] (see Table I), while non-quadratic bounds include the absolute value and linear bounds [21–23] (see Table II).

In the present paper we reformulate the bounded real, Popov, inverse, shifted bounded real [18], and shifted Popov [19] bounds as guaranteed cost inequalities. In addition, we present two new guaranteed cost bounds, namely, the shifted linear and shifted inverse bounds, which we also reformulate as linear matrix inequalities.

*Correspondence to: Dennis S. Bernstein, Department of Aerospace Engineering, University of Michigan, Ann Arbor MI, 48109-2140, U.S.A.
† E-mail: dsbaero@umich.edu

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Table I. Quadratic guaranteed cost bounds.

Bound	Reference
Bounded real	Anderson *et al.* [1], Noldus [2], Peterson and Hollot [3]
Positive real	Anderson [16], Haddad and Bernstein [6]
Popov	Haddad and Bernstein [6,17]
Shifted bounded real	Tyan and Bernstein [18]
Shifted positive real	Tyan and Bernstein [18]
Shifted Popov	Kapila *et al.* [19]
Implicit small gain	Haddad *et al.* [20]

Table II. Non-quadratic guaranteed cost bounds.

Bound	Reference
Absolute value	Chang and Peng [21]
Linear	Jain [22], Bernstein [23], Kosmidou and Bertrand [24]
Inverse	Bernstein and Haddad [4]
Double commutator	Tyan *et al.* [25]
Shifted linear	This paper
Shifted inverse	This paper

The guaranteed cost bounds that we consider are either parameter independent or parameter dependent. Parameter-independent bounds, such as the bounded real bound, use a common Lyapunov function, whereas parameter-dependent bounds, such as the Popov bound, use a family of Lyapunov functions. For polytopic uncertainty we show that the least conservative common (parameter-independent) guaranteed cost bound can be determined by solving an optimization problem involving a set of linear matrix inequalities. The interesting feature of the guaranteed cost bounds is the fact that they give rise to sets of LMIs whose dimensions are less than the dimensions of the vertex LMIs.

The contents of the paper are as follows. In Section 2 we consider the robust analysis problem in a guaranteed H_2 cost inequality framework. In Section 3 we consider the use of vertex LMI's to obtain guaranteed H_2 cost bounds. In Sections 4 and 5 we review and analyse the shifted bounded real and shifted Popov bounds, while in Sections 6 and 7 we present the shifted linear bound and shifted inverse bound. Finally, in Section 8 several examples are considered to compare the guaranteed cost and vertex LMI bounds.

Proofs can be found in Appendix A.

2. ROBUST PERFORMANCE AND GUARANTEED COST BOUNDS

Let $\mathcal{U} \subset \mathbf{R}^{n\times n}$ denote an uncertainty set and consider the uncertain $p \times m$ transfer function $G_{\Delta A}(s) = E(sI - A - \Delta A)^{-1}D$, where $A \in \mathbf{R}^{n\times n}$, $\Delta A \in \mathcal{U}$, $D \in \mathbf{R}^{n\times m}$, and $E \in \mathbf{R}^{p\times n}$. If $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$, then we define the worst-case H_2 performance by

$$J(\mathcal{U}) \triangleq \sup_{\Delta A \in \mathcal{U}} \|G_{\Delta A}\|_2. \qquad (1)$$

It follows from standard results that

$$J(\mathcal{U}) = \sup_{\Delta A \in \mathcal{U}} \operatorname{tr} P_{\Delta A} V, \tag{2}$$

where $V \triangleq DD^{\mathrm{T}}$ and $P_{\Delta A}$ is the unique, non-negative-definite solution to the Lyapunov equation

$$(A + \Delta A)^{\mathrm{T}} P_{\Delta A} + P_{\Delta A}(A + \Delta A) + R = 0, \tag{3}$$

where $R \triangleq E^{\mathrm{T}} E$.

The following definition will be used to construct bounds for $J(\mathcal{U})$.

Definition 1

Let $\mathcal{N} \subseteq \mathbf{S}^n$, $\Omega : \mathcal{N} \to \mathbf{S}^n$ and $P_0 : \mathcal{U} \to \mathbf{S}^n$. Then (Ω, P_0) is a *bounding pair* if

$$0 \leqslant P + P_0(\Delta A), \quad P \in \mathcal{N}, \;\; \Delta A \in \mathcal{U} \tag{4}$$

and

$$\Delta A^{\mathrm{T}} P + P\Delta A + (A + \Delta A)^{\mathrm{T}} P_0(\Delta A) + P_0(\Delta A)(A + \Delta A) \leqslant \Omega(P), \quad P \in \mathcal{N}, \;\; \Delta A \in \mathcal{U}. \tag{5}$$

The following result, which is slightly stronger form of Theorem 3.1 of Reference [17] provides a bound for $J(\mathcal{U})$.

Theorem 1

Let (Ω, P_0) be a bounding pair and assume there exists $P \in \mathcal{N}$ satisfying

$$A^{\mathrm{T}} P + PA + \Omega(P) + R \leqslant 0 \tag{6}$$

Then $(A + \Delta A, E)$ is detectable for all $\Delta A \in \mathcal{U}$ if and only if $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. In this case,

$$P_{\Delta A} \leqslant P + P_0(\Delta A), \quad \Delta A \in \mathcal{U}, \tag{7}$$

where $P_{\Delta A} \in \mathbf{N}^n$ is given by (3), and

$$J(\mathcal{U}) \leqslant \mathcal{J}(\mathcal{U}) \tag{8}$$

where

$$\mathcal{J}(\mathcal{U}) \triangleq \operatorname{tr} PV + \sup_{\Delta A \in \mathcal{U}} \operatorname{tr} P_0(\Delta A) V. \tag{9}$$

Remark 1

If there exists $\bar{P}_0 \in \mathbf{S}^n$ such that

$$P_0(\Delta A) \leqslant \bar{P}_0, \quad \Delta A \in \mathcal{U}, \tag{10}$$

then

$$\mathcal{J}(\mathcal{U}) \leqslant \bar{\mathcal{J}} \tag{11}$$

where

$$\bar{\mathcal{J}} \triangleq \operatorname{tr}[(P + \bar{P}_0)V]. \tag{12}$$

Remark 2

In Theorem 3.1 of Reference [17], inequality (6) appears as an equation. Inequality (6) is desirable since it permits the use of LMI techniques.

A bounding pair (Ω, P_0) is *parameter dependent* if P_0 is not constant. Alternatively, a bounding pair (Ω, P_0) is *parameter independent* if P_0 is constant. In this case, $P_0(\Delta A)$ is replaced by P_0 and $\bar{P}_0 = P_0$ so that $\bar{\mathscr{J}} = \mathscr{J}(\mathscr{U}) = \text{tr}[(P + P_0)V]$.

The remainder of the paper is concerned with the construction of bounding pairs (Ω, P_0). To construct a bounding pair (Ω, P_0) we must specify the set $\mathscr{N} \subseteq \mathbf{S}^n$ along with the functions $\Omega : \mathscr{N} \to \mathbf{S}^n$ and $P_0 : \mathscr{U} \to \mathbf{S}^n$ that satisfy (4) and (5). No other assumptions on Ω and P_0 are required. To apply Theorem 1, however, requires the existence of a solution $P \in \mathscr{N}$ to inequality (6). LMI techniques will be used to obtain such solutions.

For a given bounding pair (Ω, P_0), the following immediate result yields an equivalent bounding pair $(\hat{\Omega}, \hat{P}_0)$.

Proposition 1
Let $\Omega : \mathscr{N} \subseteq \mathbf{S}^n \to \mathbf{S}^n$, $P \in \mathscr{N}$, and $P_0 : \mathscr{U} \to \mathbf{S}^n$ satisfy (4)–(6), and let $\bar{P}_0 \in \mathbf{S}^n$ satisfy (10). Furthermore, let $\bar{\hat{P}}_0 \in \mathbf{S}^n$, and define $\hat{\mathscr{N}} \subseteq \mathbf{S}^n$, $\hat{\Omega} : \hat{\mathscr{N}} \to \mathbf{S}^n$ and $\hat{P}_0 : \mathscr{U} \to \mathbf{S}^n$ by

$$\hat{\mathscr{N}} \triangleq \mathscr{N} + \bar{P}_0 - \bar{\hat{P}}_0, \tag{13}$$

$$\hat{\Omega}(\hat{P}) \triangleq \Omega(\hat{P} - \bar{P}_0 + \bar{\hat{P}}_0) - A^{\mathrm{T}}(\bar{P}_0 - \bar{\hat{P}}_0) - (\bar{P}_0 - \bar{\hat{P}}_0)A, \quad \hat{P} \in \hat{\mathscr{N}}, \tag{14}$$

and

$$\hat{P}_0(\Delta A) \triangleq P_0(\Delta A) - \bar{P}_0 + \bar{\hat{P}}_0. \tag{15}$$

Then (4)–(6) and (10) are satisfied with $\mathscr{N}$, Ω, P, P_0, and $\bar{P}_0$ replaced by $\hat{\mathscr{N}}$, $\hat{\Omega}$, $P + \bar{P}_0 - \bar{\hat{P}}_0$, $\hat{P}_0$, and $\bar{\hat{P}}_0$. Furthermore, the bounding pairs (Ω, P_0) and $(\hat{\Omega}, \hat{P}_0)$ yield the same performance bound $\mathscr{J}(\mathscr{U})$.

