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Logarithmic Lyapunov functions for direct adaptive stabilization with normalized adaptive laws

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The problem of discrete-time and continuous-time adaptive stabilization under full-state feedback control is considered. In the discrete-time case the main result is based on a gain update law involving a step-size function. The formulation generalizes and unifies prior results based on quadratic and logarithmic Lyapunov functions. In the continuous-time case adaptive stabilization under full-state feedback using a normalized gradient algorithm is considered and Lyapunov stability is demonstrated.

1. Introduction

In adaptive control the rate of parameter estimate adaptation is a function of the parameter error and the regressor. Let $\hat{\theta}$, $\tilde{\theta}$ and φ denote the parameter estimate, parameter error and regressor, respectively. The update law $\dot{\hat{\theta}} = f(\tilde{\theta}, \varphi)$ is normalized if $f(\tilde{\theta}, \varphi)$ is radially bounded in φ (Ioannou and Sun 1996). For example $\dot{\hat{\theta}} = \varphi^{\mathrm{T}}\tilde{\theta}\varphi/(1 + \varphi^{\mathrm{T}}\varphi)$ is normalized whereas $\dot{\hat{\theta}} = \varphi^{\mathrm{T}}\tilde{\theta}\varphi$ is un-normalized. By choosing the normalizing factor $1 + \varphi^{\mathrm{T}}\varphi$ to be independent of the parameter error (as in Tao 2003, p. 102), the estimation algorithm retains its sensitivity to parameter error while reducing sensitivity to the regressor. Reduced sensitivity to the regressor helps reduce parameter drift.

In continuous time, un-normalized adaptive control laws arise naturally from Lyapunov design with a quadratic Lyapunov function (Narendra and Annaswamy 1989, Kristic *et al.* 1995, Ioannou and Sun 1996). Normalized adaptive laws are then obtained by normalizing the un-normalized laws to reduce parameter drift. For normalized control laws the Gronwall–Bellman lemma is used to prove convergence (Sastry and Bodson 1989, Ioannou and Sun 1996). However, the Lyapunov stability of normalized control laws is not addressed in the literature.

In the discrete-time setting, all estimation algorithms are normalized. Convergence of discrete-time adaptive control is demonstrated via the key technical lemma (Goodwin *et al.* 1980), which does not guarantee Lyapunov stability. The difficulty in performing Lyapunov analysis for discrete-time adaptive control algorithms has been recognized in the literature (Kanellakopolous 1994) and has been mainly attributed to the fact that, unlike in the Lyapunov derivative, the Lyapunov difference is not linear in the parameter error or the increment of the parameter error. This is not the main impediment, however, since the Lyapunov difference is quadratic in the estimation error, which is the product of the parameter error and the regressor. The negative-definite terms in the Lyapunov difference arising from the estimation algorithm cannot be used to bound the positive-definite terms arising from the plant dynamics since the positive definite terms are un-normalized while the negative definite terms are normalized. Hence, by using a quadratic Lyapunov function (Venugopal *et al.* 2003), convergence can be established, but Lyapunov stability is not guaranteed.

It turns out that the positive-definite terms can be normalized using a logarithmic Lyapunov function for the plant dynamics. This approach was taken in Haddad *et al.* (2002) and Johansson (1989, 1995). Although the bound $\log x \le x - 1$ removes the logarithm function from the analysis at an early stage, the subsequent analysis is greatly improved.

The present paper has three objectives. First, we use a logarithmic Lyapunov function to demonstrate the Lyapunov stability of continuous-time direct adaptive stabilization under full-state feedback when a normalized adaptive law is used. Second, we provide a complete development of a discrete-time adaptive stabilization algorithm for the case of full-state feedback. This development clarifies and extends the development in Haddad *et al.* (2002) by providing a more general characterization of the allowable normalizing step size and update rule. Finally, we show that, for the single input case with a specialized cost function, the main result has an optimality interpretation in terms of a gradient update direction and optimal step size. This result coincides with the main result of Venugopal *et al.* (2003), thus effectively extending the results of Venugopal *et al.* (2003) to include Lyapunov stability.

The contents of this paper are as follows. Lyapunov stability of continuous-time direct adaptive control with full-state feedback and a normalized adaptive

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law is demonstrated in § 2. In § 3 we present a Lyapunov-based proof demonstrating convergence of the plant states and boundedness of the feedback gains in discrete time. We provide explicit bounds on the adaptive step size necessary for stability. In § 4 we provide an optimal interpretation of the proposed algorithm for the single input case. Finally, § 5 presents simulation results.

2. Continuous-time adaptive stabilization

We make the following assumptions.

Assumption 1: *The input matrix B is known.*

Assumption 2: *There exists $K_s \in \mathbb{R}^{m\times n}$ such that $A_s \triangleq A + BK_s$ is known and asymptotically stable.*

Note that Assumption 2 does not require that K_s be known. It does, however, require that A_s be known. This requirement can be satisfied for the case of matched uncertainty. For example, consider the single-input system in companion form

$$A = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ a & \end{bmatrix}, \qquad B = \begin{bmatrix} 0_{(n-1)\times 1} \\ b \end{bmatrix}$$

where $a \in \mathbb{R}^{1\times n}$ and $b \in \mathbb{R}$ is non-zero. Letting $K_s = (1/b)(a_s - a)$, where $a_s \in \mathbb{R}^{1\times n}$, it follows that

$$A_s = A + BK_s = \begin{bmatrix} A_0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \frac{1}{b}(a_s - a) = \begin{bmatrix} A_0 \\ a_s \end{bmatrix}$$

where a_s is chosen such that A_s is asymptotically stable. The choice of a_s does not depend on knowledge of either a or b. The same is true for systems in multivariable canonical form.

Theorem 1: *Consider the linear system*

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Let $P \in \mathbb{R}^{n\times n}$ be the unique positive-definite matrix satisfying

$$A_s^{\mathrm{T}} P + PA_s + R = 0 \tag{2}$$

where $R \in \mathbb{R}^{n\times n}$ is positive definite, and consider the control law

$$u = Kx \tag{3}$$

with gain update

$$\dot{K} = -\frac{B^{\mathrm{T}} Pxx^{\mathrm{T}}}{1 + x^{\mathrm{T}} Px}. \tag{4}$$

Then the solution $(0, K_s)$ of the closed-loop system (1), (3) and (3) is Lyapunov stable, K converges and $x \to 0$ as $k \to \infty$.

