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Time-domain analysis of motion transmissibilities in force-driven and displacement-driven structures

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ABSTRACT

A transmissibility is a relationship between signals of the same type, where the input and the output of the transmissibility are outputs of the underlying system. Transmissibility estimates are traditionally obtained using frequency-domain methods, which are based on the assumption that the input and output signals are stationary, and thus initial conditions and transient effects are ignored. In this paper we develop a time-domain framework for SISO and MIMO transmissibilities that accounts for nonzero initial conditions for both force-driven and displacement-driven structures. We show that transmissibilities in force-driven and displacement-driven structures are equal when the locations of the forces and prescribed displacements are identical. We present three examples to illustrate this equality.

1. Introduction

Structural vibration is most commonly modeled as the displacement, velocity, or acceleration response to a force input. Assuming that the dynamics are linear, lumped models of structural vibration with multiple degrees of freedom typically have the form of matrix differential equations with inertia, damping, and stiffness coefficients [1]. In the frequency domain, these force-driven outputs are modeled by compliance, admittance, and inertance transfer functions. Alternatively, a transfer function can relate displacements at different locations on a structure. The resulting transfer function is called a *motion transmissibility* [2,3]. Velocity and acceleration signals can also be considered instead of displacements. These concepts extend directly to rotational variables, where "torque" replaces "force."

It is also possible to define a force transmissibility, and the relationship between force and motion transmissibilities is discussed in [4,5]. In the present paper, force transmissibility is not considered, and the term "transmissibility" refers to motion transmissibility.

In the most common setup, the transmissibility involves the motion of the point at which the force is prescribed. A more general notion of transmissibility arises in the case where neither of the displacement measurements coincides with the location of the applied force. This situation is of interest in applications where the applied force is unknown. Except for the case where one of the measurements is located at a node of a mode, the resulting transmissibility captures information about only the zeros (anti-resonances) in the structural response, and thus information about the modal resonances is not included in the model.

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The potential usefulness of transmissibilities for applications such as damage detection [6–9] has led to increased interest in their properties. In [10–12], transmissibilities are used to update modal models, while computation and identification of transmissibilities is discussed in [13–15]. Transmissibilities are used in [16] to analyze the effects of structural coupling. Multi-input, multi-output (MIMO) transmissibilities are considered in [17], while the effect of distributed forces is analyzed in [15]. Finally, transmissibilities play a role in "operational modal analysis" [10,18], which assumes stationary excitation.

A common feature of the treatment of transmissibilities in [1–3,5,6,9–13,17] is that these models are expressed in the frequency-domain. For identification purposes, these transfer functions can be estimated by computing the Fourier transforms of the measured signals, and thus the effect of nonzero initial conditions is ignored. To account for initial conditions, the present paper focuses on time-domain transmissibility models [19–22,24]. These models provide the foundation for time-domain identification methods.

The development of time-domain transmissibility models requires special attention to the cancellation of poles in the underlying structural model as well as the role of the initial conditions. The resulting model is not an input–output model in the usual sense, and therefore the notions of free and forced response do not apply. These issues were considered in [19–21,24] in terms of "pseudo transfer functions."

Unlike the discrete-time models given in [19–21], the results in the present paper are developed in continuous time. This setting facilitates the analysis of transmissibilities of structures. Furthermore, the present paper also considers transmissibilities arising from displacement-driven structures and shows that the force- and displacement-driven transmissibilities are equal when the locations of the force and prescribed displacement are identical. Together, these developments provide the foundation for a time-domain framework for transmissibilities that accounts for nonzero initial conditions.

The contents of the paper are as follows. In Section 2 we show a numerical comparison between time-domain and frequency-domain identification methods under nonzero initial conditions. In Section 3 and Section 4 we derive SISO and MIMO time-domain models for transmissibility operators in force-driven structures, respectively. In Section 5 we consider displacement-driven structures, while in Section 6 and Section 7 we derive SISO and MIMO time-domain models for transmissibility operators in displacement-driven structures, respectively. In Section 8 we show the equality of transmissibilities of force-driven and displacement-driven structures with identical inputs and outputs when the force and the prescribed motion are applied to the same location. We introduce examples in Section 9. Finally, we present conclusions and future research in Section 10.

2. Effects of nonzero initial conditions on estimating frequency response functions

Transmissibility estimates are traditionally obtained using frequency-domain methods [2,3,5,6,9–13,17], which are based on the assumption that the response of the system consists entirely of the forced response and thus the free response is zero. For asymptotically stable systems, the free response decays exponentially, which suggests that measurements of the forced response can be obtained by using only data obtained after the free response is approximately zero. However, as shown in the following example, at the time at which data collection begins, there is a possibly nonzero initial condition, which can degrade the accuracy of frequency-domain identification.

Example 1. Consider the discrete-time asymptotically stable system $\mathcal{S}$ with the state-space realization

$$A=\begin{bmatrix} -0.5 & 0.2 \\ 0 & 0.7 \end{bmatrix}, \quad B=\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad C=[1.25 \quad -3], \quad D=0. \tag{1}$$

Let $x(k)\in\mathbb{R}^2$ be the state vector and thus $x(0)$ is the initial state. Let $u_0\in\mathbb{R}^{1\times N}$ be a realization of a stationary white random process with the Gaussian distribution $\mathcal{N}(0,1)$. Define the input $u \triangleq [u_0 \;\, u_0]\in\mathbb{R}^{1\times 2N}$, that is, u is formed by repeating u_0. Consider zero initial conditions, that is, $x(0)=0$, and define $y(k)\triangleq Cx(k)$. If we split $y\in\mathbb{R}^{1\times 2N}$ into two halves, then the first half of y is the response of $\mathcal{S}$ due to the input u_0 and the zero initial condition $x(0)$, while the second half of y is the response of $\mathcal{S}$ due to the input u_0 and the possibly nonzero initial condition $x(N)$. Fig. 1 shows a plot of the difference $y(k)-y(k+N)$, where $k=0,\dots,N-1$ and $N=500$ time steps for a given realization u_0. Note that despite the initial condition $x(0)=0$, the difference $y(k)-y(k+N)$ is not zero due to the fact that $x(N)$, which is the initial state when data collection begins at time $k=N$, is not zero.

Next, define $Y_{N,L}\triangleq[y(N)\cdots y(N+L-1)]\in\mathbb{R}^{1\times L}$ and $U_{N,L}\triangleq[u(N)\cdots u(N+L-1)]\in\mathbb{R}^{1\times L}$, and define $M_L\triangleq 2^p$, where p is the smallest integer such that $2^p\geq L$. For all $j=1,\dots,M_L$, let $\mathcal{S}(e^{\jmath\theta_j})$ be the frequency response of $\mathcal{S}$ at frequency θ_j where $\jmath\triangleq\sqrt{-1}$. Moreover, for all $j=1,\dots,M_L$, let

$$\hat{S}_{N,L}\left(e^{\jmath\theta_j}\right)\triangleq\frac{1}{r}\sum_{i=1}^{r}\hat{S}_{N,L,i}\left(e^{\jmath\theta_j}\right), \tag{2}$$

where r is the number of runs and $\hat{S}_{N,L,i}(e^{\jmath\theta_j})$ is the estimated value of $\mathcal{S}(e^{\jmath\theta_j})$ obtained from the ith run using either frequency-domain or time-domain identification. For frequency-domain identification, $\hat{S}_{N,L,i}(e^{\jmath\theta_j})$ is obtained by finding the ratio of the cross power spectral density of $Y_{N,L}$ and $U_{N,L}$ to the power spectral density of $U_{N,L}$ for the ith run. For time-domain identification, $\hat{S}_{N,L,i}(e^{\jmath\theta_j})$ is obtained by finding the frequency response of the estimated model obtained using least

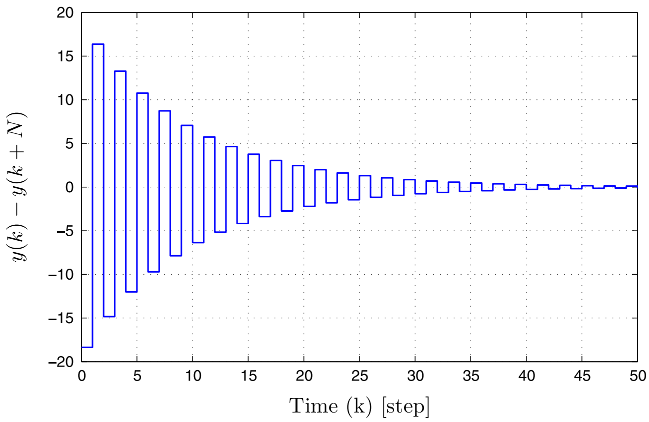

Fig. 1. Plot of the difference $y(k)-y(k+N)$ for the system $\mathcal{S}$ with the realization Eq. (1), where $k=0,\dots,50$, $N=500$, $u=[u_0\ \ u_0]$ is the input, and $x(0)=0$ is the initial state. This plot shows that the difference $y(k)-y(k+N)$ is not zero due to the fact that $x(N)$, which is the initial state of the system when we start collecting data at time $k=N$, is not zero.

squares identification with the time-domain data $U_{N,L}$ and $Y_{N,L}$. Define the error

$$e_{N,L} \triangleq \left(\sum_{j=1}^{M_L} \left(|\mathcal{S}(e^{\jmath\theta_j})| - |\hat{\mathcal{S}}_{N,L}(e^{\jmath\theta_j})| \right)^2 \right)^{1/2}. \tag{3}$$

Fig. 2 shows a plot of $e_{N,L}$ when using time-domain identification with $L=10{,}000$ time steps and N varies from 1 to 1000. Moreover, Fig. 2 shows a plot of $e_{N,L}$ when using frequency-domain identification with $L=10{,}000$ and $L=100{,}000$ time steps and N varies from 1 to 1000. The initial condition is $x(0)=[1000\ 1000]^{\mathrm{T}}$. Note from Fig. 2 that the FRF estimates obtained using time-domain identification are much better than the FRF estimates obtained using frequency-domain identification. Moreover, although we are using noise-free data, Fig. 2 shows that waiting for the free response to decay does not help the FRF estimates obtained using frequency-domain identification to converge to the true values. This is partly due to the nonzero initial condition $x(N)$, which occurs at the instant that data collection begins, and thus corrupts the estimates when using finite data sets. On the other hand, Fig. 2 shows that the FRF estimates obtained using time-domain identification are not affected by the nonzero initial conditions. It can be seen that the significance of the transients depends on the magnitude of the initial state relative to the magnitude of the state under stationary conditions.