Remark 3
If there exists $\Delta A \in \mathscr{U}$ such that $P_0(\Delta A) = 0$, then (4) implies $P \geqslant 0$ for all $P \in \mathscr{N}$, and thus without loss of generality we can assume $\mathscr{N} \subseteq \mathbf{N}^n$.

Remark 4
Let (Ω, P_0) be a parameter-independent bounding pair with $\bar{P}_0 = P_0$. Letting $\bar{\hat{P}}_0 = 0$ in Proposition 1 yields the equivalent parameter-independent bounding pair $(\hat{\Omega}, 0)$. Thus, without loss of generality, we can consider parameter-independent bounding pairs of the form $(\Omega, 0)$, where, by Remark 3, $\mathscr{N} \subseteq \mathbf{N}^n$.

In the following sections, $\mathscr{U}$ is given by either the *parametric uncertainty set*

$$\mathscr{U}_{\mathrm{p}}(\mathscr{R}) = \left\{ \Delta A\colon\ \Delta A = \sum_{i=1}^{r} \delta_i A_i,\ \text{where } (\delta_1, \ldots, \delta_r) \in \mathscr{R} \right\}, \tag{16}$$

where $\mathscr{R} \subseteq \mathbf{R}^r$ and $A_i \in \mathbf{R}^{n \times n}$, $i = 1, \ldots, r$, or the *factored uncertainty set*

$$\mathscr{U}_{\mathrm{f}}(\mathscr{F}) = \{\Delta A :\ \Delta A = B_0 F C_0,\ \text{where } F \in \mathscr{F}\}, \tag{17}$$

where $\mathscr{F} \subseteq \mathbf{R}^{l_1 \times l_2}$, $B_0 \in \mathbf{R}^{n \times l_1}$ and $C_0 \in \mathbf{R}^{l_2 \times n}$.

Note that the parametric uncertainty set $\mathcal{U}_{\mathrm{p}}(\mathcal{R})$ requires specification of the set $\mathcal{R}$, while the factored uncertainty set $\mathcal{U}_{\mathrm{f}}(\mathcal{F})$ requires specification of the set $\mathcal{F}$. These sets will be specified in later sections for each bounding pair that we consider.

Next we show that $\mathcal{U}_{\mathrm{p}}(\mathcal{R})$ is a special case of $\mathcal{U}_{\mathrm{f}}(\mathcal{F})$ for a special choice of $\mathcal{F}$. To show this, let $B_i \in \mathbf{R}^{n\times k_i}$ and $C_i \in \mathbf{R}^{k_i\times n}$ satisfy

$$A_i = B_i C_i, \quad i = 1, \ldots, r, \tag{18}$$

and define

$$B_0 \triangleq [B_1 \cdots B_r] \in \mathbf{R}^{n\times k}, \qquad C_0 \triangleq \begin{bmatrix} C_1 \\ \vdots \\ C_r \end{bmatrix} \in \mathbf{R}^{k\times n}, \tag{19}$$

where $k \triangleq \sum_{i=1}^{r} k_i$. Then

$$\Delta A = \sum_{i=1}^{r} \delta_i A_i = B_0 F C_0, \tag{20}$$

where $F = \mathrm{diag}(\delta_1 I_{k_1}, \ldots, \delta_r I_{k_r}) \in \mathbf{R}^{k\times k}$ so that $l_1 = l_2 = k$. Hence, with (18) and (19), it follows that

$$\mathcal{U}_{\mathrm{f}}(\mathcal{F}_{\mathcal{R}}) = \mathcal{U}_{\mathrm{p}}(\mathcal{R}), \tag{21}$$

where $\mathcal{U}_{\mathrm{f}}(\mathcal{F}_{\mathcal{R}})$ is the *factored parametric uncertainty set*, where

$$\mathcal{F}_{\mathcal{R}} \triangleq \{F \in \mathbf{S}^k : F = \mathrm{diag}(\delta_1 I_{k_1}, \ldots, \delta_r I_{k_r}),\ (\delta_1, \ldots, \delta_r) \in \mathcal{R}\}. \tag{22}$$

3. VERTEX LMIs FOR ROBUST PERFORMANCE

In this section linear matrix inequalities are used to construct parameter-independent bounding pairs. For $\gamma > 0$ define the *polytopic uncertainty set*

$$\mathcal{U}_{\mathrm{p}}(\mathcal{R}_\gamma) \triangleq \left\{ \Delta A:\ \Delta A = \sum_{i=1}^{r} \delta_i A_i,\ \text{where } |\delta_i| \leqslant \gamma,\ i = 1, \ldots, r \right\}, \tag{23}$$

where

$$\mathcal{R}_\gamma \triangleq \{(\delta_1, \ldots, \delta_r) : |\delta_i| \leqslant \gamma,\ i = 1, \ldots, r\}. \tag{24}$$

With (18) and (19), the *factored polytopic uncertainty set* is given by

$$\mathcal{U}_{\mathrm{f}}(\mathcal{F}_{\mathcal{R}_\gamma}) = \mathcal{U}_{\mathrm{p}}(\mathcal{R}_\gamma), \tag{25}$$

where, with $\mathcal{R} = \mathcal{R}_\gamma$ in (22),

$$\mathcal{F}_{\mathcal{R}_\gamma} = \{F \in \mathbf{S}^k :\ F = \mathrm{diag}(\delta_1 I_{k_1}, \ldots, \delta_r I_{k_r}),\ |\delta_i| \leqslant \gamma,\ i = 1, \ldots, r\}. \tag{26}$$

Lemma 1

$P \in \mathbf{N}^n$ satisfies the 2^r LMIs

$$A^{\mathrm{T}} P + PA \pm \gamma(A_1^{\mathrm{T}} P + PA_1) \pm \cdots \pm \gamma(A_r^{\mathrm{T}} P + PA_r) + R \leqslant 0 \tag{27}$$

Table III. LMI dimensions for continuous-time polytopic uncertainty bounds. For the linear and inverse families of bounds, α must be chosen separately.

Bound		Variables	Variable size	LMI dimension
Vertex LMI	Prop. 2	P	n^2	$2^r n^2$
Shifted bounded real	Prop. 8	$P,\ N,\ Y_i$	$n^2 + 2\sum k_i^2$	$(n+k)^2 + 2\sum k_i^2$
Bounded real		P	n^2	$(n+k)^2$
Shifted Popov	Prop. 11	$P,\ \bar{P}_0, \tilde{X},\ Y,\ \tilde{N},\ \tilde{H}$	$3n^2 + 3k^2$	$(n+k)^2 + (2^{r+1}+1)n^2$
	Cor. 1	$P,\ P_i,\ \tilde{X},\ Y_i,\ \tilde{N},\ \tilde{H}$	$(2r+1)n^2 + 3k^2$	$(n+k)^2 + (4r+5)n^2$
Popov		$P,\ P_i,\ \tilde{N},\ \tilde{H}$	$(r+1)n^2 + 2k^2$	$(n+k)^2 + (2r+3)n^2$
Shifted linear	Prop. 15	$P,\ N_i,\ Y$	$(r+2)n^2$	$(2^r+2)n^2$
	Cor. 2	$P,\ N,\ Y_i$	$(r+2)n^2$	$(2r+2)n^2$
Linear		P	n^2	$2n^2$
Shifted inverse	Prop. 18	$P,\ N_i,\ M_i,\ Y$	$(2r+2)n^2$	$(r+1)^2 n^2 + (2^r+1)n^2$
	Remark 12	$P,\ N_i,\ M_i,\ Y_i$	$(3r+1)n^2$	$(r+1)^2 n^2 + (2r+1)n^2$
		$P,\ N_i$	$(r+1)n^2$	$(r+1)^2 n^2 + n^2$
Inverse		P	n^2	$(r+1)^2 n^2 + n^2$

if and only if P satisfies

$$(A+\Delta A)^{\mathrm{T}} P + P(A+\Delta A) + R \leqslant 0, \quad \Delta A \in \mathcal{U}_{\mathrm{p}}(\mathcal{R}_\gamma). \tag{28}$$

The following result shows that the set of solutions to the 2^r vertex LMIs (27) gives rise to a parameter-independent bounding pair $(\Omega_{\mathrm{LMI}}, 0)$. Define

$$\mathcal{P} \triangleq \{P \in \mathbf{N}^n : P \text{ satisfies } (27)\}.$$

Proposition 2

Let $\mathcal{U} = \mathcal{U}_{\mathrm{p}}(\mathcal{R}_\gamma)$ and define $\Omega_{\mathrm{LMI}} : \mathcal{P} \to \mathbf{N}^n$ by

$$\Omega_{\mathrm{LMI}}(P) \triangleq -R - A^{\mathrm{T}}P - PA. \tag{29}$$

Then $(\Omega_{\mathrm{LMI}}, 0)$ is a bounding pair. Furthermore, $J(\mathcal{U}_{\mathrm{p}}(\mathcal{R}_\gamma)) \leqslant \operatorname{tr} PV$ for all $P \in \mathcal{P}$.

The next result shows that every bound $\mathcal{J}(\mathcal{U})$ obtainable from a parameter-independent bounding pair $(\Omega, 0)$ is also obtainable from vertex LMIs.

Proposition 3

Let $\mathcal{U} = \mathcal{U}_{\mathrm{p}}(\mathcal{R}_\gamma)$, let $(\Omega, 0)$ be a bounding pair, where $\Omega : \mathcal{N} \subseteq \mathbf{N}^n \to \mathbf{S}^n$, and assume there exists $P \in \mathcal{N}$ satisfying (6). Then P satisfies (27).

