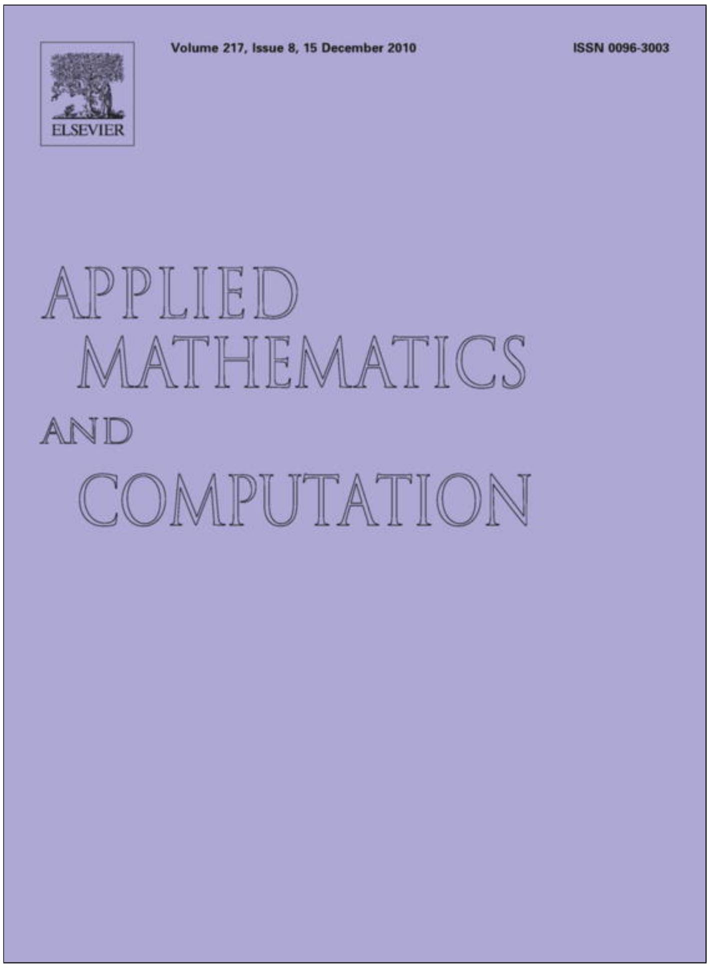
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On the equality between rank and trace of an idempotent matrix

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ABSTRACT

The paper was inspired by the question whether it is possible to derive the equality between the rank and trace of an idempotent matrix by using only the idempotency property, without referring to any further features of the matrix. It is shown that such a proof can be obtained by exploiting a general characteristic of the rank of any matrix. An original proof of this characteristic is provided, which utilizes a formula for the Moore–Penrose inverse of a partitioned matrix. Further consequences of the rank property are discussed, in particular, several additional facts are established with considerably simpler proofs than those available. Moreover, a collection of new results referring to the coincidence between rank and trace of an idempotent matrix are derived as well.

1. Introduction

It is known that the rank of an idempotent matrix (also called an oblique projector) coincides with its trace. There are several alternative proofs of this fact available in the literature, all of which refer to some further property of an idempotent matrix, and not only to the requirement that the second power of the matrix coincides with itself. This property can deal with, for example, full rank decomposition [1, Theorem 3.6.4], spectrum or Jordan form [2, Corollary 2.12], and singular value decomposition [3, Lemma 1]. In the present paper, we derive the equality between the rank and trace of an idempotent matrix without referring to any other feature of the matrix than its idempotency. This aim is achieved by exploiting a general characteristic of the rank of any matrix, whose original proof, based on a formula for the Moore–Penrose inverse of a partitioned matrix, is provided as well. Moreover, the rank characteristic enables us to derive other facts known in the literature with considerably simpler proofs than those available. Section 3 of the paper provides some additional results referring to the coincidence between the rank and trace of an idempotent matrix.

In what follows, the set of $m \times n$ complex matrices is denoted by $\mathbb{C}_{m,n}$. The symbols $\mathbf{A}^*$, $\mathcal{R}(\mathbf{A})$, and $\mathrm{rk}(\mathbf{A})$ stand for conjugate transpose, column space (range), and rank of $\mathbf{A} \in \mathbb{C}_{m,n}$, whereas $\mathrm{tr}(\mathbf{A})$ denotes trace of $\mathbf{A} \in \mathbb{C}_{n,n}$. Furthermore, $\mathbf{A}^\dagger \in \mathbb{C}_{n,m}$ is the Moore–Penrose inverse of $\mathbf{A} \in \mathbb{C}_{m,n}$, i.e., the unique solution to the equations:

$$\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}, \quad \mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger, \quad (\mathbf{A}\mathbf{A}^\dagger)^* = \mathbf{A}\mathbf{A}^\dagger, \quad (\mathbf{A}^\dagger\mathbf{A})^* = \mathbf{A}^\dagger\mathbf{A}.$$

Finally, $\mathbf{I}_n$ stands for the identity matrix of order n.