Proof: Defining $\tilde{K} \triangleq K - K_s$ and using (3) and (1) it follows that

$$\dot{x} = A_s x + B\tilde{K}x \tag{5}$$

and

$$\dot{\tilde{K}} = -\frac{B^{\mathrm{T}} Pxx^{\mathrm{T}}}{1 + x^{\mathrm{T}} Px}. \tag{6}$$

Next consider the Lyapunov candidate

$$V(x, \tilde{K}) = \ln(1 + x^{\mathrm{T}} Px) + \mathrm{tr}(\tilde{K}^{\mathrm{T}} \tilde{K}). \tag{7}$$

Then the derivative of the Lyapunov function along the closed-loop system trajectories is given by

$$\begin{aligned}
\dot{V} &= \frac{(x^{\mathrm{T}} \tilde{K}^{\mathrm{T}} B^{\mathrm{T}} + x^{\mathrm{T}} A_s^{\mathrm{T}})Px + x^{\mathrm{T}} P(A_s x + B\tilde{K}x)}{1 + x^{\mathrm{T}} Px} \\
&\quad - \mathrm{tr}\left(\frac{xx^{\mathrm{T}} PB\tilde{K}}{1 + x^{\mathrm{T}} Px}\right) - \mathrm{tr}\left(\frac{\tilde{K}^{\mathrm{T}} B^{\mathrm{T}} Pxx^{\mathrm{T}}}{1 + x^{\mathrm{T}} Px}\right) \\
&= \Big\{x^{\mathrm{T}}(A_s^{\mathrm{T}} P + PA_s)x + x^{\mathrm{T}} \tilde{K}^{\mathrm{T}} B^{\mathrm{T}} Px + x^{\mathrm{T}} PB\tilde{K}x \\
&\quad - x^{\mathrm{T}} PB\tilde{K}x - x^{\mathrm{T}} \tilde{K}^{\mathrm{T}} B^{\mathrm{T}} Px\Big\}/(1 + x^{\mathrm{T}} Px) \\
&= \frac{-x^{\mathrm{T}}(A_s^{\mathrm{T}} P + PA_s)x}{1 + x^{\mathrm{T}} Px} \\
&= \frac{-x^{\mathrm{T}} Rx}{1 + x^{\mathrm{T}} Px} \le 0.
\end{aligned}$$

Hence the closed-loop system given by (1), (3) and (4) is Lyapunov stable. Furthermore it follows from Theorem 4.4 of Khalil (1996) that $x \to 0$ as $k \to \infty$. Convergence of K follows from (4). □

Remark 1: The system (1) can also be stabilized with the un-normalized gain adaptation $\dot{K} = -B^{\mathrm{T}} Pxx^{\mathrm{T}}$ (Hong *et al.* 2001).

3. Discrete-time adaptive stabilization

Consider the discrete-time system

$$x_{k+1} = Ax_k + Bu_k, \qquad k = 1, 2, \ldots \tag{8}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. We invoke the following assumptions concerning A and B.

Assumption 3: *There exists $K_s \in \mathbb{R}^{m\times n}$ such that $A_s \triangleq A + BK_s$ is known and asymptotically stable.*

Assumption 4: *There exist $G \in \mathbb{R}^{m\times n}$, positive-definite $\Sigma \in \mathbb{R}^{m\times m}$ and $\gamma > 0$, all of which are known, such that $GB + (GB)^{\mathrm{T}}$ is positive definite and*

$$\gamma \le \lambda_{\min}\Big[(GB + B^{\mathrm{T}} G^{\mathrm{T}})\big(B^{\mathrm{T}} G^{\mathrm{T}} \Sigma GB\big)^{-1}\Big]. \tag{9}$$

Assumption 4 implies that GB is non-singular and $\operatorname{rank}(B) = m$.

For convenience define

$$\varepsilon_k(K) \triangleq x_{k+1}(K) - x_{k+1}(K_\mathrm{s})$$

where $x_{k+1}(K) = (A + BK)x_k$, the state at time $k+1$ when the gain matrix K is used at time k.

Theorem 2: *Consider the linear system* (8) *and assume that Assumption 3 and Assumption 4 are satisfied. Let* $\Gamma \in \mathbb{R}^{n\times n}$ *be positive definite, let* $\mu > 0$, *and let* $\rho : \mathbb{R}^n \to (0, \infty)$ *be such that*

$$\inf_{x\in\mathbb{R}^n} \rho(x)(1 + \mu x^\mathrm{T} x) > 0 \tag{10}$$

and

$$\sup_{x\in\mathbb{R}^n} \rho(x)x^\mathrm{T}\Gamma x < \gamma. \tag{11}$$

Furthermore, consider the adaptive feedback control law

$$u_k = K_k x_k \tag{12}$$

and gain update

$$K_{k+1} = K_k - \rho(x_k)\Sigma G\varepsilon_k(K_k)x_k^\mathrm{T}\Gamma. \tag{13}$$

Then the solution $(0, K_\mathrm{s})$ *of the closed-loop system* (8), (12) *and* (13) *is Lyapunov stable*, K_k *converges, and* $x_k \to 0$ *as* $k \to \infty$.

Proof: Defining $\tilde{K}_k \triangleq K_k - K_\mathrm{s}$ and $\tilde{u}_k \triangleq \tilde{K}_k x_k$ and using (12) and (8) it follows that

$$x_{k+1} = A_\mathrm{s} x_k + B\tilde{u}_k. \tag{14}$$

Subtracting K_s from both sides of (13) and using (14) yields

$$\tilde{K}_{k+1} = \tilde{K}_k - \rho(x_k)\Sigma GB\tilde{K}_k\, x_k\, x_k^\mathrm{T}\Gamma. \tag{15}$$

Next consider the Lyapunov candidate

$$V(x, \tilde{K}) \triangleq \ln(1 + x^\mathrm{T}Px) + a\,\mathrm{tr}\,(\Gamma^{-1}\tilde{K}^\mathrm{T}\Sigma^{-1}\tilde{K}) \tag{16}$$

where $a > 0$ and P is a positive-definite matrix chosen below. Suppressing the argument of ρ, the Lyapunov difference along the closed-loop system trajectories is given by

$$\begin{aligned}\triangle V &\triangleq V(x_{k+1}, \tilde{K}_{k+1}) - V(x_k, \tilde{K}_k)\\
&= \ln(1 + x_{k+1}^\mathrm{T}Px_{k+1})\\
&\quad + a\,\mathrm{tr}\Big[\Gamma^{-1}(\tilde{K}_k - \rho\Sigma GB\tilde{K}_k x_k x_k^\mathrm{T}\Gamma)^\mathrm{T}\Sigma^{-1}\\
&\quad \times (\tilde{K}_k - \rho\Sigma GB\tilde{K}_k x_k x_k^\mathrm{T}\Gamma)\Big]\\
&\quad - \ln(1 + x_k^\mathrm{T}Px_k) - a\,\mathrm{tr}(\Gamma^{-1}\tilde{K}_k^\mathrm{T}\Sigma^{-1}\tilde{K}_k).\end{aligned}$$