Propositions 2 and 3 show that there is an equivalence between the performance bounds obtainable from vertex LMIs and the performance bounds obtainable from parameter-independent bounding pairs $(\Omega, 0)$. This equivalence does not suggest, however, that parameter-independent bounding pairs $(\Omega, 0)$ are of no interest. Rather, as can be seen in Table III, the bounding pairs $(\Omega, 0)$ may entail LMIs that are of lower dimensionality than vertex LMIs. With this motivation in mind, we turn our attention to the construction of parameter-independent and parameter-dependent bounding pairs.

4. SHIFTED BOUNDED REAL BOUND

Define $\mathscr{F}_{\mathrm{BR}}(M) \subset \mathbf{R}^{l_1 \times l_2}$ by

$$\mathscr{F}_{\mathrm{BR}}(M) \triangleq \{F \in \mathbf{R}^{l_1 \times l_2}\colon F^{\mathrm{T}}F \leqslant M\}, \tag{30}$$

where $M \in \mathbf{N}^{l_2}$. The following result concerns the classical *bounded real bound* [3,5].

Proposition 4
Let $\mathscr{U} = \mathscr{U}_{\mathrm{f}}(\mathscr{F}_{\mathrm{BR}}(M))$, $\mathscr{N} = \mathbf{N}^n$, and

$$\Omega(P) = PB_0B_0^{\mathrm{T}}P + C_0^{\mathrm{T}}MC_0. \tag{31}$$

Then $(\Omega, 0)$ is a bounding pair.

Next, define

$$\mathscr{F}_{\mathrm{BRs}}(M, N) \triangleq \{F \in \mathbf{R}^{l_1 \times l_2} : (F + N)^{\mathrm{T}}(F + N) \leqslant M\}, \tag{32}$$

where $M \in \mathbf{N}^{l_2}$ and $N \in \mathbf{R}^{l_1 \times l_2}$. Note that $\mathscr{F}_{\mathrm{BRs}}(M, 0) = \mathscr{F}_{\mathrm{BR}}(M)$. The following result concerns the *shifted bounded real bound* [18].

Proposition 5
Let $\mathscr{U} = \mathscr{U}_{\mathrm{f}}(\mathscr{F}_{\mathrm{BRs}}(M, N))$, $\mathscr{N} = \mathbf{N}^n$, and

$$\Omega(P) = PB_0B_0^{\mathrm{T}}P - (B_0NC_0)^{\mathrm{T}}P - PB_0NC_0 + C_0^{\mathrm{T}}MC_0. \tag{33}$$

Then $(\Omega, 0)$ is a bounding pair.

The *shifted bounded real bound inequality* is given by (6) with Ω given by (33), which has the form

$$(A - B_0NC_0)^{\mathrm{T}}P + P(A - B_0NC_0) + PB_0B_0^{\mathrm{T}}P + C_0^{\mathrm{T}}MC_0 + R \leqslant 0. \tag{34}$$

Remark 5
Note that

$$\begin{aligned} A + \mathscr{U}_{\mathrm{f}}(\mathscr{F}_{\mathrm{BRs}}(M, N)) &= \{A + B_0FC_0 : F \in \mathscr{F}_{\mathrm{BRs}}(M, N)\} \\ &= \{A + B_0FC_0 : (F + N)^{\mathrm{T}}(F + N) \leqslant M\} \\ &= \{A + B_0(\hat{F} - N)C_0 : \hat{F}^{\mathrm{T}}\hat{F} \leqslant M\} \\ &= \{A_{\mathrm{s}} + B_0\hat{F}C_0 : \hat{F}^{\mathrm{T}}\hat{F} \leqslant M\} \\ &= A_{\mathrm{s}} + \mathscr{U}_{\mathrm{f}}(\mathscr{F}_{\mathrm{BR}}(M)), \end{aligned}$$

where $A_{\mathrm{s}} \triangleq A - B_0NC_0$. This identity suggests that the shifted bounded real bound is not more general than the bounded real bound. However, this is definitely not the case. Rather, the shifted bounded real bound has the form of the bounded real bound for a *shifted* nominal dynamics matrix A_s that is *different* from the original nominal dynamics matrix A. The numerical results in Section 8 show that, for the examples considered, the shifted bounded real bound is markedly less conservative than the standard bounded real bound.

Remark 6
Other factorizations can be used in place of (32). In particular, [18] uses a factorization of the form

$$\hat{\mathscr{F}}_{\mathrm{BRs}}(M_s, N) \triangleq \{F \in \mathbf{R}^{l_1 \times l_2} : (FC_0 + N)^{\mathrm{T}}(FC_0 + N) \leqslant M_s\},$$

where N and M_s are chosen to have appropriate dimension. Example 3 in Section 8 uses a factorization of this form for the shifted bounded-real bound.

Next we apply the shifted bounded real bound to the factored polytopic uncertainty set $\mathscr{U}_{\mathrm{f}}(\mathscr{F}_{\mathscr{R}_\gamma}) = \mathscr{U}_{\mathrm{p}}(\mathscr{R}_\gamma)$ with B_0, C_0 given by (18) and (19), so that $F = \mathrm{diag}(\delta_1 I_{k_1}, \ldots, \delta_r I_{k_r}) \in \mathscr{F}_{\mathscr{R}_\gamma}$. Note that if $\gamma^2 I \leqslant M$ then $\mathscr{F}_{\mathscr{R}_\gamma} \subseteq \mathscr{F}_{\mathrm{BR}}(M)$. Now let $N = \mathrm{diag}(N_1, \ldots, N_r) \in \mathbf{R}^{k \times k}$, where $N_i \in \mathbf{R}^{k_i \times k_i}$, $i = 1, \ldots, r$. Then

$$\mathscr{F}_{\mathrm{BRs}\mathscr{R}}(M, N) \triangleq \mathscr{F}_{\mathrm{BRs}}(M, N) \cap \mathscr{F}_{\mathscr{R}}$$
$$= \{F = \mathrm{diag}(\delta_1 I_{k_1}, \ldots, \delta_r I_{k_r}) : (F + N)^{\mathrm{T}}(F + N) \leqslant M, (\delta_1, \ldots, \delta_r) \in \mathscr{R}\}.$$

Proposition 6
Let M be given by

$$M = N^{\mathrm{T}}N + \gamma^2 I + Y, \tag{35}$$

where $Y \triangleq \mathrm{diag}(Y_1, \ldots, Y_r)$, and $Y_i \in \mathbf{N}^{k_i}$, $i = 1, \ldots, r$, satisfies

$$\delta_i(N_i + N_i^{\mathrm{T}}) \leqslant Y_i, \qquad |\delta_i| \leqslant \gamma, \quad i = 1, \ldots, r. \tag{36}$$

Then

$$\mathscr{F}_{\mathscr{R}_\gamma} \subseteq \mathscr{F}_{\mathrm{BRs}}(M, N). \tag{37}$$

With M given by (35), (34) becomes

$$(A - B_0NC_0)^{\mathrm{T}}P + P(A - B_0NC_0) + PB_0B_0^{\mathrm{T}}P + C_0^{\mathrm{T}}(N^{\mathrm{T}}N + \gamma^2 I + Y)C_0 + R \leqslant 0. \tag{38}$$

The next proposition gives two choices of Y_i that satisfy (36).

Proposition 7
Let $Y_i \in \mathbf{N}^{k_i}$, $i = 1, \ldots, r$, and consider the conditions

$$Y_i = \gamma |N_i + N_i^{\mathrm{T}}|, \quad i = 1, \ldots, r, \tag{39}$$

and

$$-Y_i \leqslant \gamma(N_i + N_i^{\mathrm{T}}) \leqslant Y_i, \quad i = 1, \ldots, r. \tag{40}$$

Then (39) $\Rightarrow$ (40) $\Leftrightarrow$ (36).

Next, we formulate an LMI to obtain a feasible solution $P \in \mathbf{N}^n$ to the shifted bounded real inequality (34) along with M and N. The following result follows from the equivalence of (36) and (40) as well as by using Schur complements to rewrite (38). Let $N = \mathrm{diag}(N_1, \ldots, N_r)$ and $Y = \mathrm{diag}(Y_1, \ldots, Y_r)$.

Proposition 8
Let $\mathscr{U} = \mathscr{U}_{\mathrm{f}}(\mathscr{F}_{\mathscr{R}_\gamma})$ and let $P \in \mathbf{N}^n$, $N \in \mathbf{R}^{k\times k}$, and $Y \in \mathbf{S}^n$. Then P, N, Y satisfy (40) and

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + C_0^{\mathrm{T}}(\gamma^2 I + Y)C_0 + R & PB_0 - C_0^{\mathrm{T}}N^{\mathrm{T}} \\ B_0^{\mathrm{T}}P - NC_0 & -I \end{bmatrix} \leqslant 0 \tag{41}$$

if and only if P, N, Y satisfy (36) and (38).