Some of the derivations in Section 3 are based on the matrix decomposition, originating from the singular value decomposition, established in [4, Corollary 6], which is recalled in the lemma below.

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Lemma. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be of rank r. Then there exists unitary* $\mathbf{U} \in \mathbb{C}_{n,n}$ *such that*

$$\mathbf{A} = \mathbf{U}\begin{pmatrix} \mathbf{\Sigma K} & \mathbf{\Sigma L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*, \tag{1.1}$$

where $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1\mathbf{I}_{r_1}, \ldots, \sigma_t\mathbf{I}_{r_t})$ *is the diagonal matrix of singular values of* $\mathbf{A}$, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, *and* $\mathbf{K} \in \mathbb{C}_{r,r}, \mathbf{L} \in \mathbb{C}_{r,n-r}$ *satisfy* $\mathbf{KK}^* + \mathbf{LL}^* = \mathbf{I}_r$.

With the representation established in the preceding Lemma, several useful characterizations of the matrix **A** can be derived. For instance, direct calculations show that **A** is:

(i) idempotent if and only if $\mathbf{\Sigma K} = \mathbf{I}_r$,
(ii) EP, i.e., $\mathbf{A}^\dagger\mathbf{A} = \mathbf{AA}^\dagger$, or, equivalently $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A}^*)$, if and only if $\mathbf{L} = \mathbf{0}$,
(iii) an orthogonal projector, i.e., $\mathbf{A}^2 = \mathbf{A} = \mathbf{A}^*$, if and only if $\mathbf{L} = \mathbf{0}$, $\mathbf{\Sigma} = \mathbf{I}_r$, $\mathbf{K} = \mathbf{I}_r$,
(iv) nilpotent, i.e., $\mathbf{A}^2 = \mathbf{0}$, if and only if $\mathbf{K} = \mathbf{0}$.

It is also seen that **A** is an orthogonal projector if and only if it is simultaneously idempotent and EP; for further facts on the representation (1.1) see e.g., [5, Section 1].

2. The rank property and its consequences

The theorem below states a fundamental rank property, which is known in the literature; see e.g., [6, Proposition 6.1.6]. We provide a novel proof based on a general formula for the Moore–Penrose inverse of a partitioned matrix.

Theorem. *Let* $\mathbf{A} \in \mathbb{C}_{m,n}$. *Then*

$$\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{AA}^\dagger). \tag{2.1}$$

Proof. The proof is based on the mathematical induction on the number of columns n. Subsequently, the letters **A** and **B** stand for matrices, whereas **a**,**b**,**c**, and **d** denote column vectors.

If $n = 1$, then **A** is a column vector and we denote $\mathbf{A} = \mathbf{a}$. In this case $\mathbf{a}^\dagger = \mathbf{a}^*/\mathbf{a}^*\mathbf{a}$ whenever $\mathbf{a} \neq \mathbf{0}$, and $\mathbf{a}^\dagger = \mathbf{0}$ whenever $\mathbf{a} = \mathbf{0}$. In both situations $\mathrm{rk}(\mathbf{a}) = \mathrm{tr}(\mathbf{aa}^\dagger)$.

Assume now that **A** has n columns and satisfies (2.1). We will show that $\mathrm{rk}(\mathbf{B}) = \mathrm{tr}(\mathbf{BB}^\dagger)$, where **B** is the columnwise partitioned matrix $\mathbf{B} = (\mathbf{A} : \mathbf{a})$. Applying to this matrix the formula for the Moore–Penrose inverse derived by Greville [7, Section 4] (alternatively see [6, Fact 6.5.17]) gives

$$\mathbf{B}^\dagger = \begin{pmatrix} \mathbf{A}^\dagger - \mathbf{db}^* \\ \mathbf{b}^* \end{pmatrix}, \tag{2.2}$$

where

$$\mathbf{b}^* = \begin{cases} \mathbf{c}^\dagger & \text{if } \mathbf{c} \neq \mathbf{0} \\ \gamma\mathbf{d}^*\mathbf{A}^\dagger & \text{if } \mathbf{c} = \mathbf{0}, \end{cases} \tag{2.3}$$

with

$$\mathbf{d} = \mathbf{A}^\dagger\mathbf{a}, \quad \mathbf{c} = (\mathbf{I}_m - \mathbf{AA}^\dagger)\mathbf{a}, \quad \text{and} \quad \gamma = (1 + \mathbf{d}^*\mathbf{d})^{-1}. \tag{2.4}$$