Another issue with frequency-domain identification techniques is leakage errors, which are unavoidable in the case of aperiodic random excitations [23]. Theorem 2.6 in [23] shows that leakage errors decrease as the number of samples increases, but it is not guaranteed that the leakage errors are small for finite data sets. Example 2.7 in [23] shows that leakage errors can be interpreted as a transient effect, that is, as the effect of a nonzero initial condition. Leakage errors can be avoided by using periodic excitation and measurements of an integer number of periods, which cannot be achieved if the excitation signal cannot be specified.

Motivated by the advantages of time-domain identification techniques over frequency-domain identification techniques, in the following we develop a time-domain framework for SISO and MIMO transmissibilities that accounts for nonzero initial conditions for both force-driven and displacement-driven structures.

3. SISO transmissibilities in force-driven structures

Consider a lumped force-driven structure (FDS) consisting of masses $m_1,\dots,m_n$ connected by springs modeled by

$$M\ddot{q}(t)+Kq(t)=f_b(t), \tag{4}$$

where $M \triangleq \mathrm{diag}(m_1,\dots,m_n)\in\mathbb{R}^{n\times n}$ is the positive-definite mass matrix, $K\in\mathbb{R}^{n\times n}$ is the positive-definite stiffness matrix, $q(t)\triangleq[q_1(t)\ \cdots\ q_n(t)]^{\mathrm{T}}\in\mathbb{R}^n$ is the vector of mass displacements, and $f_b(t)\triangleq bu(t)=[f_1(t)\ \cdots\ f_n(t)]^{\mathrm{T}}\in\mathbb{R}^n$ is the vector of forces, where $b\in\mathbb{R}^n$ is a nonzero vector, $u(t)$ is a scalar force, and $f_i(t)$ is the force applied to the ith mass. Let $c\in\mathbb{R}^{1\times n}$ be nonzero and consider the scalar output

$$q_{c|bu}\triangleq cq, \tag{5}$$

where $q_{c|bu}$ denotes the output cq with the driving force bu. Note that $q_{e_{i,n}^{\mathrm{T}}|bu}=e_{i,n}^{\mathrm{T}}q=q_i$, where $e_{i,n}\in\mathbb{R}^n$ is the ith unit vector.

Next, let $w_i, w_o\in\mathbb{R}^{1\times n}$ and define

$$y_i\triangleq q_{w_i|bu}=w_iq, \tag{6}$$

$$y_o\triangleq q_{w_o|bu}=w_oq. \tag{7}$$

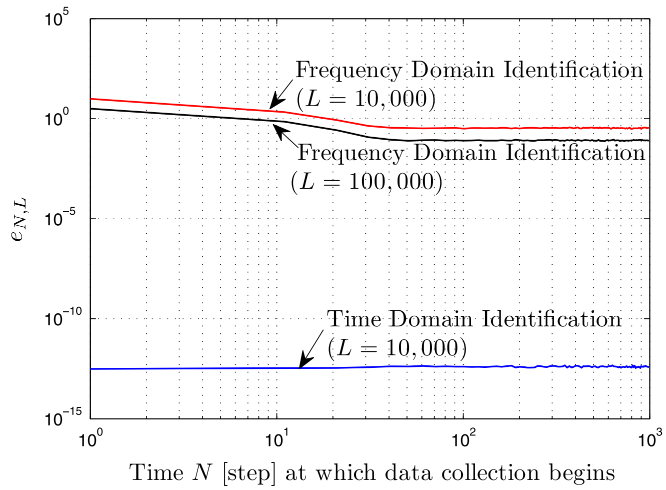

Fig. 2. Plot of $e_{N,L}$ using time-domain identification with $L=10{,}000$ time steps and frequency-domain identification with $L=10{,}000$ and $L=100{,}000$ time steps, N varies from 1 to 1000, and $r=100$ runs. The initial condition is $x(0)=[1000\ 1000]^{\mathrm{T}}$. Note that the FRF estimates obtained using time-domain identification are much better than the FRF estimates obtained using frequency-domain identification. Moreover, waiting for the free response to decay does not help the FRF estimates obtained using frequency-domain identification to converge to the true values, whereas the FRF estimates obtained using time-domain identification are not affected by the nonzero initial conditions.

The goal is to obtain a transmissibility function relating y_i and y_o that is independent of the initial conditions $q(0)$ and $\dot{q}(0)$ as well as the input u. As a first attempt at obtaining such a function, transforming Eq. (4) to the Laplace domain yields

$$(s^2M+K)\hat{q}(s)-sMq(0)-M\dot{q}(0)=b\hat{u}(s), \tag{8}$$

where $\hat{q}(s)$ and $\hat{u}(s)$ are the Laplace transforms of $q(t)$ and $u(t)$, respectively. Therefore,

$$\hat{q}(s)=(s^2M+K)^{-1}b\hat{u}(s)+(s^2M+K)^{-1}M(sq(0)+\dot{q}(0)). \tag{9}$$

It follows from Eqs. (6), (7) and (9) that the Laplace transforms of y_i and y_o are given by

$$\hat{y}_i(s)=w_i(s^2M+K)^{-1}b\hat{u}(s)+w_i(s^2M+K)^{-1}M(sq(0)+\dot{q}(0)), \tag{10}$$

$$\hat{y}_o(s)=w_o(s^2M+K)^{-1}b\hat{u}(s)+w_o(s^2M+K)^{-1}M(sq(0)+\dot{q}(0)), \tag{11}$$

respectively, and thus

$$\frac{\hat{y}_o(s)}{\hat{y}_i(s)}=\frac{w_o(s^2M+K)^{-1}b\hat{u}(s)+w_o(s^2M+K)^{-1}M(sq(0)+\dot{q}(0))}{w_i(s^2M+K)^{-1}b\hat{u}(s)+w_i(s^2M+K)^{-1}M(sq(0)+\dot{q}(0))}. \tag{12}$$

Note that, if $q(0)$ and $\dot{q}(0)$ are zero, then $\hat{u}(s)$ can be cancelled in Eq. (12), and $\hat{y}_o(s)$ and $\hat{y}_i(s)$ are related by a transmissibility that is independent of the input. However, if either $q(0)$ or $\dot{q}(0)$ is not zero, then $\hat{u}(s)$ cannot be canceled in Eq. (12), and an input-independent transmissibility cannot be obtained.

Alternatively, we consider a time-domain analysis using the differentiation operator $\mathbf{p}=\mathrm{d}/\mathrm{d}t$ instead of the Laplace variable s. It follows that Eq. (4) can be written as

$$(\mathbf{p}^2M+K)q(t)=bu(t). \tag{13}$$

Multiplying Eq. (6) by the polynomial $\delta(\mathbf{p})\triangleq\det(\mathbf{p}^2M+K)$ and using the fact that

$$\delta(\mathbf{p})I_n=\mathrm{adj}(\mathbf{p}^2M+K)(\mathbf{p}^2M+K) \tag{14}$$

yields the differential equation

$$\begin{aligned}\delta(\mathbf{p})y_i(t)&=w_i\delta(\mathbf{p})I_nq(t)\\&=w_i\mathrm{adj}(\mathbf{p}^2M+K)(\mathbf{p}^2M+K)q(t)\\&=w_i\mathrm{adj}(\mathbf{p}^2M+K)(M\ddot{q}(t)+Kq(t))\\&=w_i\mathrm{adj}(\mathbf{p}^2M+K)bu(t).\end{aligned} \tag{15}$$

Similarly,

$$\delta(\mathbf{p})y_o(t)=w_o\mathrm{adj}(\mathbf{p}^2M+K)bu(t). \tag{16}$$

For convenience, we define the notation

$$G_{w_i,b}(\mathbf{p})\triangleq w_i(\mathbf{p}^2M+K)^{-1}b, \tag{17}$$

$$G_{w_o,b}(\mathbf{p}) \triangleq w_o(\mathbf{p}^2M+K)^{-1}b. \tag{18}$$

Using Eqs. (17) and (18) we can rewrite Eqs. (15) and (16) as

$$y_i(t) = G_{w_i,b}(\mathbf{p})u(t), \tag{19}$$

$$y_o(t) = G_{w_o,b}(\mathbf{p})u(t), \tag{20}$$

respectively. Note that Eqs. (19) and (20) are interpreted as the differential equations Eqs. (15) and (16), respectively.

Note that Eqs. (10), (11), (19), and (20) include the free response due to $q(0)$ and $\dot{q}(0)$ as well as the forced response due to u. In the subsequent analysis, we omit the argument "t" where no ambiguity can arise.

Define the polynomials

$$\eta_o(\mathbf{p}) \triangleq w_o \operatorname{adj}(\mathbf{p}^2M+K)b, \tag{21}$$

$$\eta_i(\mathbf{p}) \triangleq w_i \operatorname{adj}(\mathbf{p}^2M+K)b. \tag{22}$$

If $G_{w_i,b}$ and $G_{w_o,b}$ are obtained from minimal state-space realizations, then $\delta(\mathbf{p})$ is coprime relative to both $\eta_i(\mathbf{p})$ and $\eta_o(\mathbf{p})$. Moreover, it follows from Eqs. (17) to (20) that

$$y_i = G_{w_i,b}(\mathbf{p})u = \frac{\eta_i(\mathbf{p})}{\delta(\mathbf{p})}u, \tag{23}$$

$$y_o = G_{w_o,b}(\mathbf{p})u = \frac{\eta_o(\mathbf{p})}{\delta(\mathbf{p})}u. \tag{24}$$

Next, it follows from Eqs. (23) and (24) that

$$\eta_o(\mathbf{p})\delta(\mathbf{p})y_i = \eta_o(\mathbf{p})\eta_i(\mathbf{p})u,$$

$$\eta_i(\mathbf{p})\delta(\mathbf{p})y_o = \eta_i(\mathbf{p})\eta_o(\mathbf{p})u,$$

and thus

$$\eta_i(\mathbf{p})\delta(\mathbf{p})y_o = \eta_o(\mathbf{p})\delta(\mathbf{p})y_i. \tag{25}$$

Definition 1. The *transmissibility operator* from y_i to y_o is the operator

$$\mathcal{T}^{\mathrm{F}}_{w_o,w_i|b}(\mathbf{p}) \triangleq \frac{\delta(\mathbf{p})\eta_o(\mathbf{p})}{\delta(\mathbf{p})\eta_i(\mathbf{p})}. \tag{26}$$

Hence, Eq. (25) can be written as

$$y_o = \mathcal{T}^{\mathrm{F}}_{w_o,w_i|b}(\mathbf{p})y_i. \tag{27}$$

Note that Eq. (26) is independent of the input u. Because (26) is expressed in terms of the differentiation operator $\mathbf{p}$ and not the complex number s, it is a time-domain model of the differential equation (25) and thus it accounts for nonzero initial conditions. However, Eq. (26) is not a transfer function. In the case $q(0)=0$ and $\dot{q}(0)=0$, it follows from Eq. (12) that $\mathbf{p}$ in Eq. (27) can be replaced by s to obtain

$$\hat{y}_o(s) = \mathcal{T}^{\mathrm{F}}_{w_o,w_i|b}(s)\hat{y}_i(s), \tag{28}$$

where $\mathcal{T}^{\mathrm{F}}_{w_o,w_i|b}(s)$ is a possibly improper rational function. However, if $q(0)$ or $\dot{q}(0)$ is not zero, then $\mathbf{p}$ cannot be replaced by s in Eq. (27).