Remark 7
The LMI (41) is a special case of (24) in Reference [15] with

$$\hat{Q} = C_0^{\mathrm{T}}(\gamma^2 I + Y)C_0 + R, \quad \hat{S} = C_0^{\mathrm{T}}N^{\mathrm{T}}, \quad \hat{R} = -I, \quad B = B_0, \quad C = I.$$

5. SHIFTED POPOV BOUND

Let $l_1 = l_2 = k$ and define $\mathscr{F}_{\mathrm{P}} \subset \mathbf{S}^k$, $\mathscr{H}_{\mathrm{P}} \subset \mathbf{P}^k$ and $\mathscr{N}_{\mathrm{P}} \subset \mathbf{R}^{k\times k}$ by

$$\mathscr{F}_{\mathrm{P}} \subseteq \{F \in \mathbf{S}^k : M_{\mathrm{L}} \leqslant F \leqslant M_{\mathrm{U}}\}, \tag{42}$$

$$\mathscr{H}_{\mathrm{P}} \triangleq \{H \in \mathbf{P}^k : HF = FH,\ F \in \mathscr{F}_{\mathrm{P}}\}, \tag{43}$$

$$\mathscr{N}_{\mathrm{P}} \triangleq \{N \in \mathbf{R}^{k\times k} : N^{\mathrm{T}}M_{\mathrm{L}} = M_{\mathrm{L}}N,\ N^{\mathrm{T}}F = FN,\ F \in \mathscr{F}_{\mathrm{P}}\}, \tag{44}$$

where $M_{\mathrm{L}}, M_{\mathrm{U}} \in \mathbf{S}^k$ are such that

$$M \triangleq M_{\mathrm{U}} - M_{\mathrm{L}} > 0. \tag{45}$$

The following result concerns the *Popov bound* [17].

Proposition 9
Let $\mathscr{U} = \mathscr{U}_{\mathrm{f}}(\mathscr{F}_{\mathrm{P}})$, $\mathscr{N} = \mathbf{N}^n$, $N \in \mathscr{N}_{\mathrm{P}}$ and $H \in \mathscr{H}_{\mathrm{P}}$. Assume that

$$R_0 \triangleq (HM^{-1} - NC_0B_0) + (HM^{-1} - NC_0B_0)^{\mathrm{T}} > 0 \tag{46}$$

and let

$$\begin{aligned} \Omega(P) = {} & (HC_0 + NC_0\tilde{A}_{\mathrm{P}} + B_0^{\mathrm{T}}P)^{\mathrm{T}} R_0^{-1} (HC_0 + NC_0\tilde{A}_{\mathrm{P}} + B_0^{\mathrm{T}}P) \\ & + (B_0 M_{\mathrm{L}} C_0)^{\mathrm{T}} P + PB_0 M_{\mathrm{L}} C_0, \end{aligned} \tag{47}$$

where $\tilde{A}_{\mathrm{P}} \triangleq A + B_0 M_{\mathrm{L}} C_0$, and

$$P_0(\Delta A) = C_0^{\mathrm{T}}(F - M_{\mathrm{L}})NC_0. \tag{48}$$

Then (Ω, P_0) is a bounding pair.

Remark 8
If $\mu \in \mathbf{S}^k$ satisfies $\mu \geqslant (F - M_{\mathrm{L}})N$ for all $F \in \mathscr{F}_{\mathrm{P}}$, then $\bar{P}_0 = C_0^{\mathrm{T}} \mu C_0$ satisfies (10).

The following result concerns the *shifted Popov bound* [19].

Proposition 10
Let $\mathscr{U} = \mathscr{U}_{\mathrm{f}}(\mathscr{F}_{\mathrm{P}})$, let $X \in \mathbf{R}^{k \times k}$ and $Y \in \mathbf{S}^n$ satisfy

$$B_0 X^{\mathrm{T}}(F - M_{\mathrm{L}})C_0 + C_0^{\mathrm{T}}(F - M_{\mathrm{L}})XB_0^{\mathrm{T}} \leqslant Y, \quad F \in \mathscr{F}_{\mathrm{P}} \tag{49}$$

let $\mathscr{N} = \mathbf{N}^n$, $N \in \mathscr{N}_{\mathrm{P}}$, and $H \in \mathscr{H}_{\mathrm{P}}$, let R_0 be given by (46), define

$$\begin{aligned} \Omega(P) = {} & (HC_0 + NC_0\tilde{A}_{\mathrm{P}} + B_0^{\mathrm{T}}P - XB_0^{\mathrm{T}})^{\mathrm{T}} R_0^{-1} (HC_0 + NC_0\tilde{A}_{\mathrm{P}} + B_0^{\mathrm{T}}P - XB_0^{\mathrm{T}}) \\ & + (B_0 M_{\mathrm{L}} C_0)^{\mathrm{T}} P + PB_0 M_{\mathrm{L}} C_0 + Y, \end{aligned} \tag{50}$$

and let $P_0(\Delta A)$ be given by (48). Then (Ω, P_0) is a bounding pair.

The *shifted Popov bound inequality* is given by (6) with Ω given by (50), which has the form

$$\begin{aligned} & (A + B_0 M_{\mathrm{L}} C_0)^{\mathrm{T}} P + P(A + B_0 M_{\mathrm{L}} C_0) + (HC_0 + NC_0\tilde{A}_{\mathrm{P}} + B_0^{\mathrm{T}}P - XB_0^{\mathrm{T}})^{\mathrm{T}} \\ & \times R_0^{-1}(HC_0 + NC_0\tilde{A}_{\mathrm{P}} + B_0^{\mathrm{T}}P - XB_0^{\mathrm{T}}) + Y + R \leqslant 0. \end{aligned} \tag{51}$$

Remark 9
Setting $X = 0$ and $Y = 0$ in Proposition 10 yields Proposition 9.

Next, define $\mathscr{H}_{\mathrm{Pd}} \subset \mathbf{P}^k$, $\mathscr{N}_{\mathrm{Pd}} \subset \mathbf{S}^k$ and $\hat{I}_1, \ldots, \hat{I}_r \in \mathbf{S}^k$ by

$$\mathscr{H}_{\mathrm{Pd}} \triangleq \{H \in \mathbf{P}^k : H = \operatorname{diag}(H_1, \ldots, H_r),\ H_i \in \mathbf{P}^{k_i},\ i = 1, \ldots, r\}, \tag{52}$$

$$\mathscr{N}_{\mathrm{Pd}} \triangleq \{N \in \mathbf{S}^k : N = \operatorname{diag}(N_1, \ldots, N_r),\ N_i \in \mathbf{S}^{k_i},\ i = 1, \ldots, r\}, \tag{53}$$

and

$$\hat{I}_i \triangleq \operatorname{diag}(0_{k_1}, \ldots, 0_{k_{i-1}}, I_{k_i}, 0_{k_{i+1}}, \ldots, 0_{k_r}),$$

where $k = \sum_{i=1}^{r} k_i$. Let $-M_{\mathrm{L}} = M_{\mathrm{U}} = \gamma I$ and let $\mathscr{F}_P \subseteq \mathscr{F}_{\mathscr{R}}$. Then $\mathscr{F}_{\mathrm{P}} = \mathscr{F}_{\mathscr{R}_\gamma}$. The following result provides an LMI satisfying (49) and (6) with $\mathscr{U} = \mathscr{U}_{\mathrm{f}}(\mathscr{F}_{\mathscr{R}_\gamma}) = \mathscr{U}_{\mathrm{p}}(\mathscr{R}_\gamma)$ and with Ω given by (50).

Proposition 11
Let $\mathscr{U} = \mathscr{U}_{\mathrm{p}}(\mathscr{R}_\gamma)$, and let $\tilde{\mathscr{N}}$ denote the set of $(P, X, Y, N, H) \in \mathbf{N}^n \times \mathbf{R}^{k \times k} \times \mathbf{S}^n \times \mathscr{N}_{P\gamma} \times \mathscr{H}_{\mathrm{Pd}}$ satisfying

$$\begin{bmatrix} A_{\mathrm{P}}^{\mathrm{T}}P + PA_{\mathrm{P}} + Y + R & C_0^{\mathrm{T}}H + \tilde{A}_{\mathrm{P}}^{\mathrm{T}}C_0^{\mathrm{T}}N + PB_0 - B_0X^{\mathrm{T}} \\ HC_0 + NC_0\tilde{A}_{\mathrm{P}} + B_0^{\mathrm{T}}P - XB_0^{\mathrm{T}} & NC_0B_0 + B_0^{\mathrm{T}}C_0^{\mathrm{T}}N - \gamma^{-1}H \end{bmatrix} < 0 \tag{54}$$

and the 2^{r+1} LMIs

$$\pm\,\gamma(B_0X^{\mathrm{T}}C_0 + C_0^{\mathrm{T}}XB_0^{\mathrm{T}}) \pm \gamma(B_0X^{\mathrm{T}}\hat{I}_1C_0 + C_0^{\mathrm{T}}\hat{I}_1XB_0^{\mathrm{T}})$$
$$\pm \cdots \pm \gamma(B_0X^{\mathrm{T}}\hat{I}_rC_0 + C_0^{\mathrm{T}}\hat{I}_rXB_0^{\mathrm{T}}) \leqslant Y. \tag{55}$$

Then (6), with Ω given by (50), and (49) are satisfied for all $(P, X, Y, N, H) \in \tilde{\mathcal{N}}$. Furthermore, if $\bar{P}_0 \in \mathbf{S}^n$ satisfies the 2^{r+1} LMIs

$$\gamma C_0^{\mathrm{T}}(\pm I \pm \hat{I}_1 \pm \hat{I}_2 \pm \cdots \pm \hat{I}_r)NC_0 \leqslant \bar{P}_0, \tag{56}$$

then (10) is satisfied.