Omitting dependence on the time step k and letting $\nu > \lambda_{\max}(B^\mathrm{T}PB)$ it follows that

$$\begin{aligned}\triangle V &= \ln\left[\frac{1 + (A_\mathrm{s}x + B\tilde{u})^\mathrm{T}P(A_\mathrm{s}x + B\tilde{u}) + x^\mathrm{T}Px - x^\mathrm{T}Px}{1 + x^\mathrm{T}Px}\right] - a\rho\,\mathrm{tr}(\Gamma^{-1}\tilde{K}^\mathrm{T}GB\tilde{K}xx^\mathrm{T}\Gamma)\\
&\quad - a\rho\,\mathrm{tr}(xx^\mathrm{T}\tilde{K}^\mathrm{T}B^\mathrm{T}G^\mathrm{T}\tilde{K}) + a\rho^2\,\mathrm{tr}(xx^\mathrm{T}\tilde{K}^\mathrm{T}B^\mathrm{T}G^\mathrm{T}\Sigma GB\tilde{K}xx^\mathrm{T}\Gamma)\\
&\le \ln\left[\frac{1 + x^\mathrm{T}A_\mathrm{s}^\mathrm{T}PA_\mathrm{s}x + x^\mathrm{T}A_\mathrm{s}^\mathrm{T}PB\tilde{u} + \tilde{u}^\mathrm{T}B^\mathrm{T}PA_\mathrm{s}x + \nu\tilde{u}^\mathrm{T}\tilde{u} + x^\mathrm{T}Px - x^\mathrm{T}Px}{1 + x^\mathrm{T}Px}\right]\\
&\quad - a\rho\,\mathrm{tr}(\tilde{K}^\mathrm{T}GB\tilde{K}xx^\mathrm{T}) - a\rho\,\mathrm{tr}(\tilde{K}^\mathrm{T}B^\mathrm{T}G^\mathrm{T}\tilde{K}xx^\mathrm{T}) + a\rho^2\,\mathrm{tr}(\Gamma xx^\mathrm{T}\tilde{K}^\mathrm{T}B^\mathrm{T}G^\mathrm{T}\Sigma GB\tilde{K}xx^\mathrm{T})\\
&= \ln\left[\frac{1 + x^\mathrm{T}(A_\mathrm{s}^\mathrm{T}PA_\mathrm{s} - P)x + x^\mathrm{T}A_\mathrm{s}^\mathrm{T}PB\tilde{u} + \tilde{u}^\mathrm{T}B^\mathrm{T}PA_\mathrm{s}x + \nu\tilde{u}^\mathrm{T}\tilde{u} + x^\mathrm{T}Px}{1 + x^\mathrm{T}Px}\right]\\
&\quad - a\rho\;\mathrm{tr}(x^\mathrm{T}\tilde{K}^\mathrm{T}GB\tilde{K}x) - a\rho\,\mathrm{tr}(x^\mathrm{T}\tilde{K}^\mathrm{T}B^\mathrm{T}G^\mathrm{T}\tilde{K}x) + a\rho^2\,\mathrm{tr}(x^\mathrm{T}\Gamma xx^\mathrm{T}\tilde{K}^\mathrm{T}B^\mathrm{T}G^\mathrm{T}\Sigma GB\tilde{K}x)\\
&= \ln\left[1 + \frac{x^\mathrm{T}(A_\mathrm{s}^\mathrm{T}PA_\mathrm{s} - P)x + x^\mathrm{T}A_\mathrm{s}^\mathrm{T}PB\tilde{u} + \tilde{u}^\mathrm{T}B^\mathrm{T}PA_\mathrm{s}x + \nu\tilde{u}^\mathrm{T}\tilde{u}}{1 + x^\mathrm{T}Px}\right]\\
&\quad - a\rho x^\mathrm{T}\tilde{K}^\mathrm{T}GB\tilde{K}x - a\rho x^\mathrm{T}\tilde{K}^\mathrm{T}B^\mathrm{T}G^\mathrm{T}\tilde{K}x + a\rho^2 x^\mathrm{T}\Gamma xx^\mathrm{T}\tilde{K}^\mathrm{T}B^\mathrm{T}G^\mathrm{T}\Sigma GB\tilde{K}x\\
&\le \frac{x^\mathrm{T}(A_\mathrm{s}^\mathrm{T}PA_\mathrm{s} - P)x + x^\mathrm{T}A_\mathrm{s}^\mathrm{T}PB\tilde{u} + \tilde{u}^\mathrm{T}B^\mathrm{T}PA_\mathrm{s}x + \nu\tilde{u}^\mathrm{T}\tilde{u}}{1 + x^\mathrm{T}Px}\\
&\quad - a\rho x^\mathrm{T}\tilde{K}^\mathrm{T}GB\tilde{K}x - a\rho x^\mathrm{T}\tilde{K}^\mathrm{T}B^\mathrm{T}G^\mathrm{T}\tilde{K}x + a\rho^2 x^\mathrm{T}\Gamma xx^\mathrm{T}\tilde{K}^\mathrm{T}B^\mathrm{T}G^\mathrm{T}\Sigma GB\tilde{K}x.\end{aligned} \tag{17}$$

Let $P \in \mathbb{R}^{n\times n}$ be the unique positive-definite matrix satisfying

$$A_s^T P A_s - P + \hat{R} = 0$$

where $\hat{R} \in \mathbb{R}^{n\times n}$ is positive definite and chosen such that $\lambda_{\min}(P) = \mu$. Next, let $\hat{\xi} \in (0,1)$ and let $\xi > 0$ be sufficiently small such that

$$4\xi A_s^T P B B^T P A_s < \hat{\xi}\hat{R}.$$

Hence

$$A_s^T P A_s - P + R + 4\xi A_s^T P B B^T P A_s < 0 \tag{18}$$

where $R \triangleq (1-\hat{\xi})\hat{R}$ is positive definite. Therefore, equation (17) and (18) imply

$$\begin{aligned}\triangle V \le \Big\{ & - x^T R x - x^T 4\xi A_s^T P B B^T P A_s x + x^T A_s^T P B \tilde{u} \\ & + \tilde{u}^T B^T P A_s x + \nu \tilde{u}^T \tilde{u} \Big\}/(1 + x^T P x) \\ & - a\rho x^T \tilde{K}^T G B \tilde{K} x - a\rho x^T \tilde{K}^T B^T G^T \tilde{K} x \\ & + a\rho^2 x^T \Gamma x x^T \tilde{K}^T B^T G^T \Sigma G B \tilde{K} x.\end{aligned}$$