Unlike common factors in the complex number s, common factors in the differentiation operator $\mathbf{p}$ cannot always be cancelled, as shown in the following example.

Example 2. Consider the signals $y_i(t)=1$ and $y_o(t)=1+e^{-t}$. Operating on $y_i(t)$ and $y_o(t)$ with $\mathbf{p}+1$ yields $(\mathbf{p}+1)y_i(t) = \dot{y}_i(t)+y_i(t) = 1 = \dot{y}_o(t)+y_o(t) = (\mathbf{p}+1)y_o(t)$. Hence $(\mathbf{p}+1)y_i = (\mathbf{p}+1)y_o$. However, $y_i \neq y_o$.

Despite Example 2, the following theorem shows that the common factor $\delta(\mathbf{p})$ in (26) can be cancelled without excluding any solutions of Eq. (25).

Theorem 1. *y_i and y_o satisfy*

$$y_o = \frac{\eta_o(\mathbf{p})}{\eta_i(\mathbf{p})}y_i. \tag{29}$$

Proof. See [24]. □

It follows from Theorem 1 that

$$y_o = \mathcal{T}^{\mathrm{F}}_{w_o,w_i|b}(\mathbf{p})y_i, \tag{30}$$

where the transmissibility operator in Eq. (26) is redefined as

$$\mathcal{T}^{\mathrm{F}}_{w_o,w_i|b}(\mathbf{p}) \triangleq \frac{\eta_o(\mathbf{p})}{\eta_i(\mathbf{p})} = \frac{w_o\ \mathrm{adj}(\mathbf{p}^2M+K)b}{w_i\ \mathrm{adj}(\mathbf{p}^2M+K)b}. \tag{31}$$

Note that $\mathcal{T}^{\mathrm{F}}_{w_o,w_i|b}(\mathbf{p})$ is not necessarily proper, and the polynomials $w_o\ \mathrm{adj}(\mathbf{p}^2M+K)b$ and $w_i\ \mathrm{adj}(\mathbf{p}^2M+K)b$ are not necessarily coprime.

4. MIMO transmissibilities in force-driven structures

Consider the lumped MIMO force-driven structure

$$M\ddot{q}(t)+\mathrm{K}q(t) = F_B(t), \tag{32}$$

where M, K, and q are as defined in Eq. (4), and

$$F_B \triangleq Bu(t), \tag{33}$$

where

$$B \triangleq [b_1 \ \cdots \ b_m], \quad u(t) \triangleq [u_1(t) \ \cdots \ u_m(t)]^{\mathrm{T}}, \tag{34}$$

and, for all $i \in \{1, \ldots, m\}$, $b_i \in \mathbb{R}^n$ and u_i is a scalar force.

Consider p outputs for Eq. (32). Let $W_i \in \mathbb{R}^{m\times n}, W_o \in \mathbb{R}^{(p-m)\times n}$ and define

$$y_i \triangleq q_{W_i|bu} = W_i q \in \mathbb{R}^m, \tag{35}$$

$$y_o \triangleq q_{W_o|bu} = W_o q \in \mathbb{R}^{p-m}. \tag{36}$$

The goal is to obtain a transmissibility function relating y_i and y_o that is independent of both the initial conditions $q(0)$ and $\dot{q}(0)$, as well as the input u.

Multiplying Eqs. (35) and (36) by $\delta(\mathbf{p})$ and following the procedure used to derive Eqs. (15) and (16) yields

$$\delta(\mathbf{p})y_i = W_i\ \mathrm{adj}(\mathbf{p}^2M+K)Bu, \tag{37}$$

$$\delta(\mathbf{p})y_o = W_o\ \mathrm{adj}(\mathbf{p}^2M+K)Bu. \tag{38}$$

For convenience, we define

$$G_{W_i,B}(\mathbf{p}) \triangleq W_i(\mathbf{p}^2M+K)^{-1}B, \tag{39}$$

$$G_{W_o,B}(\mathbf{p}) \triangleq W_o(\mathbf{p}^2M+K)^{-1}B, \tag{40}$$

and rewrite Eqs. (37) and (38) as

$$y_i = G_{W_i,B}(\mathbf{p})u, \quad y_o = G_{W_o,B}(\mathbf{p})u, \tag{41}$$

respectively, which are interpreted as the differential Eqs. (37) and (38), respectively. Note that Eq. (41) includes the free response due to $q(0)$ and $\dot{q}(0)$ as well as the forced response due to u.

Defining the polynomial matrices

$$\Gamma_i(\mathbf{p}) \triangleq W_i\ \mathrm{adj}(\mathbf{p}^2M+K)B \in \mathbb{R}^{m\times m}[\mathbf{p}], \tag{42}$$

$$\Gamma_o(\mathbf{p}) \triangleq W_o\ \mathrm{adj}(\mathbf{p}^2M+K)B \in \mathbb{R}^{(p-m)\times m}[\mathbf{p}], \tag{43}$$

we can rewrite Eqs. (37) and (38) as

$$\delta(\mathbf{p})y_i = \Gamma_i(\mathbf{p})u, \tag{44}$$

$$\delta(\mathbf{p})y_o = \Gamma_o(\mathbf{p})u, \tag{45}$$

respectively. Multiplying Eq. (44) by adj $\Gamma_i(\mathbf{p})$ from the left yields

$$\delta(\mathbf{p})\ \mathrm{adj}\ \Gamma_i(\mathbf{p})y_i = [\mathrm{adj}\ \Gamma_i(\mathbf{p})]\Gamma_i(\mathbf{p})u = \det\Gamma_i(\mathbf{p})u. \tag{46}$$

Next, multiplying Eq. (45) by det $\Gamma_i(\mathbf{p})$ yields

$$[\det\ \Gamma_i(\mathbf{p})]\delta(\mathbf{p})y_o = [\det\Gamma_i(\mathbf{p})]\Gamma_o(\mathbf{p})u. \tag{47}$$

Substituting the left hand side of Eq. (46) in Eq. (47) yields

$$\delta(\mathbf{p})\det\Gamma_i(\mathbf{p})y_o=\delta(\mathbf{p})\Gamma_o(\mathbf{p})\mathrm{adj}\,\Gamma_i(\mathbf{p})y_i. \tag{48}$$

Definition 2. Assume that $\det\Gamma_i(\mathbf{p})$ is not the zero polynomial. Then, the *transmissibility operator* from y_i to y_o is the operator

$$\mathcal{T}^{\mathrm{F}}_{W_o,W_i|B}(\mathbf{p})\triangleq\frac{\delta(\mathbf{p})}{\delta(\mathbf{p})\det\Gamma_i(\mathbf{p})}\Gamma_o(\mathbf{p})\mathrm{adj}\,\Gamma_i(\mathbf{p})=\frac{\delta(\mathbf{p})}{\delta(\mathbf{p})}\Gamma_o(\mathbf{p})\Gamma_i^{-1}(\mathbf{p}). \tag{49}$$

Note that Eq. (49) is independent of the input u and the initial condition $q(0)$ and $\dot{q}(0)$. Using Eq. (49), the differential Eq. (48) can be written as

$$y_o=\mathcal{T}^{\mathrm{F}}_{W_o,W_i|B}(\mathbf{p})y_i. \tag{50}$$

The following theorem shows that the common factor $\delta(\mathbf{p})$ in Eq. (49) can be cancelled without excluding any solutions of Eq. (48).

Theorem 2. *Assume that* $\det\Gamma_i(\mathbf{p})$ *is not the zero polynomial. Then,* y_i *and* y_o *satisfy*

$$y_o=\frac{1}{\det\Gamma_i(\mathbf{p})}\Gamma_o(\mathbf{p})\left[\mathrm{adj}\,\Gamma_i(\mathbf{p})\right]y_i=\Gamma_o(\mathbf{p})\Gamma_i^{-1}(\mathbf{p})y_i. \tag{51}$$

Proof. See [24]. □

It follows from Theorem 2 that

$$y_o=\mathcal{T}^{\mathrm{F}}_{W_o,W_i|B}(\mathbf{p})y_i, \tag{52}$$

where the transmissibility operator Eq. (49) is redefined as

$$\mathcal{T}^{\mathrm{F}}_{W_o,W_i|B}(\mathbf{p})\triangleq\Gamma_o(\mathbf{p})\Gamma_i^{-1}(\mathbf{p}). \tag{53}$$

Note that each entry of $\mathcal{T}^{\mathrm{F}}_{W_o,W_i|B}(\mathbf{p})$ is a rational operator that is not necessarily proper and whose numerator and denominator are not necessarily coprime.