Corollary 1
Let $X \in \mathbf{R}^{k\times k}$ and let $Y_1, \ldots, Y_{r+1} \in \mathbf{S}^n$ satisfy

$$-Y_i \leqslant \gamma B_0X^{\mathrm{T}}\hat{I}_iC_0 + \gamma C_0^{\mathrm{T}}X\hat{I}_iB_0^{\mathrm{T}} \leqslant Y_i, \quad i = 1, \ldots, r, \tag{57}$$

$$-Y_{r+1} \leqslant \gamma B_0X^{\mathrm{T}}C_0 + \gamma C_0^{\mathrm{T}}XB_0^{\mathrm{T}} \leqslant Y_{r+1}, \tag{58}$$

let $N \in \mathcal{N}_{\mathrm{P}\gamma}$ and let $P_1, \ldots, P_{r+1} \in \mathbf{S}^n$ satisfy

$$-P_i \leqslant \gamma C_0^{\mathrm{T}}\hat{I}_iNC_0 \leqslant P_i, \quad i = 1, \ldots, r, \tag{59}$$

$$-P_{r+1} \leqslant \gamma C_0^{\mathrm{T}}NC_0 \leqslant P_{r+1}. \tag{60}$$

Then $Y = \sum_{i=1}^{r+1} Y_i$ satisfies (49). Finally, let $P_0(\Delta A)$ be given by (48). Then $\bar{P}_0 = \sum_{i=1}^{r+1} P_i$ satisfies (10).

6. SHIFTED LINEAR BOUND

In this section we consider the *linear bound* [22–24].

Proposition 12
Let $\mathcal{U} = \mathcal{U}_{\mathrm{p}}(\mathcal{R}_\gamma)$, $\alpha > 0$ and $\mathcal{N} = \mathbf{N}^n$, and define

$$\Omega(P) = \alpha rP + \frac{\gamma^2}{\alpha}\sum_{i=1}^{r} A_i^{\mathrm{T}}PA_i. \tag{61}$$

Then $(\Omega, 0)$ is a bounding pair.

Next, the *shifted linear bound* is obtained.

Proposition 13
Let $\mathcal{U} = \mathcal{U}_{\mathrm{p}}(\mathcal{R}_\gamma)$ and $\alpha > 0$, let $N_1, \ldots, N_r$, $Y \in \mathbf{S}^n$ satisfy

$$\sum_{i=1}^{r} \delta_i(A_i^{\mathrm{T}}N_i + N_i^{\mathrm{T}}A_i) \leqslant Y, \quad |\delta_i| \leqslant \gamma, \;\; i = 1, \ldots r, \tag{62}$$

and define $\mathcal{N} = \mathbf{N}^n \cap \left(\bigcap_{i=1}^{r}[\mathbf{N}^n + N_i]\right)$ and

$$\Omega(P) = \sum_{i=1}^{r}\left[\alpha(P - N_i) + \frac{\gamma^2}{\alpha}A_i^{\mathrm{T}}(P - N_i)A_i\right] + Y. \tag{63}$$

Then $(\Omega, 0)$ is a bounding pair.

Remark 10
Setting $N_1 = \cdots = N_r = 0$ and $Y = 0$ in Proposition 13 yields Proposition 12.

The *shifted linear bound inequality* is given by (6) with Ω given by (63), which has the form

$$A^{\mathrm{T}}P + PA + \sum_{i=1}^{r} \left[\alpha(P - N_i) + \frac{\gamma^2}{\alpha} A_i^{\mathrm{T}}(P - N_i)A_i\right] + Y + R \leqslant 0. \tag{64}$$

The next result provides a method for computing P satisfying

$$A^{\mathrm{T}}P + PA + \sum_{i=1}^{r} \left[\alpha(P - N_i) + \frac{\gamma^2}{\alpha} A_i^{\mathrm{T}}(P - N_i)A_i\right] + Y + R = 0. \tag{65}$$

Proposition 14
Let $\alpha > 0$, $N_1, \ldots, N_r \in \mathbf{S}^n$, and $Y = \gamma \sum_{i=1}^{r} |A_i^{\mathrm{T}} N_i + N_i^{\mathrm{T}} A_i|$. Suppose

$$\mathscr{A} \triangleq \left(A + \frac{\alpha\tau}{2} I\right) \oplus \left(A + \frac{\alpha r}{2} I\right) + \frac{\gamma^2}{\alpha} \sum_{i=1}^{r} (A_i \otimes A_i) \tag{66}$$

is invertible. Then (65) has the unique solution

$$P = -\mathrm{vec}^{-1}(\mathscr{A}^{-\mathrm{T}}\, \mathrm{vec}\, R_0), \tag{67}$$

where

$$R_0 \triangleq \sum_{i=1}^{r} \left[\gamma |A_i^{\mathrm{T}} N_i + N_i A_i| - \alpha N_i - \frac{\gamma^2}{\alpha} A_i^{\mathrm{T}} N_i A_i\right] + R.$$

If, in addition, $\mathscr{A}$ is asymptotically stable and R_0 is non-negative definite, then $P \geqslant 0$.

Remark 11
The last statement of Proposition 14 follows from techniques used in Reference [26].

Proposition 15
Let $\mathscr{U} = \mathscr{U}_{\mathrm{p}}(\mathscr{R}_\gamma)$, let $\alpha > 0$, and let $\tilde{\mathscr{N}}$ denote the set of $(P, N_1, \ldots, N_r, Y) \in \mathbf{N}^n \times (\mathbf{S}^n)^{r+1}$ satisfying

$$A^{\mathrm{T}}P + PA + \sum_{i=1}^{r} \left[\alpha(P - N_i) + \frac{\gamma^2}{\alpha} A_i^{\mathrm{T}}(P - N_i)A_i\right] + Y + R \leqslant 0 \tag{68}$$

and the 2^r LMIs

$$\pm \gamma(A_i^{\mathrm{T}} N_1 + N_1 A_1) \pm \cdots \pm \gamma(A_r^{\mathrm{T}} N_r + N_r A_r) \leqslant Y. \tag{69}$$

Then (62) and (63) are satisfied for all $(P, N_1, \ldots, N_r, Y) \in \tilde{\mathscr{N}}$.

Letting $N_1 = \cdots = N_r = N$ in Proposition 13 yields the following specialization of the shifted linear bound.

Corollary 2
Let $\alpha > 0$ and let $N, Y_1, \ldots, Y_r \in \mathbf{S}^n$ satisfy

$$\delta_i(A_i^{\mathrm{T}} N + NA_i) \leqslant Y_i, \quad |\delta_i| \leqslant \gamma, \quad i = 1, \ldots, r, \tag{70}$$

and define $\mathcal{N} = \mathbf{N}^n \cap (\mathbf{N}^n + N)$ and

$$\Omega(P) = \alpha r(P - N) + \sum_{i=1}^{r} \frac{\gamma^2}{\alpha} A_i^{\mathrm{T}}(P - N)A_i + \sum_{i=1}^{r} Y_i. \tag{71}$$

Then $(\Omega, 0)$ is a bounding pair.

7. SHIFTED INVERSE BOUND

The following result concerns the *inverse bound* [4].

Proposition 16
Let $\mathcal{U} = \mathcal{U}_{\mathrm{p}}(\mathcal{R}_\gamma)$, $\alpha > 0$ and $\mathcal{N} = \mathbf{P}^n$, and define

$$\Omega(P) = \alpha\gamma rP + \frac{\gamma}{4\alpha} \sum_{i=1}^{r} (A_i^{\mathrm{T}}P + PA_i)P^{-1}(A_i^{\mathrm{T}}P + PA_i). \tag{72}$$

Then $(\Omega, 0)$ is a bounding pair.

The *inverse bound inequality*, which is given by (6) with Ω given by (72), has the form

$$A^{\mathrm{T}}P + PA + \alpha\gamma rP + \frac{\gamma}{4\alpha} \sum_{i=1}^{r} (A_i^{\mathrm{T}}P + PA_i)P^{-1}(A_i^{\mathrm{T}}P + PA_i) + R \leqslant 0. \tag{73}$$

Equation (73) can be written as

$$\tilde{A}_{\mathrm{inv}}^{\mathrm{T}}P + P\tilde{A}_{\mathrm{inv}} + \frac{\gamma}{4\alpha} \sum_{i=1}^{r} (A_i^{\mathrm{T}}PA_i + PA_iP^{-1}A_i^{\mathrm{T}}P) + R \leqslant 0, \tag{74}$$

where $\tilde{A}_{\mathrm{inv}} \triangleq A + \alpha r\frac{\gamma}{2}I + \frac{\gamma}{4\alpha}\sum_{i=1}^{r} A_i^2$.

Next, the *shifted inverse bound* is obtained.

Proposition 17
Let $\mathcal{U} = \mathcal{U}_{\mathrm{p}}(\mathcal{R}_\gamma)$ and $\alpha > 0$, and let $M_{1i}, M_{2i} \in \mathbf{R}^{n\times n}$, $i = 1, \ldots, r$, and $Y \in \mathbf{S}^n$ satisfy

$$\sum_{i=1}^{r} \frac{\delta_i}{2}[A_i^{\mathrm{T}}(M_{1i} + M_{2i}^{\mathrm{T}}) + (M_{2i} + M_{1i}^{\mathrm{T}})A_i] \leqslant Y, \quad |\delta_i| \leqslant \gamma, \ \ i = 1, \ldots, r. \tag{75}$$

Let $N_1, \ldots, N_r \in \mathbf{S}^n$, and define

$$\mathcal{N} = \mathbf{N}^n \cap \left(\bigcap_{i=1}^{r} [\mathbf{P}^n + N_i] \right) \tag{76}$$

and

$$\Omega(P) = \sum_{i=1}^{r} \left(\frac{\gamma}{4\alpha}[A_i^{\mathrm{T}}(P - M_{1i}) + (P - M_{2i})A_i](P - N_i)^{-1} \right.$$
$$\left. \times [A_i^{\mathrm{T}}(P - M_{1i}) + (P - M_{2i})A_i]^{\mathrm{T}} + \alpha\gamma(P - N_i) \right) + Y. \tag{77}$$

Then $(\Omega, 0)$ is a bounding pair.