Adding and subtracting

$$\frac{1}{4\xi}\frac{\tilde{u}^T\tilde{u}}{1 + x^T P x}$$

yields

$$\begin{aligned}\triangle V \le & -\frac{x^T R x}{1+x^T P x} + \frac{(1/4\xi)\,\tilde{u}^T\tilde{u} + \nu\tilde{u}^T\tilde{u}}{1+x^T P x} \\ & - a\rho x^T \tilde{K}^T G B \tilde{K} x - a\rho x^T \tilde{K}^T B^T G^T \tilde{K} x + a\rho^2 x^T \Gamma x x^T \tilde{K}^T B^T G^T \Sigma G B \tilde{K} x \\ & - \frac{x^T 4\xi A_s^T P B B^T P A_s x - \tilde{u}^T B^T P A_s x - x^T A_s^T P B \tilde{u} + (1/4\xi)\tilde{u}^T\tilde{u}}{1+x^T P x} \\ = & -\frac{x^T R x}{1+x^T P x} + \frac{(1/4\xi)\tilde{u}^T\tilde{u} + \nu\tilde{u}^T\tilde{u}}{1+x^T P x} \\ & - a\rho x^T \tilde{K}^T G B \tilde{K} x - a\rho x^T \tilde{K}^T B^T G^T \tilde{K} x + a\rho^2 x^T \Gamma x x^T \tilde{K}^T B^T G^T \Sigma G B \tilde{K} x \\ & - \frac{[2\xi x^T A_s^T P B - \tfrac{1}{2}\tilde{u}^T \quad - x^T A_s^T P B + (1/4\xi)\tilde{u}^T]}{1+x^T P x}\begin{bmatrix} 2B^T P A_s x \\ \tilde{u}\end{bmatrix} \\ = & -\frac{x^T R x}{1+x^T P x} + \frac{(1/4\xi + \nu)\tilde{u}^T\tilde{u}}{1+x^T P x} \\ & - a\rho x^T \tilde{K}^T G B \tilde{K} x - a\rho x^T \tilde{K}^T B^T G^T \tilde{K} x + a\rho^2 x^T \Gamma x x^T \tilde{K}^T B^T G^T \Sigma G B \tilde{K} x \\ & - \frac{[2x^T A_s^T P B \quad \tilde{u}^T]}{1+x^T P x}\begin{bmatrix} \xi I_m & -\tfrac{1}{2} I_m \\ -\tfrac{1}{2} I_m & (1/4\xi) I_m\end{bmatrix}\begin{bmatrix} 2B^T P A_s x \\ \tilde{u}\end{bmatrix} \\ \le & -\frac{x^T R x}{1+x^T P x} + \frac{(1/4\xi + \nu) x^T \tilde{K}^T \tilde{K} x}{1+x^T P x} \\ & - a\rho x^T \tilde{K}^T G B \tilde{K} x - a\rho x^T \tilde{K}^T B^T G^T \tilde{K} x + a\rho^2 x^T \Gamma x x^T \tilde{K}^T B^T G^T \Sigma G B \tilde{K} x \\ = & -\frac{x^T R x}{1+x^T P x} - a x^T \tilde{K}^T \Psi(x) \tilde{K} x + \frac{\delta x^T \tilde{K}^T \tilde{K} x}{1+x^T P x}\end{aligned} \tag{19}$$

where

$$\Psi(x) \triangleq \rho(x)\big[Z - \rho(x) x^T \Gamma x N\big] \tag{20}$$

and where $Z \triangleq GB + (GB)^T$, $N \triangleq (GB)^T \Sigma G B$, and $\delta \triangleq (1/4\xi) + \nu$.

Next we show that there exists $a > 0$ such that, for all $x \in \mathbb{R}^n$

$$(1 + x^T P x)\Psi(x) \ge \frac{\delta}{a} I. \tag{21}$$

Since $\lambda_{\min}(P) = \mu$, it follows from (9), (10), and (11) that there exist $\tau_1 > 0$ and $\tau_2 > 0$ such that, for all $x \in \mathbb{R}^n$

$$\rho(x)(1 + \mu x^T x) \ge \tau_1 \tag{22}$$

and

$$\rho(x) x^T \Gamma x N \le \lambda_{\min}(Z N^{-1}) N - \tau_2 N. \tag{23}$$

Since

$$\lambda_{\min}(N^{-1/2} Z N^{-1/2}) I \le N^{-1/2} Z N^{-1/2}$$

it follows that

$$\lambda_{\min}(Z N^{-1}) N = N^{1/2} \lambda_{\min}(N^{-1/2} Z N^{-1/2}) N^{1/2} \le Z. \tag{24}$$

Now (23) and (24) imply that, for all $x \in \mathbb{R}^n$

$$\rho(x) x^T \Gamma x N \le Z - \tau_2 N. \tag{25}$$

From (22) and (25) it follows that, for all $x \in \mathbb{R}^n$

$$\rho(x)(1 + x^{\mathrm{T}}Px)\big[Z - \rho(x)x^{\mathrm{T}}\Gamma xN\big] \geq \tau_1\tau_2 N. \quad (26)$$

Letting $a \geq \delta/\tau_1\tau_2\lambda_{\min}(N)$ implies $\tau_1\tau_2 N \geq \delta/a$. Consequently $\Psi(x) \geq (\delta/a)I$ for all $x \in \mathbb{R}^n$, and therefore it follows from (19) that, for all $x \in \mathbb{R}^n$

$$\Delta V \leq -\frac{x^{\mathrm{T}}Rx}{x^{\mathrm{T}}Px} \leq 0.$$

Hence the equilibrium $(0, K_{\mathrm{s}})$ is Lyapunov stable. Since $V(x, \tilde{K})$ is radially unbounded, it follows that K_k is bounded for all $x_0 \in \mathbb{R}^n$, $K_0 \in \mathbb{R}^{m\times n}$. Furthermore, it follows from Theorem 4 that $x_k \to 0$ as $k \to \infty$. Convergence of K_k follows from (13). □

Remark 2: Making G and Σ large causes γ, and thus ρ, to be small.