Let us determine $\mathbf{BB}^\dagger$ separately in the two cases characterized by the two specifications of the vector **b** provided in (2.3). In the first of them, when $\mathbf{c} \neq \mathbf{0}$, which is equivalent, by the middle condition in (2.4), to $\mathbf{a} \notin \mathcal{R}(\mathbf{A})$, it follows that

$$\mathbf{BB}^\dagger = (\mathbf{A} : \mathbf{a})\begin{pmatrix} \mathbf{A}^\dagger - \mathbf{dc}^\dagger \\ \mathbf{c}^\dagger \end{pmatrix} = \mathbf{AA}^\dagger - \mathbf{Adc}^\dagger + \mathbf{ac}^\dagger = \mathbf{AA}^\dagger - \mathbf{AA}^\dagger\mathbf{ac}^\dagger + \mathbf{ac}^\dagger = \mathbf{AA}^\dagger + (\mathbf{I}_m - \mathbf{AA}^\dagger)\mathbf{ac}^\dagger = \mathbf{AA}^\dagger + \mathbf{cc}^\dagger.$$

Hence, $\mathrm{tr}(\mathbf{BB}^\dagger) = \mathrm{tr}(\mathbf{AA}^\dagger) + \mathrm{tr}(\mathbf{cc}^\dagger)$. Since $\mathrm{tr}(\mathbf{cc}^\dagger) = 1$ and, by the induction hypothesis, $\mathrm{tr}(\mathbf{AA}^\dagger) = \mathrm{rk}(\mathbf{A})$, we thus have $\mathrm{tr}(\mathbf{BB}^\dagger) = \mathrm{rk}(\mathbf{A}) + 1 = \mathrm{rk}(\mathbf{B})$, where the last equality is a consequence of the assumption $\mathbf{a} \notin \mathcal{R}(\mathbf{A})$.

In the second case, when $\mathbf{c} = \mathbf{0}$, or, equivalently, when $\mathbf{a} \in \mathcal{R}(\mathbf{A})$, we get

$$\mathbf{BB}^\dagger = (\mathbf{A} : \mathbf{a})\begin{pmatrix} \mathbf{A}^\dagger - \gamma\mathbf{dd}^*\mathbf{A}^\dagger \\ \gamma\mathbf{d}^*\mathbf{A}^\dagger \end{pmatrix} = \mathbf{AA}^\dagger - \gamma\mathbf{Add}^*\mathbf{A}^\dagger + \gamma\mathbf{ad}^*\mathbf{A}^\dagger.$$

Utilizing, on the one hand, the fact that the trace of a product of conformable matrices is invariant with respect to the cyclical permutations of those matrices, and, on the other hand, the relationships $\mathbf{A}^\dagger\mathbf{Ad} = \mathbf{A}^\dagger\mathbf{a} = \mathbf{d}$, originating from the first condition in (2.4), gives

$$\mathrm{tr}(\mathbf{BB}^{\dagger}) = \mathrm{tr}(\mathbf{AA}^{\dagger}) - \gamma\mathbf{d}^*\mathbf{A}^{\dagger}\mathbf{Ad} + \gamma\mathbf{d}^*\mathbf{A}^{\dagger}\mathbf{a} = \mathrm{tr}(\mathbf{AA}^{\dagger}) - \gamma\mathbf{d}^*\mathbf{d} + \gamma\mathbf{d}^*\mathbf{d} = \mathrm{tr}(\mathbf{AA}^{\dagger}).$$

In consequence, by the induction hypothesis, $\mathrm{tr}(\mathbf{BB}^{\dagger}) = \mathrm{rk}(\mathbf{A})$. Noting that $\mathbf{a} \in \mathcal{R}(\mathbf{A})$ implies $\mathrm{rk}(\mathbf{B}) = \mathrm{rk}(\mathbf{A})$, we arrive at $\mathrm{tr}(\mathbf{BB}^{\dagger}) = \mathrm{rk}(\mathbf{B})$. □

It is worth mentioning that Theorem can also be established by virtue of the singular value decomposition, or its particular version given in Lemma.

Observe that as a byproduct of the iterative procedure for calculating the Moore–Penrose inverse based on the relationships (2.2)–(2.4), we obtain a basis of the column space of $\mathbf{A} \in \mathbb{C}_{m,n}$. Take the first nonzero column of $\mathbf{A}$, seen from the left. Then, make iterative steps related to the remaining columns of $\mathbf{A}$, in each of them verifying whether the corresponding vector $\mathbf{c}$ calculated by virtue of (2.4) is zero or not. If it is not, the column belongs to the set of $\mathrm{rk}(\mathbf{A})$ vectors constituting a basis of $\mathcal{R}(\mathbf{A})$.

In particular, when $\mathbf{A}$ is square and nonsingular, the procedure described in the proof of Theorem yields the inverse of $\mathbf{A}$.