Substituting $\Omega(P)$ into (6) yields the *shifted inverse bound inequality*

$$A^{\mathrm{T}}P + PA + \sum_{i=1}^{r}\left(\frac{\gamma}{4\alpha}[A_i^{\mathrm{T}}(P-M_{1i}) + (P-M_{2i})A_i](P-N_i)^{-1}\right.$$
$$\left.\times[A_i^{\mathrm{T}}(P-M_{1i}) + (P-M_{2i})A_i]^{\mathrm{T}} - \alpha\gamma N_i\right) + \alpha\gamma rP + Y + R \leqslant 0. \quad (78)$$

Let $M_{i1} = M_{2i} = M_i \in \mathbf{S}^n$, $i = 1, \ldots, r$. The next result uses LMIs to find $P \in \mathbf{P}^n$ and $M_1, \ldots, M_r, N_1, \ldots, N_r$ and $Y \in \mathbf{S}^n$ satisfying (75) and (78).

Proposition 18
Let $\mathscr{U} = \mathscr{U}_{\mathrm{p}}(\mathscr{R}_\gamma)$ and let $\alpha > 0$. Let $\tilde{\mathscr{N}}$ denote the set of $(P, N_1, \ldots, N_r, M_1, \ldots, M_r, Y) \in \mathbf{N}^n \times (\mathbf{S}^n)^{2r+1}$ satisfying

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + \alpha\gamma\sum_{i=1}^{r}(P-N_i) + Y + R & A_1^{\mathrm{T}}(P-M_1) + (P-M_1)A_1 & \cdots & A_r^{\mathrm{T}}(P-M_r) + (P-M_r)A_r \\ A_1^{\mathrm{T}}(P-M_1) + (P-M_1)A_1 & -\frac{4\alpha}{\gamma}(P-N_1) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ A_r^{\mathrm{T}}(P-M_r) + (P-M_r)A_r & 0 & 0 & -\frac{4\alpha}{\gamma}(P-N_r) \end{bmatrix} < 0 \quad (79)$$

and the 2^r LMIs

$$\pm\gamma(A_1^{\mathrm{T}}M_1 + M_1A_1) \pm \cdots \pm \gamma(A_r^{\mathrm{T}}M_r + M_rA_r) \leqslant Y. \quad (80)$$

Then (75) and (78) are satisfied for all $(P, N_1, \ldots, N_r, M_1, \ldots, M_r, Y) \in \tilde{\mathscr{N}}$.

Remark 12
As in Corollary 1, (80) can be recast as $2rn^2$ constraints.

Corollary 3
Assume $A + A^{\mathrm{T}} < 0$ and $A_i + A_i^{\mathrm{T}} = 0$, let $\beta > 0$ satisfy $\beta(A + A^{\mathrm{T}}) + R < 0$, let $\alpha > 0$, and define $N_i = (\alpha\gamma r)^{-1}[\beta(A + A^{\mathrm{T}}) + R] + \beta I$ and $M_{1i} = M_{2i} = 0$, $i = 1, \ldots, r$. Then $P = \beta I$ satisfies (78).

8. EXAMPLES

In this section we use LMI methods to calculate solutions along with optimal scalings for the linear, bounded real, inverse, and Popov bounds, as well as their shifted counterparts. In the case of the inverse and linear bounds, the α scalings must be chosen separately. In Example 1 through Example 3, vertex LMIs were used to obtain the best parameter-independent bound from Proposition 2 (marked LMI), along with the Popov and shifted Popov bounds.

Example 1
Let

$$A = \begin{bmatrix} -0.0002 & 0.2208 & 0 & 0 \\ -0.2208 & -0.0002 & 0 & 0 \\ 0 & 0 & -0.0103 & 1.4322 \\ 0 & 0 & -1.4322 & -0.0103 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0_{2\times2} & I_2 \\ -I_2 & 0_{2\times2} \end{bmatrix},$$

where the uncertainty represents modal coupling. Furthermore, let $R = I_4$ and $V = I_4$, and let B_1 and C_1 be given by

$$B_1 = \begin{bmatrix} I_2 & 0_{2\times2} \\ 0_{2\times2} & -I_2 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 0_{2\times2} & I_2 \\ I_2 & 0_{2\times2} \end{bmatrix}.$$

Each plot in Figure 1 shows the exact worst-case performance along with the LMI bound given by Proposition 2. As can be seen in Figure 1(a), the bounded real bound given by Proposition 4 guarantees stability for $|\delta| < 0.0003$. Applying Proposition 8, the shifted bounded real bound is shown in Figure 1(a) and guarantees stability for all $\delta \in \mathbf{R}$. Next, the Popov bound

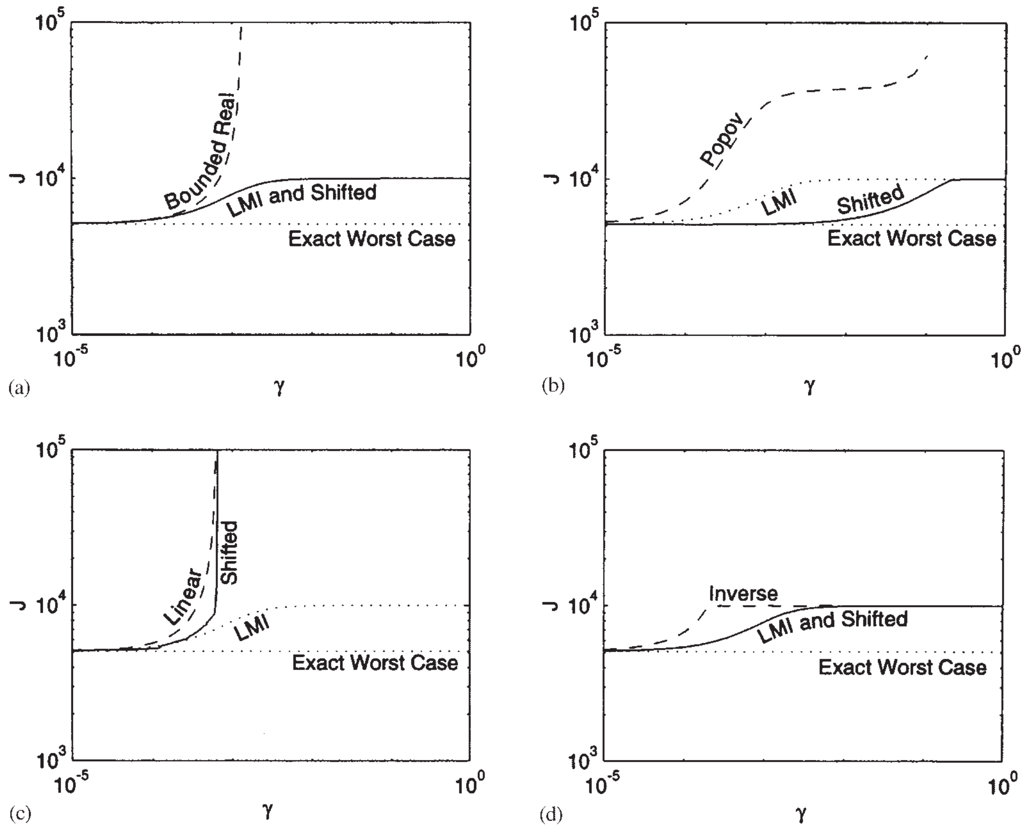

Figure 1. Comparison of shifted, unshifted, and vertex LMI bounds for Example 1

in Figure 1(b) guarantees stability for $|\delta| < 0.1$, while the shifted Popov bound using Proposition 11 is less conservative than the parameter-independent LMI bound given by Proposition 2. The linear bound in Figure 1(c) guarantees stability for $|\delta| < 6 \times 10^{-4}$, while the shifted linear bound is less conservative for $|\delta| < 6 \times 10^{-4}$. Finally, the inverse bound in Figure 1(d) guarantees stability for all $\delta \in \mathbf{R}$, while the shifted inverse bound with $M_1 = 0$ coincides with the vertex LMI bound given by Proposition 2. It can be seen that the shifted bounds provide significant improvement over their classical counterparts.

Example 2
Let

$$A = \begin{bmatrix} -0.005 & 1 \\ -1 & -0.005 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.001 & 10 \\ -10 & 0.001 \end{bmatrix},$$
$$R = \begin{bmatrix} 0.25 & 0.12 \\ 0.12 & 2.5 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 \\ 0 & 12 \end{bmatrix}.$$

Although the uncertainty is nearly skew symmetric, it is destabilizing. Figure 2 shows the exact worst case performance. Now let B_1 and C_1 is given by

$$B_1 = 0.001 I_2, \qquad C_1 = \begin{bmatrix} 1 & 10000 \\ -10000 & 1 \end{bmatrix}.$$

With this choice, the bounded real and linear bounds guarantee stability for $|\delta| < 4 \times 10^{-4}$, while the shifted linear bound yields an improved robust performance bound for $|\delta| < 4 \times 10^{-4}$. The shifted Popov bound, using Proposition 11, guarantees stability for $|\delta| < 5$ and is less conservative than the vertex LMI bound. The shifted inverse bound guarantees stability for $|\delta| < 5$ and coincides with the vertex LMI bound.