5. Modeling displacement-driven structures

Consider a displacement-driven structure (DDS), where m_k is the driven mass, and thus

$$q_k(t)=q_{k,d}(t), \tag{54}$$

where $q_{k,d}(t)$ is the prescribed motion of m_k. This prescribed motion requires applying a suitable force as in Eq. (4). Removing the kth equation from Eq. (4) yields

$$M_{[k,\cdot]}\ddot{q}(t)+K_{[k,\cdot]}q(t)=0, \tag{55}$$

where $M_{[k,\cdot]}\in\mathbb{R}^{(n-1)\times n}$ and $K_{[k,\cdot]}\in\mathbb{R}^{(n-1)\times n}$ are M and K, respectively, with the kth row removed. It follows that Eq. (55) can be written as

$$M_{[k,k]}\ddot{q}_{[k]}+K_{[k,k]}q_{[k]}=-K_{[k,\cdot]}e_{k,n}q_{k,d}, \tag{56}$$

where $M_{[k,k]}\in\mathbb{R}^{(n-1)\times(n-1)}$ and $K_{[k,k]}\in\mathbb{R}^{(n-1)\times(n-1)}$ are M and K, respectively, with both the kth row and kth column removed, and $q_{[k]}$ is q with the kth row removed. Writing Eq. (56) in terms of the differentiation operator $\mathbf{p}$ yields

$$(\mathbf{p}^2M_{[k,k]}+K_{[k,k]})q_{[k]}=-K_{[k,\cdot]}e_{k,n}q_{k,d}. \tag{57}$$

Suppose now that d masses are displacement-driven, where $1\le d\le n-2$, and let $D\triangleq\{k_1,\dots,k_d\}$ be the set of displacement-driven masses. Then, using the same procedure used to obtain Eq. (56) we obtain

$$(\mathbf{p}^2M_{[D,D]}+K_{[D,D]})q_{[D]}=-K_{[D,\cdot]}[e_{k_1,n}\ \cdots\ e_{k_d,n}]\begin{bmatrix}q_{k_1,d}\\ \vdots\\ q_{k_d,d}\end{bmatrix}, \tag{58}$$

where $M_{[D,D]}\in\mathbb{R}^{(n-d)\times(n-d)}$ and $K_{[D,D]}\in\mathbb{R}^{(n-d)\times(n-d)}$ are M and K with rows $k_1,\dots,k_d$ removed and columns $k_1,\dots,k_d$ removed, $K_{[D,\cdot]}$ is K with rows $k_1,\dots,k_d$ removed, and $q_{[D]}$ is q with rows $k_1,\dots,k_d$ removed.

6. SISO transmissibilities in displacement-driven structures

Define the output

$$q_{d,c|e_{k,n}} \triangleq cI_{n_{[\cdot,k]}}q_{[k]}, \tag{59}$$

where $I_{n_{[\cdot,k]}} \in \mathbb{R}^{n\times(n-1)}$ is the identity matrix $I_n \in \mathbb{R}^{n\times n}$ with the kth column removed. Thus, $q_{d,c|e_{k,n}}$ is a linear combination of all position states $q_i, i=1,\dots,n, i\neq k$, assuming that the kth mass is displacement-driven. Let $w_i, w_o \in \mathbb{R}^{1\times n}$ and define

$$y_{i,d} \triangleq q_{d,w_i|e_{k,n}} = w_iI_{n_{[\cdot,k]}}q_{[k]}, \tag{60}$$

$$y_{o,d} \triangleq q_{d,w_o|e_{k,n}} = w_oI_{n_{[\cdot,k]}}q_{[k]}. \tag{61}$$

Following the procedure used to derive Eqs. (15) and (16) we can show that

$$\delta_d(\mathbf{p})y_{i,d} = -w_iI_{n_{[\cdot,k]}}\mathrm{adj}(\mathbf{p}^2M_{[k,k]}+K_{[k,k]})K_{[k,\cdot]}e_{k,n}q_{k,d}, \tag{62}$$

$$\delta_d(\mathbf{p})y_{o,d} = -w_oI_{n_{[\cdot,k]}}\mathrm{adj}(\mathbf{p}^2M_{[k,k]}+K_{[k,k]})K_{[k,\cdot]}e_{k,n}q_{k,d}, \tag{63}$$

where $\delta_d(\mathbf{p}) \triangleq \det(\mathbf{p}^2M_{[k,k]}+K_{[k,k]})$. For convenience, we define the notation

$$G_{d,w_i,e_{k,n}}(\mathbf{p}) \triangleq -w_iI_{n_{[\cdot,k]}}(\mathbf{p}^2M_{[k,k]}+K_{[k,k]})^{-1}K_{[k,\cdot]}e_{k,n}, \tag{64}$$

$$G_{d,w_o,e_{k,n}}(\mathbf{p}) \triangleq -w_oI_{n_{[\cdot,k]}}(\mathbf{p}^2M_{[k,k]}+K_{[k,k]})^{-1}K_{[k,\cdot]}e_{k,n}. \tag{65}$$

Using Eqs. (64) and (65) we can rewrite Eqs. (62) and (63) as

$$y_{i,d} = G_{d,w_i,e_{k,n}}(\mathbf{p})q_{k,d} = \frac{\eta_{i,d}(\mathbf{p})}{\delta_d(\mathbf{p})}q_{k,d}, \tag{66}$$

$$y_{o,d} = G_{d,w_o,e_{k,n}}(\mathbf{p})q_{k,d} = \frac{\eta_{o,d}(\mathbf{p})}{\delta_d(\mathbf{p})}q_{k,d}, \tag{67}$$

respectively, where

$$\eta_{i,d}(\mathbf{p}) \triangleq -w_iI_{n_{[\cdot,k]}}\mathrm{adj}(\mathbf{p}^2M_{[k,k]}+K_{[k,k]})K_{[k,\cdot]}e_{k,n}, \tag{68}$$

$$\eta_{o,d}(\mathbf{p}) \triangleq -w_oI_{n_{[\cdot,k]}}\mathrm{adj}(\mathbf{p}^2M_{[k,k]}+K_{[k,k]})K_{[k,\cdot]}e_{k,n}, \tag{69}$$

are polynomials in $\mathbf{p}$. It follows from (66) and (67) that

$$\eta_{o,d}(\mathbf{p})\delta_d(\mathbf{p})y_{i,d} = \eta_{o,d}(\mathbf{p})\eta_{i,d}(\mathbf{p})q_{k,d},$$

$$\eta_{i,d}(\mathbf{p})\delta_d(\mathbf{p})y_{o,d} = \eta_{i,d}(\mathbf{p})\eta_{o,d}(\mathbf{p})q_{k,d},$$

and thus

$$\eta_{i,d}(\mathbf{p})\delta_d(\mathbf{p})y_{o,d} = \eta_{o,d}(\mathbf{p})\delta_d(\mathbf{p})y_{i,d}. \tag{70}$$

Definition 3. The *transmissibility operator* from $y_{i,d}$ to $y_{o,d}$ is the operator

$$\mathcal{T}^{\mathrm{D}}_{w_o,w_i|e_{k,n}}(\mathbf{p}) \triangleq \frac{\delta_d(\mathbf{p})\eta_{o,d}(\mathbf{p})}{\delta_d(\mathbf{p})\eta_{i,d}(\mathbf{p})}.$$

Hence, Eq. (70) can be written as

$$y_{o,d} = \mathcal{T}^{\mathrm{D}}_{w_o,w_i|e_{k,n}}(\mathbf{p})y_{i,d}. \tag{71}$$

As in Section 3, it can be shown that $\delta_d(\mathbf{p})$ can be cancelled without excluding any solutions of Eq. (70), that is, $\mathcal{T}^{\mathrm{D}}_{w_o,w_i|e_{k,n}}(\mathbf{p})$ in Eq. (71) can be redefined as

$$\mathcal{T}^{\mathrm{D}}_{w_o,w_i|e_{k,n}}(\mathbf{p}) \triangleq \frac{\eta_{o,d}(\mathbf{p})}{\eta_{i,d}(\mathbf{p})} = \frac{w_oI_{n_{[\cdot,k]}}\mathrm{adj}(\mathbf{p}^2M_{[k,k]}+K_{[k,k]})K_{[k,\cdot]}e_{k,n}}{w_iI_{n_{[\cdot,k]}}\mathrm{adj}(\mathbf{p}^2M_{[k,k]}+K_{[k,k]})K_{[k,\cdot]}e_{k,n}}. \tag{72}$$

Note that $\mathcal{T}^{\mathrm{D}}_{w_o,w_i|e_{k,n}}(\mathbf{p})$ is not necessarily proper, and the polynomials $\eta_{o,d}(\mathbf{p})$ and $\eta_{i,d}(\mathbf{p})$ are not necessarily coprime.

7. MIMO transmissibilities in displacement-driven structures

Consider a DDS, where $m_{k_1}, \ldots, m_{k_d}$ are the displacement-driven masses, $1 \le d \le n-2$. Define the output $q_{d,C|e_{D,n}} \in \mathbb{R}^p$ by

$$q_{d,C|e_{D,n}} \triangleq CI_{n_{[\cdot,k]}} q_{[D]}, \tag{73}$$

where $C \in \mathbb{R}^{p \times n}$, $D \triangleq \{k_1, \ldots, k_d\}$, and $e_{D,n} \triangleq [e_{k_1,n} \ \ldots \ e_{k_d,n}]$. Hence, Eq. (73) is a vector whose components are linear combinations of all $q_i, i \in \{1, \ldots, n\} \backslash D$. Let $W_i \in \mathbb{R}^{d \times n}, W_o \in \mathbb{R}^{(p-d) \times n}$ and define

$$y_{i,d} \triangleq q_{d,W_i|e_{D,n}} = W_i I_{n_{[\cdot,D]}} q_{[D]}, \tag{74}$$

$$y_{o,d} \triangleq q_{d,W_o|e_{D,n}} = W_o I_{n_{[\cdot,D]}} q_{[D]}. \tag{75}$$