Proposition 1: *ρ: $\mathbb{R}^n \to (0, \infty)$ satisfies* (10) *and* (11) *if and only if there exist $\tau_1 > 0$ and $\tau_2 > 0$ such that*

$$\rho(0) \geq \tau_1 \quad (27)$$

and, for all non-zero $x \in \mathbb{R}^n$

$$\frac{\tau_1}{1 + \mu x^{\mathrm{T}}x} \leq \rho(x) \leq \frac{\gamma - \tau_2}{x^{\mathrm{T}}\Gamma x}. \quad (28)$$

Remark 3: The parameter μ can be made arbitrarily large by scaling $\hat{R}$.

Remark 4: The step-size function ρ need not be continuous. For example, let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, $\alpha > 0$, $\lambda_{\max}(\Gamma) < \beta < \mu$, and $\kappa_1, \kappa_2 \in (0, \gamma)$. Then

$$\rho(x) \triangleq \begin{cases} \kappa_1/(1 + \beta x^{\mathrm{T}}x) & \text{if } \|x\| \leq \alpha \\ \kappa_2/(1 + \beta x^{\mathrm{T}}x) & \text{if } \|x\| > \alpha \end{cases} \quad (29)$$

satisfies (28) with $0 < \tau_1 < \min\{\kappa_1, \kappa_2\}$ and $0 < \tau_2 < \min\{\gamma - \kappa_1, \gamma - \kappa_2\}$.

Letting $(\cdot)^{\dagger}$ denote the Moore–Penrose generalized inverse, the following specialization of Theorem 2 is given in Haddad *et al.* (2003).

Corollary 1: *Consider the linear system* (8) *and assume that Assumption 3 is satisfied. Let $Q \in \mathbb{R}^{m\times m}$ be positive definite such that $\lambda_{\max}(Q) < 2$ and consider the adaptive feedback control law* (12) *and gain update*

$$K_{k+1} = K_k - QB^{\dagger}\varepsilon_k(K_k)x_k^{\dagger}. \quad (30)$$

Then the solution $(0, K_{\mathrm{s}})$ of the closed-loop system (8), (12) *and* (30) *is Lyapunov stable, K_k is bounded and $x_k \to 0$ as $k \to \infty$.*

Proof: The result is a special case of Theorem 2 with $\Sigma = I_m$, $\Gamma = I_n$, $G = QB^{\dagger}$, $\mu = 1$, $\gamma \in (0, 2/\lambda_{\max}(Q))$ and

$$\rho(x) \triangleq \begin{cases} 1/x^{\mathrm{T}}x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Remark 5: Comparing the continuous-time gain update (4) and the discrete-time gain update (15) we note that the former achieves stabilization by tracking the zero state, while the latter stabilizes by tracking the states of the reference model $x_{k+1} = A_{\mathrm{s}}x_k$.

4. Optimal adaptive stabilization

In this section we consider the single input case $m = 1$ and provide an optimality interpretation for a special case of Theorem 2. The following result extends Theorem 1 of Venugopal *et al.* (2003).

Theorem 3: *Consider the system* (8) *with $m = 1$, assume that Assumption 3 is satisfied, and define the cost function*

$$J_k(K_k) \triangleq \|K_k - K_{\mathrm{s}}\|_{\mathrm{F}}^2. \quad (31)$$

Let $E \in \mathbb{R}^{n\times n}$ be positive definite and consider the adaptive feedback control law (12) *and gain update*

$$K_{k+1}(\beta_k) = K_k - \beta_k B^{\mathrm{T}}E\varepsilon_k(K_k)x_k^{\mathrm{T}} \quad (32)$$

where

$$\beta_k \triangleq \begin{cases} \dfrac{\|E^{1/2}\varepsilon_k(K_k)\|_2^2}{\|B^{\mathrm{T}}E\varepsilon_k(K_k)x_k^{\mathrm{T}}\|_{\mathrm{F}}^2} & \text{if } x_k \neq 0 \\ 1 & \text{if } x_k = 0. \end{cases} \quad (33)$$

Then the equilibrium solution $(0, K_{\mathrm{s}})$ of the closed-loop system (8), (12) *and* (32) *is Lyapunov stable, K_k is bounded, and $x_k \to 0$ as $k \to \infty$. Furthermore, if $\varepsilon_k(K_k) = 0$, then $K_{k+1} = K_k$, and if $\varepsilon_k(K_k) \neq 0$, then the cost function $J_{k+1}(K_{k+1}(\beta))$ is minimized by $\beta = \beta_k$ with the minimum value*

$$J_{k+1}(K_{k+1}(\beta_k)) = J_k(K_k) - \frac{\|E^{1/2}\varepsilon_k(K_k)\|_2^4}{\|B^{\mathrm{T}}E\varepsilon_k(K_k)x_k^{\mathrm{T}}\|_{\mathrm{F}}^2}. \quad (34)$$

Proof: If $\varepsilon_k(K_k) = 0$ then it follows from (32) that $K_{k+1} = K_k$. Now assume $\varepsilon_k(K_k) \neq 0$. Then it follows from (31) that

$$\begin{aligned} J_{k+1}(K_{k+1}) &= \|K_{k+1} - K_{\mathrm{s}}\|_{\mathrm{F}}^2 \\ &= \|\tilde{K}_k - \beta_k B^{\mathrm{T}}E\varepsilon_k(K_k)x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 \\ &= J_k + \beta_k^2\|B^{\mathrm{T}}E\varepsilon_k(K_k)x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 \\ &\quad - 2\beta_k\,\mathrm{tr}[B^{\mathrm{T}}E\varepsilon_k(K_k)x_k^{\mathrm{T}}\tilde{K}_k^{\mathrm{T}}] \\ &= J_k + \beta_k^2\|B^{\mathrm{T}}E\varepsilon_k(K_k)x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 - 2\beta_k\|E^{1/2}\varepsilon_k(K_k)\|_2^2 \end{aligned}$$

$$= J_k + \beta_k\left(\beta_k - 2\frac{\|E^{1/2}\varepsilon_k(K_k)\|_2^2}{\|B^{\mathrm{T}}E\varepsilon_k(K_k)x_k^{\mathrm{T}}\|_{\mathrm{F}}^2}\right) \times \|B^{\mathrm{T}}E\varepsilon_k(K_k)x_k^{\mathrm{T}}\|_{\mathrm{F}}^2$$

$$= J_k + \left[\left(\beta_k - \frac{\|E^{1/2}\varepsilon_k(K_k)\|_2^2}{\|B^{\mathrm{T}}E\varepsilon_k(K_k)x_k^{\mathrm{T}}\|_{\mathrm{F}}^2}\right)^2 - \frac{\|E^{1/2}\varepsilon_k(K_k)\|_2^4}{\|B^{\mathrm{T}}E\varepsilon_k(K_k)x_k^{\mathrm{T}}\|_{\mathrm{F}}^4}\right]\|B^{\mathrm{T}}E\varepsilon_k(K_k)x_k^{\mathrm{T}}\|_{\mathrm{F}}^2.$$