Example 3
Here we consider several variations of Example 1 with two uncertain parameters. First, consider the non-destabilizing skew-symmetric uncertainties

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{81}$$

and let B_0 and C_0 be given by (19), where

$$B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{82}$$

Figure 3(a) shows the performance bound given by the shifted bounded real bound and the shifted Popov bound, which coincide with the vertex LMI bound given by Proposition 2.

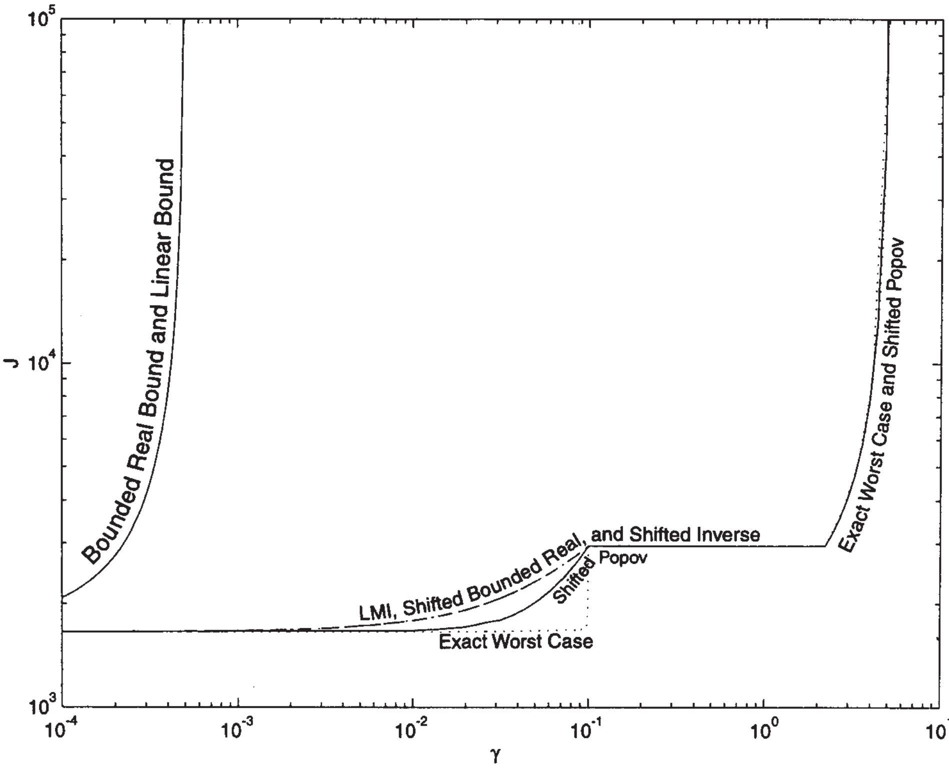

Figure 2. Performance bounds for Example 2 for a destabilizing uncertainty.

Next, consider the symmetric and skew-symmetric uncertainties

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{83}$$

and let B_0 and C_0 be given by (19), where

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \\ 0 & 0.1 \end{bmatrix},$$

and C_1, B_2 and C_2 are given in (82). In this case, the first uncertain parameter is destabilizing. Figure 3(b) shows the performance bound given by the shifted bounded real bound, which coincides with the vertex LMI bound, and the shifted Popov bound, which does slightly better for higher levels of uncertainty.

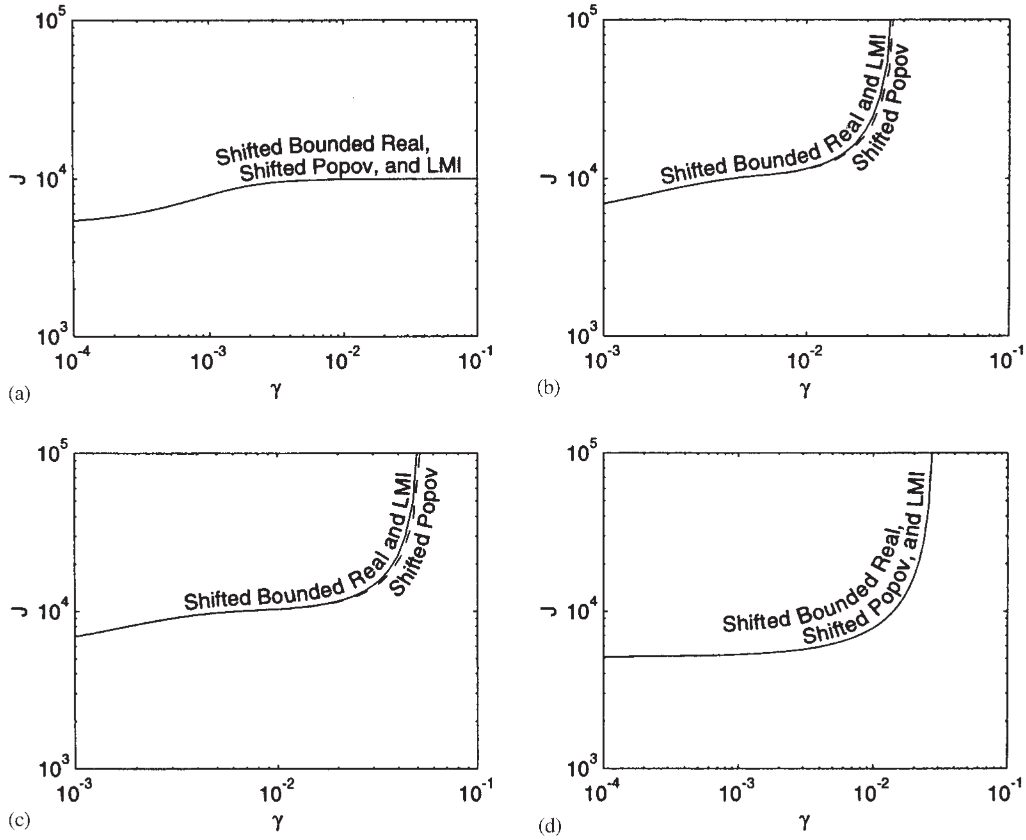

Figure 3. Performance bounds for Example 3 comparing the shifted bounded real, shifted Popov and vertex LMI bounds with two uncertain parameters: (a) two skew-symmetric uncertainties, (b) symmetric and skew-symmetric uncertainty, (c) nilpotent and skew-symmetric uncertainty, (d) two nilpotent uncertainties.

Next, consider the nilpotent and skew-symmetric uncertainties

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{84}$$

and let B_0 and C_0 be given by (19), where

$$B_1 = \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0 \end{bmatrix}, \qquad C_1 = [0 \quad 0 \quad 0 \quad 1],$$

and B_2 and C_2 are given in (82). In this case, the first uncertain parameter is destabilizing. Figure 3(c) shows the performance bound given by the shifted bounded real bound, which coincides with the vertex LMI bound, and the shifted Popov bound, which does slightly better for higher levels of uncertainty.

Finally, consider the nilpotent uncertainties

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{85}$$

and let B_0 and C_0 be given by (19), where

$$B_1 = \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0 \end{bmatrix}, \quad C_1 = [0 \quad 0 \quad 0 \quad 1], \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0.01 \\ 0 \end{bmatrix}, \quad C_2 = [1 \quad 0 \quad 0 \quad 0].$$

In this case, both uncertain parameters are destabilizing. Figure 3(d) shows the performance bound given by the shifted bounded real bound and the shifted Popov bound, which coincide with the vertex LMI bound.

9. CONCLUSIONS

In this paper the shifted bounded real bound [18], the shifted linear bound, the shifted inverse bound, and the shifted Popov bound [19] have been considered. These bounds were compared with the bounded real bound, the linear bound, the inverse bound, the Popov bound, and the vertex LMI bound. It was shown that these shifted bounds can be recast as guaranteed cost inequalities described by LMIs. For several examples, it was shown that the shifted bounded real bound and shifted inverse bound are comparable to the best possible parameter-independent bound given by Proposition 2. It has also been shown for several numerical examples that the shifted Popov bound, which is a parameter-dependent bound, may be less conservative than parameter-independent bounds.

Table III lists the various bounds discussed in this paper. The dimensionality of each LMI is compared along with the number of variables required. As can be seen from Table III, there are tradeoffs between the dimension of the constraints and the size of the free variables. The bounded real bound is a more conservative bound and has a lower dimension than the shifted bounded real bound. Similarly, the shifted bounded real bound has a lower dimension than the less conservative Popov and shifted Popov bounds.