Following the procedure used to derive Eqs. (15) and (16) yields

$$\Delta_d(\mathbf{p}) y_{i,d} = -W_i I_{n_{[\cdot,D]}} \mathrm{adj}\left(\mathbf{p}^2 M_{[D,D]} + K_{[D,D]}\right) K_{[D,\cdot]} e_{D,n} q_{D,d}, \tag{76}$$

$$\Delta_d(\mathbf{p}) y_{o,d} = -W_o I_{n_{[\cdot,D]}} \mathrm{adj}\left(\mathbf{p}^2 M_{[D,D]} + K_{[D,D]}\right) K_{[D,\cdot]} e_{D,n} q_{D,d}, \tag{77}$$

where $\Delta_d(\mathbf{p}) \triangleq \det\left(\mathbf{p}^2 M_{[D,D]} + K_{[D,D]}\right) \in \mathbb{R}[\mathbf{p}]$ and $q_{D,d} \triangleq [q_{k_1} \cdots q_{k_d}]^{\mathrm{T}} \in \mathbb{R}^d$. Using the notation

$$G_{d,W_i,e_{D,n}}(\mathbf{p}) \triangleq -W_i I_{n_{[\cdot,D]}} \left(\mathbf{p}^2 M_{[D,D]} + K_{[D,D]}\right)^{-1} K_{[D,\cdot]} e_{D,n}, \tag{78}$$

$$G_{d,W_o,e_{D,n}}(\mathbf{p}) \triangleq -W_o I_{n_{[\cdot,D]}} \left(\mathbf{p}^2 M_{[D,D]} + K_{[D,D]}\right)^{-1} K_{[D,\cdot]} e_{D,n}, \tag{79}$$

we can rewrite (76) and (77) as

$$y_{i,d} = G_{d,W_i,e_{D,n}}(\mathbf{p}) q_{D,d}, \tag{80}$$

$$y_{o,d} = G_{d,W_o,e_{D,n}}(\mathbf{p}) q_{D,d}, \tag{81}$$

which are interpreted as the differential Eqs. (76) and (77), respectively. Note that Eqs. (80) and (81) include the free response due to $q_{[D]}(0)$ and $\dot{q}_{[D]}(0)$ as well as the forced response due to $q_{D,d}$. Defining

$$\Gamma_{i,d}(\mathbf{p}) \triangleq -W_i I_{n_{[\cdot,D]}} \mathrm{adj}\left(\mathbf{p}^2 M_{[D,D]} + K_{[D,D]}\right) K_{[D,\cdot]} e_{D,n} \in \mathbb{R}^{d \times d}[\mathbf{p}], \tag{82}$$

$$\Gamma_{o,d}(\mathbf{p}) \triangleq -W_o I_{n_{[\cdot,D]}} \mathrm{adj}\left(\mathbf{p}^2 M_{[D,D]} + K_{[D,D]}\right) K_{[D,\cdot]} e_{D,n} \in \mathbb{R}^{(p-d) \times d}[\mathbf{p}], \tag{83}$$

we can rewrite Eqs. (76) and (77) as

$$\Delta_d(\mathbf{p}) y_{i,d} = \Gamma_{i,d}(\mathbf{p}) q_{D,d}, \tag{84}$$

$$\Delta_d(\mathbf{p}) y_{o,d} = \Gamma_{o,d}(\mathbf{p}) q_{D,d}. \tag{85}$$

Multiplying Eq. (84) by adj $\Gamma_{i,d}(\mathbf{p})$ from the left yields

$$\mathrm{adj}\ \Gamma_{i,d}(\mathbf{p}) \Delta_d(\mathbf{p}) y_{i,d} = \mathrm{adj}\ \Gamma_{i,d}(\mathbf{p}) \Gamma_{i,d}(\mathbf{p}) q_{D,d} = \det \Gamma_{i,d}(\mathbf{p}) q_{D,d}. \tag{86}$$

Next, multiplying Eq. (85) by det $\Gamma_{i,d}(\mathbf{p})$ yields

$$\left[\det \Gamma_{i,d}(\mathbf{p})\right] \Delta_d(\mathbf{p}) y_{o,d} = \left[\det \Gamma_{i,d}(\mathbf{p})\right] \Gamma_{o,d}(\mathbf{p}) q_{D,d}. \tag{87}$$

Substituting the left hand side of Eq. (86) into Eq. (87) yields

$$\Delta_d(\mathbf{p}) \det \Gamma_{i,d}(\mathbf{p}) y_{o,d} = \Delta_d(\mathbf{p}) \Gamma_{o,d}(\mathbf{p}) \mathrm{adj}\ \Gamma_{i,d}(\mathbf{p}) y_{i,d}. \tag{88}$$

Definition 4. Assume that det $\Gamma_{i,d}(\mathbf{p})$ is not the zero polynomial. The *transmissibility operator* from $y_{i,d}$ to $y_{o,d}$ is the operator

$$\mathcal{T}^{\mathrm{D}}_{W_o,W_i|e_{D,n}}(\mathbf{p}) \triangleq \frac{\Delta_d(\mathbf{p})}{\Delta_d(\mathbf{p}) \det \Gamma_{i,d}(\mathbf{p})} \Gamma_{o,d}(\mathbf{p}) \mathrm{adj} \Gamma_{i,d}(\mathbf{p}) = \frac{\Delta_d(\mathbf{p})}{\Delta_d(\mathbf{p})} \Gamma_{o,d}(\mathbf{p}) \Gamma_{i,d}^{-1}(\mathbf{p}).$$

Hence, Eq. (88) can be written as

$$y_{o,d} = \mathcal{T}^{\mathrm{D}}_{W_o,W_i|e_{D,n}}(\mathbf{p}) y_{i,d}. \tag{89}$$

As in Section 4, it can be shown that $\Delta_d(\mathbf{p})$ can be cancelled without excluding any solutions of Eq. (88), that is, $\mathcal{T}^{\mathrm{D}}_{W_o,W_i|e_{D,n}}(\mathbf{p})$ in Eq. (89) can be redefined as

$$\mathcal{T}^{\mathrm{D}}_{W_o,W_i|e_{D,n}}(\mathbf{p}) \triangleq \Gamma_{o,d}(\mathbf{p}) \Gamma_{i,d}^{-1}(\mathbf{p}). \tag{90}$$

8. Equality of motion transmissibilities in force-driven and displacement-driven structures

8.1. Equality of SISO motion transmissibilities in force-driven and displacement-driven structures

Define $w_{o,k}$ and $w_{i,k}$ to be w_o and w_i, respectively, with the kth component replaced by zero. The following result shows that the SISO transmissibilities of force-driven and displacement-driven structures with identical inputs and outputs and with the force and the prescribed motion applied to the same location are identical. This result is somewhat surprising since the specified displacement of a mass could be perceived as introducing a node.

Theorem 3. *The SISO force-driven and displacement-driven transmissibilities are equal, that is,*

$$\mathcal{T}^{\mathrm{F}}_{w_{o,k},w_{i,k}|e_{k,n}}(\mathbf{p}) = \mathcal{T}^{\mathrm{D}}_{w_{o,k},w_{i,k}|e_{k,n}}(\mathbf{p}). \tag{91}$$

Proof. It follows from Eq. (72) that

$$\mathcal{T}^{\mathrm{D}}_{w_o,w_i|e_{k,n}}(\mathbf{p}) = \frac{w_o I_{n_{[\cdot,k]}} \mathrm{adj}\left(\mathbf{p}^2 M_{[k,k]} + K_{[k,k]}\right) K_{[k,\cdot]} e_{k,n}}{w_i I_{n_{[\cdot,k]}} \mathrm{adj}\left(\mathbf{p}^2 M_{[k,k]} + K_{[k,k]}\right) K_{[k,\cdot]} e_{k,n}}. \tag{92}$$

From Eq. (A.1) in Appendix A we have

$$w_{o,k} I_{n_{[\cdot,k]}} \mathrm{adj}\left(\mathbf{p}^2 M_{[k,k]} + K_{[k,k]}\right) K_{[k,\cdot]} e_{k,n} = -w_{o,k} \mathrm{adj}\left(\mathbf{p}^2 M + K\right) e_{k,n}, \tag{93}$$

$$w_{i,k} I_{n_{[\cdot,k]}} \mathrm{adj}\left(\mathbf{p}^2 M_{[k,k]} + K_{[k,k]}\right) K_{[k,\cdot]} e_{k,n} = -w_{i,k} \mathrm{adj}\left(\mathbf{p}^2 M + K\right) e_{k,n}. \tag{94}$$

Using Eqs. (93) and (94), Eq. (92) yields

$$\mathcal{T}^{\mathrm{D}}_{w_{o,k},w_{i,k}|e_{k,n}}(\mathbf{p}) = \frac{w_{o,k} \mathrm{adj}\left(\mathbf{p}^2 M + K\right) e_{k,n}}{w_{i,k} \mathrm{adj}\left(\mathbf{p}^2 M + K\right) e_{k,n}}. \tag{95}$$

Replacing w_o, w_i, and b in Eq. (31) with $w_{o,k}$, $w_{i,k}$, and $e_{k,n}$, respectively, yields

$$\mathcal{T}^{\mathrm{F}}_{w_{o,k},w_{i,k}|e_{k,n}}(\mathbf{p}) = \frac{w_{o,k} \mathrm{adj}(\mathbf{p}^2 M + K) e_{k,n}}{w_{i,k} \mathrm{adj}(\mathbf{p}^2 M + K) e_{k,n}}. \tag{96}$$

Hence, Eqs. (95) and (96) yield Eq. (91). □

8.2. Equality of MIMO motion transmissibilities in force-driven and displacement-driven structures

Define $W_{o,D}$ and $W_{i,D}$ to be W_o and W_i, respectively, with the $k_1^{\mathrm{th}}, \ldots, k_d^{\mathrm{th}}$ columns replaced by zero. The following result shows that the MIMO transmissibilities of force-driven and displacement-driven structures with identical inputs and outputs and with the forces and prescribed motions applied to the same locations are identical.