Therefore, the adaptive step size given by (33) minimizes $J_{k+1}(K_{k+1}(\beta_k))$ with minimum value given by (34). Furthermore, from (31) it follows that

$$\beta_k = \frac{\|E^{1/2}\varepsilon_k(K_k)\|_2^2}{\|B^{\mathrm{T}}E\varepsilon_k(K_k)x_k^{\mathrm{T}}\|_{\mathrm{F}}^2} = \frac{\|E^{1/2}B\tilde{K}_kx_k\|_2^2}{\|B^{\mathrm{T}}EB\tilde{K}_kx_kx_k^{\mathrm{T}}\|_{\mathrm{F}}^2} = \frac{x_k^{\mathrm{T}}\tilde{K}_k^{\mathrm{T}}B^{\mathrm{T}}EB\tilde{K}_kx_k}{\mathrm{tr}[x_kx_k^{\mathrm{T}}\tilde{K}_k^{\mathrm{T}}(B^{\mathrm{T}}EB)^2\tilde{K}_kx_kx_k^{\mathrm{T}}]} = \frac{1}{x_k^{\mathrm{T}}x_kB^{\mathrm{T}}EB}.$$

To prove stability we invoke Theorem 2 with $\Sigma = I_m$, $\Gamma = I_n$, $G = B^{\mathrm{T}}E$, $\mu = 1$, $\gamma \in (0, 2/B^{\mathrm{T}}EB)$, and

$$\rho(x) \triangleq \begin{cases} \dfrac{1}{x^{\mathrm{T}}xB^{\mathrm{T}}EB} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases} \qquad \square$$

Remark 6: Consider the one-step cost function $\hat{J}_k(K_k) \triangleq \frac{1}{2}\varepsilon_k^{\mathrm{T}}(K_k)E\varepsilon_k(K_k)$. Then the gradient of $\hat{J}_k$ is given by

$$\left.\frac{\partial \hat{J}_k}{\partial K}\right|_{K_k} = B^{\mathrm{T}}E\varepsilon_k(K_k)x_k^{\mathrm{T}} \tag{35}$$

which is the direction of the gain update in (32). Hence the gain update (32) is a gradient update with optimal step size (33).

Remark 7: Theorem 1 of Venugopal *et al.* (2003) is based on a quadratic Lyapunov function that yields convergence of x_k but not Lyapunov stability as given by Theorem 2. Theorem 3 extends Theorem 1 of Venugopal *et al.* (2003) by guaranteeing Lyapunov stability of the equilibrium $(0, K_s)$.

5. Examples

Example 1: In this example we compare the performance of normalized and un-normalized state feedback control laws for a system of the form (1). Consider the unstable continuous-time system in controllable canonical form

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.25 & 1.875 & -4.375 & 3.75 \end{bmatrix}x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}u$$

with $x_0 = [1\ \ 1\ \ 1\ \ 1]^{\mathrm{T}}$. We assume that the fourth row of A is unknown, B is known and

$$A_s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -4 & -3 \end{bmatrix}x.$$

The closed-loop system response with both normalized and un-normalized control laws is shown in figure 1 and figure 2. The poor transient response for the normalized control law is due to the fact that it stabilizes the system (1) by tracking the zero state. Consequently the tracking error and the regressor are the same and normalization with respect to the regressor limits the maximum rate of adaptation.

Example 2: Consider the unstable single input system in controllable canonical form

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}x_k + \begin{bmatrix} 0 \\ \alpha \end{bmatrix}u_k, \qquad x_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

where the second row of A is unknown and it is known that $\alpha \in (0, 0.5]$. Let $G = [0\ \ 1]$ and $\Sigma = 1$. Then $\gamma \in (0, 2/\max\{\alpha\}) \subseteq (0, 4)$. Now let $\Gamma = I_2$

$$\rho(x) = \begin{cases} 2/x^{\mathrm{T}}x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

and

$$A_s = \begin{bmatrix} 0 & 1 \\ 0.25 & 0 \end{bmatrix}.$$

The closed-loop system response with $\alpha = 0.38$ is shown in figure 3.

Example 3: Consider the unstable two input system in controllable canonical form

$$x_{k+1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.25 & -1.5 & 0.31 & -2 \\ 0 & 0 & 0 & 1 \\ 0.53 & 0.35 & 0.75 & -0.7 \end{bmatrix}x_k + \begin{bmatrix} 0 & 0 \\ \alpha_1 & 0 \\ 0 & 0 \\ 0 & \alpha_2 \end{bmatrix}u_k$$