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APPENDIX A

Proof of Lemma 1

For arbitrary $x \in \mathbf{R}^n$, define $f_x : \mathscr{R} \to \mathbf{R}$ by $f_x(\delta) = x^{\mathrm{T}}[(A + \sum_{i=1}^{r} \delta_i A_i)^{\mathrm{T}} P + P(A + \sum_{i=1}^{r} \delta_i A_i) + R]x$. Note that $\mathscr{R}$ defined by (24) is the convex hull of the corner points $\mathscr{R}_{\mathrm{H}}$ of a cube in $\mathbf{R}^r$. By convexity of f_x, $f_{x|\mathscr{R}} \leqslant 0$ if and only if $f_{x|\mathscr{R}_{\mathrm{H}}} \leqslant 0$. Since x is arbitrary, the result follows. □

Proof of Proposition 2

Let $P \in \mathscr{P}$. Thus, from Lemma 1, an immediate consequence of (28) is

$$\begin{aligned} 0 &\leqslant -R - A^{\mathrm{T}}P - PA - \Delta A^{\mathrm{T}}P - P\Delta A \\ &= \Omega_{\mathrm{LMI}}(P) - \Delta A^{\mathrm{T}}P - P\Delta A. \end{aligned}$$

Therefore $(\Omega_{\mathrm{LMI}}, 0)$ is a bounding pair. □

Proof of Proposition 3

Since $(\Omega, 0)$ is a bounded pair and P satisfies (6) then

$$(A + \Delta A)^{\mathrm{T}}P + P(A + \Delta A) + R \leqslant A^{\mathrm{T}}P + PA + \Omega(P) + R \leqslant 0.$$

By Lemma 1, P satisfies (27). Therefore $P \in \mathscr{P}$, and (28) follows as an immediate consequence. □

Proof of Proposition 7

To prove (39) $\Rightarrow$ (40), suppose (39) is satisfied. Then it follows that, for $i = 1, \ldots, r$,

$$-Y_i = -\gamma|N_i^{\mathrm{T}}C_i + C_i^{\mathrm{T}}N_i| \leqslant \gamma(N_i^{\mathrm{T}}C_i + C_i^{\mathrm{T}}N_i) \leqslant \gamma|N_i^{\mathrm{T}}C_i + C_i^{\mathrm{T}}N_i| = Y_i,$$

thus (40) is satisfied. Finally, to prove that (40)$\Leftrightarrow$(36), methods from the proof of Lemma 1 can be used. □

Proof of Proposition 11

First, using the technique used in the proof of Lemma 1, it can be shown that if $\bar{P}_0$ satisfies (56), then $\bar{P}_0 \geqslant P_0(\Delta A)$ where $P_0(\Delta A)$ is given by (48). Similarly, it can be shown that if Y satisfies (55), then Y also satisfies (49). To show that (54) implies (6) with $\Omega(P)$ given by (50), premultiply and postmultiply (54) by S and S^{T}, where S is given by

$$S = \begin{bmatrix} I & (C_0^{\mathrm{T}}H + A_{\mathrm{P}}^{\mathrm{T}}C_0^{\mathrm{T}}N + PB_0 - B_0X^{\mathrm{T}})R_0^{-1} \\ 0 & I \end{bmatrix},$$

and

$$R_0 = \gamma^{-1}H - NC_0B_0 - (NC_0B_0)^{\mathrm{T}} > 0.$$

Now from Theorem 1, it follows that $J(\mathscr{U}) \leqslant \operatorname{tr}(P + \bar{P}_0)V$. □

Proof of Corollary 1

The proof uses some of the techniques used in the proof of Lemma 1. □

Proof of Proposition 13
Note that

$$\begin{aligned}
0 &\leqslant \sum_{i=1}^{r} \alpha^{-1}(\delta_i A_i^{\mathrm{T}} - \alpha I)(P - N_i)(\delta_i A_i - \alpha I) \\
&= \sum_{i=1}^{r} [\alpha(P - N_i) + \alpha^{-1}\delta_i^2 A_i^{\mathrm{T}}(P - N_i)A_i + \delta_i(A_i^{\mathrm{T}} N_i + N_i A_i)] - \Delta A^{\mathrm{T}} P - P\Delta A \\
&\leqslant \sum_{i=1}^{r} [\alpha(P - N_i) + \alpha^{-1}\gamma^2 A_i^{\mathrm{T}}(P - N_i)A_i] + Y - \Delta A^{\mathrm{T}} P - P\Delta A \\
&= \Omega(P) - \Delta A^{\mathrm{T}} P - P\Delta A.
\end{aligned}$$

□

Proof of Proposition 15
The proof uses some of the techniques used in the proof of Lemma 1. □

Proof of Proposition 17
Note that

$$\begin{aligned}
0 &\leqslant \gamma\alpha \sum_{i=1}^{r} \left[\frac{\delta_i}{\gamma}(P - N_i) - \frac{1}{2\alpha}(A_i^{\mathrm{T}}[P - M_{1i}] + [P - M_{2i}]A_i)\right] \\
&\quad (P - N_i)^{-1}\left[\frac{\delta_i}{\gamma}(P - N_i) - \frac{1}{2\alpha}(A_i^{\mathrm{T}}[P - M_{1i}] + [P - M_{2i}]A_i)\right]^{\mathrm{T}} \\
&= \sum_{i=1}^{r} \left[\frac{\gamma}{4\alpha}(A_i^{\mathrm{T}}[P - M_{1i}] + [P - M_{2i}]A_i)(P - N_i)^{-1}(A_i^{\mathrm{T}}[P - M_{1i}]\right. \\
&\quad + [P - M_{2i}]A_i)^{\mathrm{T}} + \frac{\alpha}{\gamma}\delta_i^2(P - N_i) - \frac{\delta_i}{2}(A_i^{\mathrm{T}}[2P - M_{1i} - M_{2i}^{\mathrm{T}}] \\
&\quad \left. + [2P - M_{2i} - M_{1i}^{\mathrm{T}}]A_i)\right] \\
&= \sum_{i=1}^{r} \left[\frac{\gamma}{4\alpha}(A_i^{\mathrm{T}}[P - M_{1i}] + [P - M_{2i}]A_i)(P - N_i)^{-1}(A_i^{\mathrm{T}}[P - M_{1i}]\right. \\
&\quad \left. + [P - M_{2i}]A_i)^{\mathrm{T}} + \frac{\alpha}{\gamma}\delta_i^2(P - N_i) - \frac{\delta_i}{2}(A_i^{\mathrm{T}}[M_{1i} + M_{2i}^{\mathrm{T}}] + [M_{2i} + M_{1i}^{\mathrm{T}}]A_i)\right] \\
&\quad - \Delta A^{\mathrm{T}} P - P\Delta A \\
&\leqslant \sum_{i=1}^{r} \left[\frac{\gamma}{4\alpha}(A_i^{\mathrm{T}}[P - M_{1i}] + [P - M_{2i}]A_i)(P - N_i)^{-1}(A_i^{\mathrm{T}}[P - M_{1i}]\right. \\
&\quad \left. + [P - M_{2i}]A_i)^{\mathrm{T}} + \alpha\gamma(P - N_i)\right] + Y - \Delta A^{\mathrm{T}} P - P\Delta A \\
&= \Omega(P) - \Delta A^{\mathrm{T}} P - P\Delta A.
\end{aligned}$$

□

Proof of Proposition 18

Let $P \geqslant 0$, $\alpha > 0$, $N_1, \ldots, N_r$, $M_1, \ldots, M_r \in \mathbf{S}^n$ satisfy (79). To show that (79) is equivalent to (78) with $P \in \mathcal{N}$, premultiply and postmultiply (79) by S and S^{T}, where S is given by

$$S = \begin{bmatrix} I & \frac{\gamma}{4\alpha}[A_1^{\mathrm{T}}(P - M_1) + (P - M_1)A_1](P - N_1)^{-1} & \cdots & \frac{\gamma}{4\alpha}[A_r^{\mathrm{T}}(P - M_r) + (P - M_r)A_r](P - N_r)^{-1} \\ 0 & I & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \qquad \square$$

Proof of Corollary 3

Suppose $P = \beta I$. Then (78) becomes

$$\begin{aligned} 0 &= A^{\mathrm{T}}P + PA + \sum_{i=1}^{r} \alpha\gamma(P - N_i) + R \\ &\quad + \frac{\gamma}{4\alpha} \sum_{i=1}^{r} [A_i^{\mathrm{T}}(P - M_{1i}) + (P - M_{2i})A_i](P - N_i)^{-1}[A_i^{\mathrm{T}}(P - M_{1i}) + (P - M_{2i})A_i]^{\mathrm{T}} \\ &= \beta(A^{\mathrm{T}} + A) + \sum_{i=1}^{r} \alpha\gamma\left(\beta I - \frac{1}{\alpha\gamma}\left[\frac{\beta}{r}(A + A^{\mathrm{T}}) + R\right] - \beta I\right) + R \\ &= \beta(A^{\mathrm{T}} + A) - \frac{\beta}{r}\sum_{i=1}^{r} (A + A^{\mathrm{T}}). \qquad \square \end{aligned}$$

APPENDIX B: NOMENCLATURE

$\mathbf{R}^d$	$d \times 1$ real column vectors
$\mathbf{R}^{m\times n}$	$m \times n$ real matrices
$I_n, 0_n, \mathbf{S}^n$	$n \times n$ identity matrix, $n \times n$ zero matrix, $n \times n$ symmetric matrices
$\mathbf{N}^n, \mathbf{P}^n$	$n \times n$ non-negative-definite matrices, $n \times n$ positive-definite matrices
$A \leqslant B,\ A < B$	$B - A$ is non-negative definite, $B - A$ is positive definite
tr	trace
$\lvert H\rvert$	$(HH^{\mathrm{T}})^{1/2}$, where $H \in \mathbf{R}^{k_1\times k_2}$
vec, $\oplus$, $\otimes$	column stacking operator, Kronecker sum, Kronecker product
$[G, H]$	$GH - HG$

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