Theorem 4. *The MIMO force-driven and displacement driven transmissibilities are equal, that is,*

$$\mathcal{T}^{\mathrm{F}}_{W_{o,D},W_{i,D}|e_{D,n}}(\mathbf{p}) = \mathcal{T}^{\mathrm{D}}_{W_{o,D},W_{i,D}|e_{D,n}}(\mathbf{p}). \tag{97}$$

Proof. It follows from Eqs. (82), (83) and (90) that

$$\begin{aligned} &\mathcal{T}^{\mathrm{D}}_{W_{o,D},W_{i,D}|e_{D,n}}(\mathbf{p}) = \Gamma_{o,d}(\mathbf{p}) \Gamma_{i,d}^{-1}(\mathbf{p}) \\ &= W_o I_{n_{[\cdot,D]}} \mathrm{adj}\left(\mathbf{p}^2 M_{[D,D]} + K_{[D,D]}\right) K_{[D,\cdot]} e_{D,n} (W_i I_{n_{[\cdot,D]}} \mathrm{adj}(\mathbf{p}^2 M_{[D,D]} + K_{[D,D]}) K_{[D,\cdot]} e_{D,n})^{-1}. \end{aligned} \tag{98}$$

Using Eq. (A.2) in Appendix A, we have

$$\begin{aligned} &W_o I_{n_{[\cdot,D]}} \mathrm{adj}\left(\mathbf{p}^2 M_{[D,D]} + K_{[D,D]}\right) K_{[D,\cdot]} e_{D,n} (W_i I_{n_{[\cdot,D]}} \mathrm{adj}\left(\mathbf{p}^2 M_{[D,D]} + K_{[D,D]}\right) K_{[D,\cdot]} e_{D,n})^{-1} \\ &= W_{o,D} \mathrm{adj}\left(\mathbf{p}^2 M + K\right) e_{D,n} (W_{i,D} \mathrm{adj}(\mathbf{p}^2 M + K) e_{D,n})^{-1}. \end{aligned} \tag{99}$$

Therefore, Eq. (98) becomes

$$\mathcal{T}^{\mathrm{D}}_{W_{o,D},W_{i,D}|e_{D,n}}(\mathbf{p}) = W_{o,D} \mathrm{adj}(\mathbf{p}^2 M + K) e_{D,n} (W_{i,D} \mathrm{adj}(\mathbf{p}^2 M + K) e_{D,n})^{-1}. \tag{100}$$

Next, replacing W_o, W_i, and B in Eqs. (42) and (43) with $W_{o,D}$, $W_{i,D}$, and $e_{D,n}$, respectively, Eq. (53) becomes

$$\begin{aligned} &\mathcal{T}^{\mathrm{F}}_{W_{o,D},W_{i,D}|e_{D,n}}(\mathbf{p}) = \Gamma_o(\mathbf{p}) \Gamma_i^{-1}(\mathbf{p}) \\ &= W_{o,D} \mathrm{adj}(\mathbf{p}^2 M + K) e_{D,n} (W_{i,D} \mathrm{adj}(\mathbf{p}^2 M + K) e_{D,n})^{-1}. \end{aligned} \tag{101}$$

Comparing Eq. (100) with Eq. (101) yields Eq. (97). □

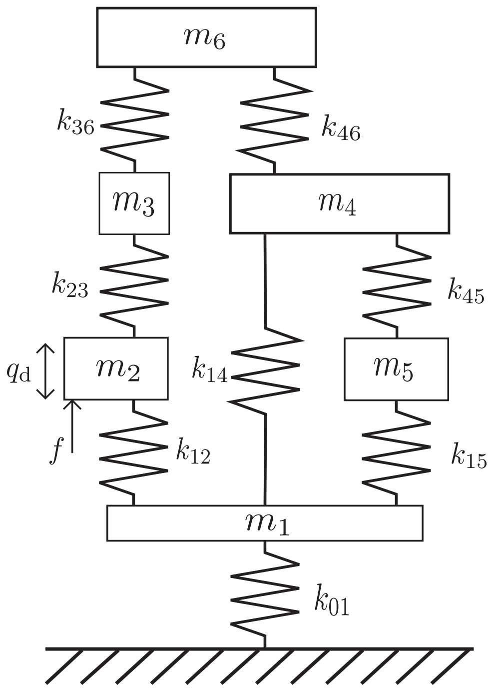

Fig. 3. Mass–spring system for Example 3, where $m_1=m_2=m_3=m_4=m_5=m_6=1$ kg and $k_{01}=k_{12}=k_{14}=k_{15}=k_{23}=k_{36}=k_{45}=k_{46}=1$ N/m. m_2 is either force-driven by the force f or displacement-driven with the prescribed motion q_d.

9. Numerical examples

In this section we present three examples to illustrate the equality of transmissibilities in force-driven and displacement-driven structures.

Example 3. Consider the mass-spring system shown in Fig. 3, where $m_1=m_2=m_3=m_4=m_5=m_6=1$ kg and $k_{01}=k_{12}=k_{14}=k_{15}=k_{23}=k_{36}=k_{45}=k_{46}=1$ N/m. We force-driven m_2 and consider the transmissibility from q_1 to q_6. Then we displacement-driven m_2 and consider the transmissibility from q_1 to q_6. Note that $M=I_6$, $M_{[2,2]}=I_5$,

$$K=\begin{bmatrix} 4 & -1 & 0 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{bmatrix}, \quad K_{[2,2]}=\begin{bmatrix} 4 & 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1 \\ -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{bmatrix}. \tag{102}$$

It follows that

$$\operatorname{adj}(\mathbf{p}^2M+K)e_{2,6}=\begin{bmatrix} \mathbf{p}^8+9\mathbf{p}^6+27\mathbf{p}^4+32\mathbf{p}^2+14 \\ \mathbf{p}^{10}+13\mathbf{p}^8+61\mathbf{p}^6+124\mathbf{p}^4+102\mathbf{p}^2+25 \\ \mathbf{p}^8+11\mathbf{p}^6+40\mathbf{p}^4+54\mathbf{p}^2+22 \\ \mathbf{p}^6+8\mathbf{p}^4+21\mathbf{p}^2+16 \\ \mathbf{p}^6+8\mathbf{p}^4+19\mathbf{p}^2+15 \\ \mathbf{p}^6+10\mathbf{p}^4+28\mathbf{p}^2+19 \end{bmatrix}, \tag{103}$$

$$\operatorname{adj}(\mathbf{p}^2M_{[2,2]}+K_{[2,2]})K_{[2,\cdot]}e_{2,6}=-\begin{bmatrix} \mathbf{p}^8+9\mathbf{p}^6+27\mathbf{p}^4+32\mathbf{p}^2+14 \\ \mathbf{p}^8+11\mathbf{p}^6+40\mathbf{p}^4+54\mathbf{p}^2+22 \\ \mathbf{p}^6+8\mathbf{p}^4+21\mathbf{p}^2+16 \\ \mathbf{p}^6+8\mathbf{p}^4+19\mathbf{p}^2+15 \\ \mathbf{p}^6+10\mathbf{p}^4+28\mathbf{p}^2+19 \end{bmatrix}. \tag{104}$$

Next, it follows from Eq. (31) with $w_o = e_{6,6}^{\mathrm{T}}, w_i = e_{1,6}^{\mathrm{T}}$, and $b = e_{2,6}$ that

$$\mathcal{T}^{\mathrm{F}}_{e_{6,6}^{\mathrm{T}}, e_{1,6}^{\mathrm{T}}|e_{2,6}}(\mathbf{p}) = \frac{\mathbf{p}^6 + 10\mathbf{p}^4 + 28\mathbf{p}^2 + 19}{\mathbf{p}^8 + 9\mathbf{p}^6 + 27\mathbf{p}^4 + 32\mathbf{p}^2 + 14}. \tag{105}$$

Similarly, it follows from Eq. (72) with $w_o = e_{6,6}^{\mathrm{T}}, w_i = e_{1,6}^{\mathrm{T}}$, and $b = e_{2,6}$ that

$$\mathcal{T}^{\mathrm{D}}_{e_{6,6}^{\mathrm{T}}, e_{1,6}^{\mathrm{T}}|e_{2,6}}(\mathbf{p}) = \frac{\mathbf{p}^6 + 10\mathbf{p}^4 + 28\mathbf{p}^2 + 19}{\mathbf{p}^8 + 9\mathbf{p}^6 + 27\mathbf{p}^4 + 32\mathbf{p}^2 + 14}. \tag{106}$$

Hence,

$$\mathcal{T}^{\mathrm{F}}_{e_{6,6}^{\mathrm{T}}, e_{1,6}^{\mathrm{T}}|e_{2,6}}(\mathbf{p}) = \mathcal{T}^{\mathrm{D}}_{e_{6,6}^{\mathrm{T}}, e_{1,6}^{\mathrm{T}}|e_{2,6}}(\mathbf{p}).$$

Example 4. Consider a simply supported beam with a uniform density ρ per unit length, modulus of elasticity E, moment of inertia I, length L, and rectangular cross section with area A. We consider first the force-driven case by applying a concentrated transverse force at the location x_a, where $0 < x_a < L$. Let $y(t,x)$ denote the displacement of the beam from its equilibrium shape, and let $\delta(x - x_a)f(t)$ denote the external force. The beam is modeled by

$$\frac{\partial^4}{\partial x^4} y(t,x) + \frac{\rho A}{EI}\frac{\partial^2}{\partial t^2} y(t,x) = \delta(x - x_a)f(t). \tag{107}$$

Let

$$y(t,x) = \sum_{i=1}^{\infty} q_i(t) v_i(x), \tag{108}$$

where q_i is the modal coordinate corresponding to the mode shape $v_i(x) = \sin(i\pi x/L)$. Substituting Eq. (108) in Eq. (107) and taking the inner product of both sides of the resulting equation with $v_i(x_a)$ yields

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = b_i f(t), \quad i = 1, 2, 3, \ldots, \tag{109}$$

where $\omega_i = i^2\pi^2/L^2\sqrt{EI/\rho A}$ is the modal frequency corresponding to $v_i(x)$ and $b_i \triangleq v_i(x_a)$. Defining

$$q(t) \triangleq [q_1(t) \ \cdots \ q_r(t)]^{\mathrm{T}}, \quad b \triangleq [b_1 \ \cdots \ b_r]^{\mathrm{T}}, \tag{110}$$

it follows from Eq. (109) that

$$\ddot{q}(t) + \Omega^2 q(t) = bf(t), \tag{111}$$

where $\Omega^2 \triangleq \mathrm{diag}(\omega_1^2, \ldots, \omega_r^2)$.

In the displacement-driven case we assume that the interior point x_a moves with the specified displacement $q_d(t,x_a) = \sum_{i=1}^{r} q_i(t) v_i(x_a)$. We define the coordinates

$$\hat{q}(t) \triangleq S^{-\mathrm{T}} q(t), \tag{112}$$

where

$$S \triangleq \begin{bmatrix} I_{r-1} & & 0_{(r-1)\times 1} \\ v_1(x_a) & \cdots & v_r(x_a). \end{bmatrix}^{-\mathrm{T}}, \tag{113}$$

where to ensure nonsingularity we assume that $v_r(x_a) \neq 0$. Then, the resulting coordinates are

$$\hat{q}(t) = [q_1(t) \ \cdots \ q_{r-1}(t) \ \ q_d(t,x_a)]^{\mathrm{T}}. \tag{114}$$

Using Eqs. (112) and (111) yields

$$\hat{M}\ddot{\hat{q}}(t) + \hat{K}\hat{q}(t) = \hat{B}f(t), \tag{115}$$

where $\hat{M} \triangleq SS^{\mathrm{T}}$, $\hat{K} \triangleq S\Omega^2 S^{\mathrm{T}}$, $\hat{B} = Sb \triangleq e_{n,n}$.