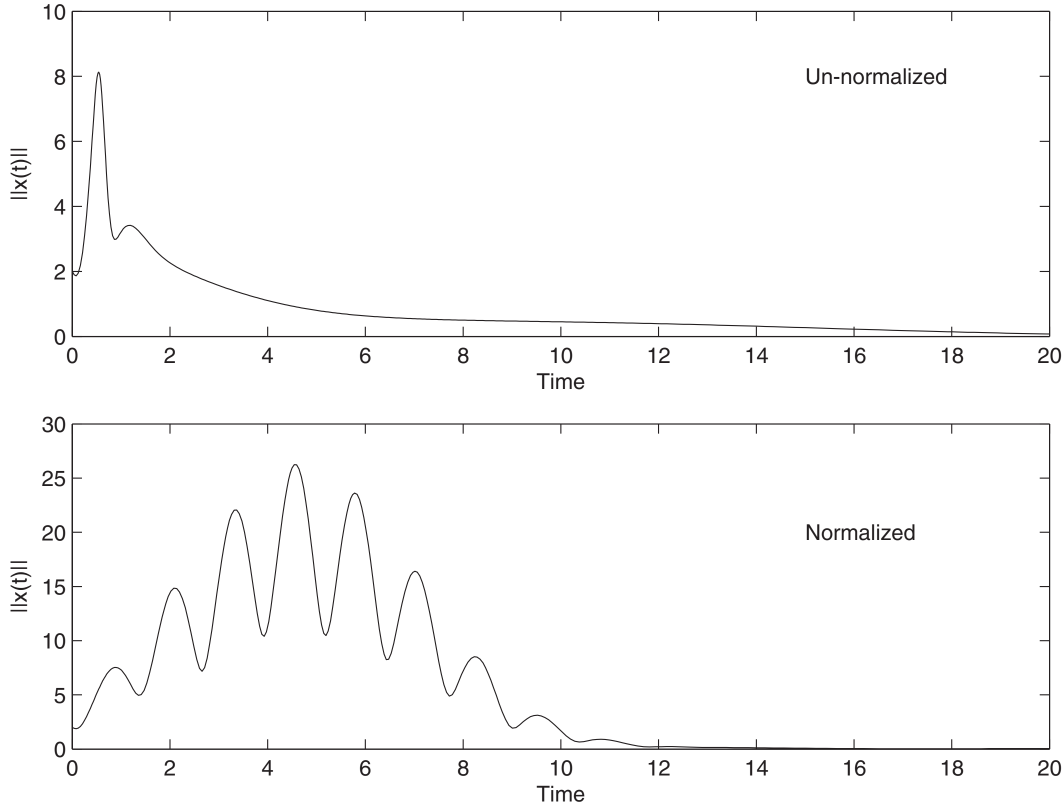

Figure 1. Norm of states and control signal versus time for Example 2.

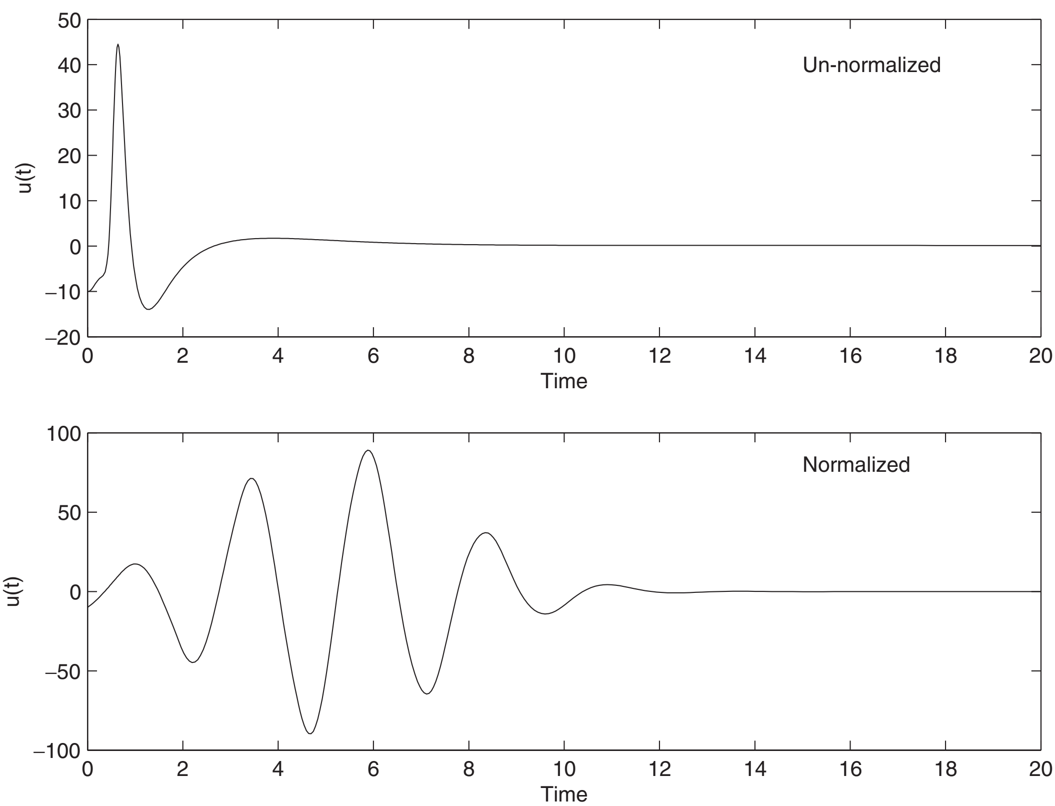

Figure 2. Norm of states and control signal versus time for Example 3.

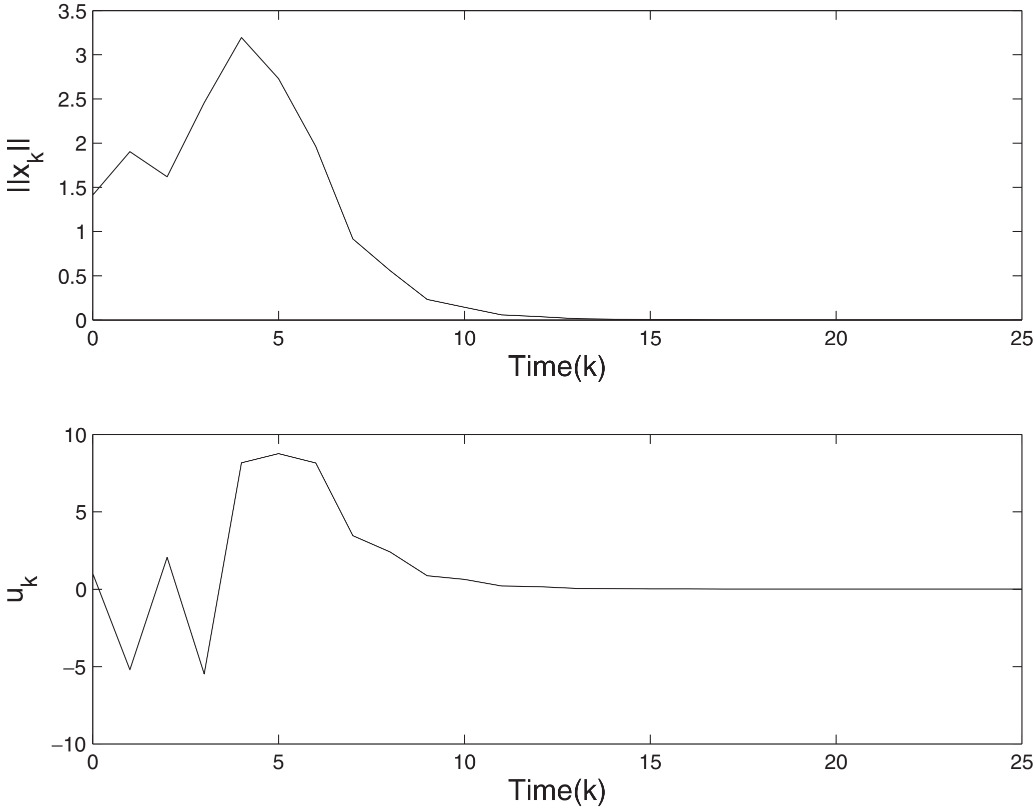

Figure 3. Norm of states versus time for Example 1.

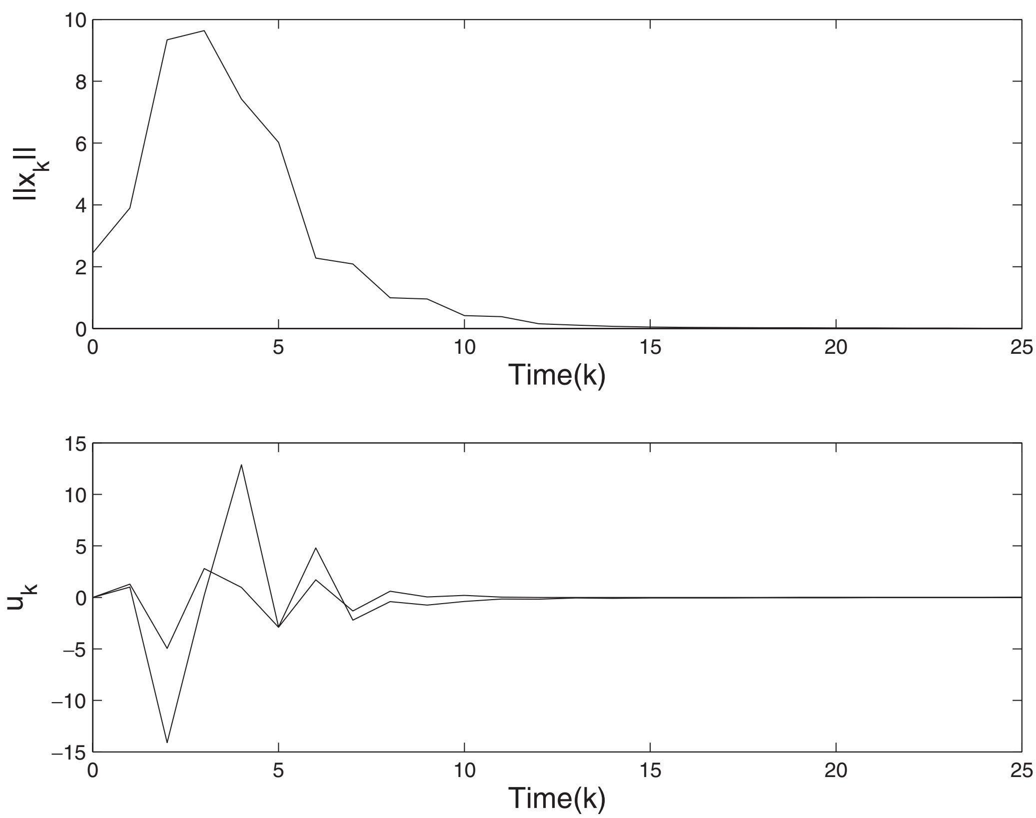

Figure 4. Norm of control signal versus time for Example 1.

with $x_0 = [2\ \ 0\ \ 1\ \ -1]^{\mathrm{T}}$. We assume that the second and fourth rows of A are unknown and it is known that $\alpha_1, \alpha_2 \in (0, 1]$. Let $\Sigma = I_2$ and

$$G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\gamma \in (0, \min\{2/\max\{\alpha_1\}, 2/\max\{\alpha_2\}\}) \subseteq (0, 2)$. Now let $\Gamma = I_4$

$$\rho(x) = \begin{cases} 1/x^{\mathrm{T}}x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

and

$$A_{\mathrm{s}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.2 & 0.2 & 0.13 & 0.16 \\ 0 & 0 & 0 & 1 \\ 0.1 & 0.15 & 0.3 & 0.3 \end{bmatrix}.$$

The closed-loop system response with $\alpha_1 = 0.7$ and $\alpha_2 = 0.95$ is shown in figure 4.

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Appendix

The following is a discrete-time version of Theorem 4.4 of Khalil (1996).

Theorem 4: *Let $\mathbb{N}$ denote the set of non-negative integers, let $\mathcal{D} \subset \mathbb{R}^n$ be an open neighbourhood of the origin, $W_1\colon \mathcal{D} \to (0, \infty)$, $W_2\colon \mathcal{D} \to (0, \infty)$, $W\colon \mathcal{D} \to [0, \infty)$, $f\colon \mathbb{N} \times \mathcal{D} \to \mathbb{R}^n$ and $V\colon \mathbb{N} \times \mathcal{D} \to \mathbb{R}$ be such that $W_1(x) \leq V(k, x) \leq W_2(x)$ for all $k \in \mathbb{N}$ and $x \in \mathcal{D}$ and $\Delta V \triangleq V(k+1, f(k, x)) - V(k, x) \leq -W(x)$ for all $k \in \mathbb{N}$ and $x \in \mathcal{D}$. Furthermore, define $B_r \triangleq \{x \in \mathcal{D}\colon \|x\| < r\}$ and let $\tau_r < \min_{\|x\|=r} W_1(x)$. Then all solutions of $x_{k+1} = f(k, x_k)$ with $x_0 \in \{x \in B_r\colon W_2(x) \leq \tau_r\}$ are bounded and satisfy $W(x_k) \to 0$ as $k \to \infty$.*

Proof: Choose $\tau_r > 0$ such that $\tau_r < \min_{\|x\|=r} W_1(x)$. Then $\{x \in B_r\colon W_2(x) \leq \tau_r\} \subset \{x \in B_r\colon V(k, x) \leq \tau_r\} \subset \{x \in B_r\colon W_1(x) \leq \tau_r\}$. Let $x_0 \in \{x \in B_r\colon W_2(x) \leq \tau_r\}$. Then $x \in \{x \in B_r\colon V(k, x) \leq \tau_r\}$ for all $k \geq 0$. Hence $\|x(t)\| < r$ for all $k \geq 0$. Since $V(k, x)$ is bounded from below, the limit

$$\lim_{k \to \infty} \sum_{\zeta=0}^{k} W(x(\zeta)) \leq -\lim_{k \to \infty} \sum_{\zeta=0}^{k} \Delta V(\zeta, x(\zeta)) = V(0, x_0) - \lim_{k \to \infty} V(k, x_k)$$

exists. This implies that $W(x_k) \to 0$ as $k \to \infty$. □

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