Driving x_a with a prescribed motion requires applying a suitable force as in Eq. (107). As in Section 5 we remove the rth equation of Eq. (115) and manipulate the remaining equations to make $q_d(t,x_a)$ the input. Therefore, Eq. (115) becomes

$$\hat{M}_{[r,r]}\ddot{q}_{[r]} + \hat{K}_{[r,r]} q_{[r]} = -\hat{K}_{[r,\cdot]} e_{k,n} q_d(t,x_a). \tag{116}$$

Suppose that $E = 200$ GPa, $L = 100$ mm, $h = 10$ mm, $w = 1$ mm, $x_a = 83.3$ mm, and $x_s = 21.1$ mm. The transmissibility from x_a to x_s for the force-driven beam is given by

$$\begin{aligned}\mathcal{T}^{\mathrm{F}}_{v^{\mathrm{T}}(x_s,r), v^{\mathrm{T}}(x_a,r)|v(x_a)} &= \frac{v^{\mathrm{T}}(x_s,r)\mathrm{adj}(\mathbf{p}^2\hat{M} + \hat{K})v(x_a)}{v^{\mathrm{T}}(x_a,r)\mathrm{adj}(\mathbf{p}^2\hat{M} + \hat{K})v(x_a)} \\ &= \frac{\mathbf{p}^6 + 156.4\mathbf{p}^4 - 1.814\times 10^4\mathbf{p}^2 + 3.454\times 10^6}{63.38\mathbf{p}^6 + 1.426\times 10^4\mathbf{p}^4 + 8.057\times 10^5\mathbf{p}^2 + 9.591\times 10^6},\end{aligned} \tag{117}$$

where $v^{\mathrm{T}}(x_s, r)$ and $v^{\mathrm{T}}(x_a, r)$ denote $v^{\mathrm{T}}(x_s)$ and $v^{\mathrm{T}}(x_a)$, respectively, after setting the rth component of $v^{\mathrm{T}}(x_s, r)$ and $v^{\mathrm{T}}(x_a, r)$ to zero as suggested by Theorem 3. Next, with a prescribed motion at x_a, the transmissibility from x_a to x_s is given by

$$\mathcal{T}^{\mathrm{D}}_{v^{\mathrm{T}}(x_s,r),v^{\mathrm{T}}(x_a,r)|v^{\mathrm{T}}(x_a,r)} = \frac{v^{\mathrm{T}}(x_s,r)I_{[r,\cdot]}\mathrm{adj}(\mathbf{p}^2\hat{M}_{[r,r]}+\hat{K}_{[r,r]})K_{[r,\cdot]}e^{\mathrm{T}}_{r,r}}{v^{\mathrm{T}}(x_a,r)I_{[r,\cdot]}\mathrm{adj}(\mathbf{p}^2\hat{M}_{[r,r]}+\hat{K}_{[r,r]})K_{[r,\cdot]}e^{\mathrm{T}}_{r,r}}
= \frac{\mathbf{p}^6+156.4\mathbf{p}^4-1.814\times10^4\mathbf{p}^2+3.454\times10^6}{63.38\mathbf{p}^6+1.426\times10^4\mathbf{p}^4+8.057\times10^5\mathbf{p}^2+9.591\times10^6},$$

which is equivalent to Eq. (117).

Example 5. Consider the mass–spring system shown in Fig. 3, where $m_1=m_2=m_3=m_4=m_5=m_6=1$ kg and $k_{01}=k_{12}=k_{14}=k_{15}=k_{23}=k_{36}=k_{45}=k_{46}=1$ N/m. We force-drive m_2 and m_3 and consider the transmissibility from $[q_1\ q_4]^{\mathrm{T}}$ to $[q_5\ q_6]^{\mathrm{T}}$. Then we displacement-drive m_2 and m_3 and consider the transmissibility from $[q_1\ q_4]^{\mathrm{T}}$ to $[q_5\ q_6]^{\mathrm{T}}$. Note that $D=\{2,3\}$, $M=I_6$, $M_{[D,D]}=I_4$, $W_o=[e_{5,6}\ e_{6,6}]^{\mathrm{T}}$, and $W_i=[e_{1,6}\ e_{4,6}]^{\mathrm{T}}$. Hence, we have

$$K=\begin{bmatrix}4&-1&0&-1&-1&0\\-1&2&-1&0&0&0\\0&-1&2&0&0&-1\\-1&0&0&3&-1&-1\\-1&0&0&-1&2&0\\0&0&-1&-1&0&2\end{bmatrix},\quad K_{[D,D]}=\begin{bmatrix}4&-1&-1&0\\-1&3&-1&-1\\-1&-1&2&0\\0&-1&0&2\end{bmatrix}. \tag{118}$$

It follows that

$$\mathrm{adj}(\mathbf{p}^2M+K)[e_{2,6}\ \ e_{3,6}]=\begin{bmatrix}\mathbf{p}^8+9\mathbf{p}^6+27\mathbf{p}^4+32\mathbf{p}^2+14 & \mathbf{p}^6+8\mathbf{p}^4+19\mathbf{p}^2+14\\ \mathbf{p}^{10}+13\mathbf{p}^8+61\mathbf{p}^6+124\mathbf{p}^4+102\mathbf{p}^2+25 & \mathbf{p}^8+11\mathbf{p}^6+40\mathbf{p}^4+54\mathbf{p}^2+22\\ \mathbf{p}^8+11\mathbf{p}^6+40\mathbf{p}^4+54\mathbf{p}^2+22 & \mathbf{p}^{10}+13\mathbf{p}^8+61\mathbf{p}^6+126\mathbf{p}^4+111\mathbf{p}^2+30\\ \mathbf{p}^6+8\mathbf{p}^4+21\mathbf{p}^2+16 & \mathbf{p}^6+9\mathbf{p}^4+23\mathbf{p}^2+18\\ \mathbf{p}^6+8\mathbf{p}^4+19\mathbf{p}^2+15 & 2\mathbf{p}^4+13\mathbf{p}^2+16\\ \mathbf{p}^6+10\mathbf{p}^4+28\mathbf{p}^2+19 & \mathbf{p}^8+11\mathbf{p}^6+40\mathbf{p}^4+55\mathbf{p}^2+24\end{bmatrix}. \tag{119}$$

Using Eqs. (42) and (43) we have

$$\Gamma_i(\mathbf{p})=W_i\mathrm{adj}(\mathbf{p}^2M+K)[e_{2,6}\ \ e_{3,6}]=\begin{bmatrix}\mathbf{p}^8+9\mathbf{p}^6+27\mathbf{p}^4+32\mathbf{p}^2+14 & \mathbf{p}^6+8\mathbf{p}^4+19\mathbf{p}^2+14\\ \mathbf{p}^6+8\mathbf{p}^4+21\mathbf{p}^2+16 & \mathbf{p}^6+9\mathbf{p}^4+23\mathbf{p}^2+18\end{bmatrix}, \tag{120}$$

$$\Gamma_o(\mathbf{p})=W_o\mathrm{adj}(\mathbf{p}^2M+K)[e_{2,6}\ \ e_{3,6}]=\begin{bmatrix}\mathbf{p}^6+8\mathbf{p}^4+19\mathbf{p}^2+15 & 2\mathbf{p}^4+13\mathbf{p}^2+16\\ \mathbf{p}^6+10\mathbf{p}^4+28\mathbf{p}^2+19 & \mathbf{p}^8+11\mathbf{p}^6+40\mathbf{p}^4+55\mathbf{p}^2+24\end{bmatrix}. \tag{121}$$

Moreover,

$$\mathrm{adj}(\mathbf{p}^2M_{[D,D]}+K_{[D,D]})K_{[D,\cdot]}[e_{2,6}\ \ e_{3,6}]=-\begin{bmatrix}\mathbf{p}^6+7\mathbf{p}^4+14\mathbf{p}^2+8 & \mathbf{p}^2+3\\ \mathbf{p}^4+5\mathbf{p}^2+6 & \mathbf{p}^4+6\mathbf{p}^2+7\\ \mathbf{p}^4+6\mathbf{p}^2+7 & \mathbf{p}^2+5\\ \mathbf{p}^2+3 & \mathbf{p}^6+9\mathbf{p}^4+23\mathbf{p}^2+13\end{bmatrix}. \tag{122}$$

It follows from Eqs. (82) and (83) that

$$\Gamma_{i,d}=-W_iI_{[\cdot,D]}\mathrm{adj}(\mathbf{p}^2M_{[D,D]}+K_{[D,D]})K_{[D,\cdot]}[e_{2,6}\ \ e_{3,6}]=\begin{bmatrix}\mathbf{p}^6+7\mathbf{p}^4+14\mathbf{p}^2+8 & \mathbf{p}^2+3\\ \mathbf{p}^4+5\mathbf{p}^2+6 & \mathbf{p}^4+6\mathbf{p}^2+7\end{bmatrix}, \tag{123}$$

$$\Gamma_{o,d}=-W_oI_{[\cdot,D]}\mathrm{adj}(\mathbf{p}^2M_{[D,D]}+K_{[D,D]})K_{[D,\cdot]}[e_{2,6}\ \ e_{3,6}]=\begin{bmatrix}\mathbf{p}^4+6\mathbf{p}^2+7 & \mathbf{p}^2+5\\ \mathbf{p}^2+3 & \mathbf{p}^6+9\mathbf{p}^4+23\mathbf{p}^2+13\end{bmatrix}. \tag{124}$$

Therefore,

$$\begin{aligned}\det\Gamma_{i,d}(\mathbf{p})\Gamma_o(\mathbf{p})\mathrm{adj}\,\Gamma_i(\mathbf{p}) &= (\mathbf{p}^{10}+13\mathbf{p}^8+62\mathbf{p}^6+133\mathbf{p}^4+125\mathbf{p}^2+38)\\ &\cdot\begin{bmatrix}\mathbf{p}^6+8\mathbf{p}^4+19\mathbf{p}^2+15 & 2\mathbf{p}^4+13\mathbf{p}^2+16\\ \mathbf{p}^6+10\mathbf{p}^4+28\mathbf{p}^2+19 & \mathbf{p}^8+11\mathbf{p}^6+40\mathbf{p}^4+55\mathbf{p}^2+24\end{bmatrix}\begin{bmatrix}\mathbf{p}^6+9\mathbf{p}^4+23\mathbf{p}^2+18 & -\mathbf{p}^6-8\mathbf{p}^4-19\mathbf{p}^2-14\\ -\mathbf{p}^6-8\mathbf{p}^4-21\mathbf{p}^2-16 & \mathbf{p}^8+9\mathbf{p}^6+27\mathbf{p}^4+32\mathbf{p}^2+14\end{bmatrix}\\ &=\begin{bmatrix}A_{1,1}(\mathbf{p}) & A_{1,2}(\mathbf{p})\\ A_{2,1}(\mathbf{p}) & A_{2,2}(\mathbf{p})\end{bmatrix},\end{aligned} \tag{125}$$

where

$$A_{1,1}(\mathbf{p}) = \mathbf{p}^{22} + 28\mathbf{p}^{20} + 342\mathbf{p}^{18} + 2394\mathbf{p}^{16} + 10,611\mathbf{p}^{14} + 31,052\mathbf{p}^{12} + 60,672\mathbf{p}^{10} + 78,167\mathbf{p}^{8} + 63,850\mathbf{p}^{6} + 30,491\mathbf{p}^{4} + 7184\mathbf{p}^{2} + 532, \quad (126)$$

$$A_{1,2}(\mathbf{p}) = A_{1,1}(\mathbf{p}), \quad (127)$$

$$A_{2,1}(\mathbf{p}) = -\mathbf{p}^{24} - 31\mathbf{p}^{22} - 426\mathbf{p}^{20} - 3420\mathbf{p}^{18} - 17793\mathbf{p}^{16} - 62,885\mathbf{p}^{14} - 153,828\mathbf{p}^{12} - 260,183\mathbf{p}^{10} - 298,351\mathbf{p}^{8} - 222,041\mathbf{p}^{6} - 98,657\mathbf{p}^{4} - 22,084\mathbf{p}^{2} - 1596, \quad (128)$$

$$A_{2,2}(\mathbf{p}) = \mathbf{p}^{26} + 33\mathbf{p}^{24} + 487\mathbf{p}^{22} + 4244\mathbf{p}^{20} + 24,291\mathbf{p}^{18} + 96,077\mathbf{p}^{16} + 268,987\mathbf{p}^{14} + 536,787\mathbf{p}^{12} + 758,045\mathbf{p}^{10} + 740,576\mathbf{p}^{8} + 478,889\mathbf{p}^{6} + 188,907\mathbf{p}^{4} + 38,580\mathbf{p}^{2} + 2660. \quad (129)$$

Moreover,

$$\det \Gamma_i(\mathbf{p})\Gamma_{o,d}(\mathbf{p})\mathrm{adj}\,\Gamma_{i,d}(\mathbf{p}) = (\mathbf{p}^{14} + 17\mathbf{p}^{12} + 115\mathbf{p}^{10} + 396\mathbf{p}^{8} + 735\mathbf{p}^{6} + 709\mathbf{p}^{4} + 300\mathbf{p}^{2} + 28) \cdot \begin{bmatrix} \mathbf{p}^4 + 6\mathbf{p}^2 + 7 & \mathbf{p}^2 + 5 \\ \mathbf{p}^2 + 3 & \mathbf{p}^6 + 9\mathbf{p}^4 + 23\mathbf{p}^2 + 13 \end{bmatrix} \begin{bmatrix} \mathbf{p}^4 + 6\mathbf{p}^2 + 7 & -\mathbf{p}^2 - 3 \\ -\mathbf{p}^4 - 5\mathbf{p}^2 - 6 & \mathbf{p}^6 + 7\mathbf{p}^4 + 14\mathbf{p}^2 + 8 \end{bmatrix} = \begin{bmatrix} A_{d,1,1}(\mathbf{p}) & A_{d,1,2}(\mathbf{p}) \\ A_{d,2,1}(\mathbf{p}) & A_{d,2,2}(\mathbf{p}) \end{bmatrix}, \quad (130)$$

where

$$A_{d,1,1}(\mathbf{p}) = \mathbf{p}^{22} + 28\mathbf{p}^{20} + 342\mathbf{p}^{18} + 2394\mathbf{p}^{16} + 10,611\mathbf{p}^{14} + 31,052\mathbf{p}^{12} + 60,672\mathbf{p}^{10} + 78,167\mathbf{p}^{8} + 63,850\mathbf{p}^{6} + 30,491\mathbf{p}^{4} + 7184\mathbf{p}^{2} + 532, \quad (131)$$

$$A_{d,1,2}(\mathbf{p}) = A_{d,1,1}(\mathbf{p}), \quad (132)$$

$$A_{d,2,1}(\mathbf{p}) = -\mathbf{p}^{24} - 31\mathbf{p}^{22} - 426\mathbf{p}^{20} - 3420\mathbf{p}^{18} - 17,793\mathbf{p}^{16} - 62,885\mathbf{p}^{14} - 153,828\mathbf{p}^{12} - 260,183\mathbf{p}^{10} - 298,351\mathbf{p}^{8} - 222,041\mathbf{p}^{6} - 98,657\mathbf{p}^{4} - 22,084\mathbf{p}^{2} - 1596, \quad (133)$$

$$A_{d,2,2}(\mathbf{p}) = \mathbf{p}^{26} + 33\mathbf{p}^{24} + 487\mathbf{p}^{22} + 4244\mathbf{p}^{20} + 24,291\mathbf{p}^{18} + 96,077\mathbf{p}^{16} + 268,987\mathbf{p}^{14} + 536,787\mathbf{p}^{12} + 758,045\mathbf{p}^{10} + 740,576\mathbf{p}^{8} + 478,889\mathbf{p}^{6} + 188,907\mathbf{p}^{4} + 38,580\mathbf{p}^{2} + 2660. \quad (134)$$

Comparing Eqs. (126), (127), (128), and (129) with Eqs. (131), (132), (133), and (134), respectively, yields,

$$A_{1,1} = A_{d,1,1}, \quad A_{1,2} = A_{d,1,2}, \quad A_{2,1} = A_{d,2,1}, \quad A_{2,2} = A_{d,2,2}. \quad (135)$$

Therefore, it follows from Eqs. (125) and (130) that

$$\det \Gamma_{i,d}(\mathbf{p})\Gamma_o(\mathbf{p})\mathrm{adj}\,\Gamma_i(\mathbf{p}) = \det \Gamma_i(\mathbf{p})\Gamma_{o,d}(\mathbf{p})\mathrm{adj}\,\Gamma_{i,d}(\mathbf{p}). \quad (136)$$

That is,

$$\mathcal{T}^{\mathrm{F}}_{W_o,W_i|e_{D,n}}(\mathbf{p}) = \mathcal{T}^{\mathrm{D}}_{W_o,W_i|e_{D,n}}(\mathbf{p}), \quad (137)$$

which confirms Theorem 4.

10. Conclusions and future research

Transmissibility estimates are traditionally obtained using only frequency-domain methods, which are based on the assumption that the input and output signals are stationary, and thus initial conditions and transient effects are either assumed to be absent or are ignored. We showed that ignoring the initial conditions and transient effects can degrade the transmissibility estimates in the frequency-domain. Moreover, we showed that frequency-domain identification techniques cannot give exact estimates with finite data sets. Therefore, we developed a time-domain framework for SISO and MIMO transmissibilities that accounts for nonzero initial conditions for both force-driven and displacement-driven structures. It was shown that if the locations of the forces and prescribed displacements are identical, then the SISO and MIMO force- and displacement-driven transmissibilities are equal. Numerical examples for a mass–spring system and a simply supported beam were presented to illustrate the equality of transmissibilities in force-driven and displacement-driven structures.

The time-domain transmissibility models developed in this paper are intended to facilitate the use of time-domain identification methods. Preliminary results in this direction are given in [19–21].

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Appendix A. Adjugate identities

Let $A \in \mathbb{C}^{n\times n}$, let $A^{\mathrm{A}} \in \mathbb{C}^{n\times n}$ denote the adjugate of A, and let $A_{(i,j)} \in \mathbb{C}$ denote the (i,j) entry of A. Let $D \triangleq \{k_1, \ldots, k_d\}$ where $1 \le d \le n-2$ and $k_i \in \{1, \ldots, n\}$ for all $i=1,\ldots,d$. Let $A_{[D,\cdot]} \in \mathbb{C}^{(n-d)\times n}$ denote A with rows $k_1, \ldots, k_d$ removed and let $A_{[D,D]} \in \mathbb{C}^{(n-d)\times(n-d)}$ denote A with rows $k_1, \ldots, k_d$ removed and columns $k_1, \ldots, k_d$ removed. Finally, Let $e_{D,n} \triangleq [e_{k_1,n} \; \ldots \; e_{k_d,n}] \in \mathbb{C}^{n\times d}$ where $e_{i,n} \in \mathbb{C}^n$ denotes the ith unit vector.

Adjugate identity 1: For all $i \in \{1, \ldots, n\}$,

$$\left[(A^{\mathrm{A}})_{[i,\cdot]} + (A_{[i,i]})^{\mathrm{A}} A_{[i,\cdot]}\right] e_{i,n} = 0_{(n-1)\times 1}. \tag{A.1}$$

Proof. See [25]. □

Adjugate identity 2: Let $C \in \mathbb{C}^{d\times n}$ and define $R \triangleq A^{\mathrm{A}} e_{D,n} \in \mathbb{C}^{n\times d}$ and $S \triangleq (A_{[D,D]})^{\mathrm{A}} A_{[D,\cdot]} e_{D,n} \in \mathbb{C}^{(n-d)\times d}$. Let $CR \in \mathbb{C}^{d\times d}$ and $C_{[\cdot,D]} S \in \mathbb{C}^{d\times d}$ be nonsingular where $C_{[\cdot,D]} \in \mathbb{C}^{d\times(n-d)}$ denotes C with columns $k_1, \ldots, k_d$ removed. Then,

$$I_{n,[D,\cdot]} R (CR)^{-1} = S (C_{[\cdot,D]} S)^{-1}, \tag{A.2}$$

where $I_n \in \mathbb{C}^{n\times n}$ is the identity matrix and $I_{n,[D,\cdot]} \in \mathbb{C}^{(n-d)\times n}$ denotes I_n with rows $k_1, \ldots, k_d$ removed.

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