The two corollaries below recall some important facts known in the literature. These results are accompanied with simple proofs based on Theorem.

Corollary 1. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$ *be idempotent. Then* $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$.

Proof. The property of the trace ensures that $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{AA}^{\dagger}\mathbf{A}) = \mathrm{tr}(\mathbf{A}^2\mathbf{A}^{\dagger})$. Since $\mathbf{A}$ is idempotent, we, thus, obtain $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{AA}^{\dagger})$, whence the assertion follows. □

Below we reestablish known rank characteristics. According to our knowledge, all the proofs of these results available by now are relatively involved in comparison with those provided below; cf. e.g., [1, Theorem 3.8.2].

Corollary 2. *Let* $\mathbf{A} \in \mathbb{C}_{m,n}$. *Then* $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{A}^*) = \mathrm{rk}(\mathbf{A}^{\dagger}) = \mathrm{rk}(\mathbf{AA}^*)$.

Proof. First note that

$$\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{AA}^{\dagger}) = \mathrm{tr}(\mathbf{A}^{\dagger}\mathbf{A}) = \mathrm{tr}[(\mathbf{A}^{\dagger}\mathbf{A})^*] = \mathrm{tr}[\mathbf{A}^*(\mathbf{A}^{\dagger})^*].$$

Hence, in light of $(\mathbf{A}^{\dagger})^* = (\mathbf{A}^*)^{\dagger}$, we arrive at $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}[\mathbf{A}^*(\mathbf{A}^*)^{\dagger}] = \mathrm{rk}(\mathbf{A}^*)$. Furthermore, since $(\mathbf{A}^{\dagger})^{\dagger} = \mathbf{A}$, it is seen that

$$\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{AA}^{\dagger}) = \mathrm{tr}[\mathbf{A}^{\dagger}(\mathbf{A}^{\dagger})^{\dagger}] = \mathrm{rk}(\mathbf{A}^{\dagger}).$$

Finally, by $\mathbf{A}^{\dagger} = \mathbf{A}^*(\mathbf{AA}^*)^{\dagger}$ (see e.g., [6, Proposition 6.1.6]), we have

$$\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{AA}^{\dagger}) = \mathrm{tr}[\mathbf{AA}^*(\mathbf{AA}^*)^{\dagger}] = \mathrm{rk}(\mathbf{AA}^*),$$

which completes the proof. □

3. Further characterizations

In the context of Corollary 1, it is of interest to inquire what additional property, besides $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$, should be possessed by $\mathbf{A}$ in order to imply $\mathbf{A}^2 = \mathbf{A}$. Two such properties are identified in the proposition below, which provides three characterizations known in the literature. The characterizations given in the points (i) and (iii) therein, published as problem [8] and [9, Theorem 2.4], respectively, are not easily accessible (and, thus, not widely known), whence we provide their complete proofs. The proof referring to the point (ii) can be found in [10, Theorem 2], where, however, only sufficiency is shown.

Proposition 1. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$. *Then* $\mathbf{A}$ *is idempotent if and only if any of the following conjunctions holds*:

(i) $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$ *and* $\mathrm{rk}(\mathbf{I}_n - \mathbf{A}) = \mathrm{tr}(\mathbf{I}_n - \mathbf{A})$,
(ii) $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$ *and* $\mathbf{A}^s = \mathbf{A}^t$ *for some* $s, t \in \mathbb{N}, s \neq t$,
(iii) $\mathrm{rk}(\mathbf{A}) \leqslant \mathrm{tr}(\mathbf{A})$ *and* $\mathrm{rk}(\mathbf{I}_n - \mathbf{A}) \leqslant \mathrm{tr}(\mathbf{I}_n - \mathbf{A})$.

Proof. For the proof concerning the conjunction (i), first recall that the idempotency of $\mathbf{A}$ is equivalent to the idempotency of $\mathbf{I}_n - \mathbf{A}$. Thus, the necessity of (i) is directly seen. To show sufficiency, note that the equalities given in (i) entail

$$\mathrm{rk}(\mathbf{I}_n - \mathbf{A}) = \mathrm{tr}(\mathbf{I}_n - \mathbf{A}) = n - \mathrm{tr}(\mathbf{A}) = n - \mathrm{rk}(\mathbf{A}).$$

Since the relationship $\mathrm{rk}(\mathbf{I}_n - \mathbf{A}) = n - \mathrm{rk}(\mathbf{A})$ is a known necessary and sufficient condition for $\mathbf{A}^2 = \mathbf{A}$ (see e.g., [11, Theorem 2]), the present part of the proof is completed.

In view of the above, only the sufficiency of the conditions given in point (iii) of the proposition is to be shown. Its proof was inspired by [9, Theorem 2.4], and utilizes the decomposition established in Lemma. Note that the inequality $\mathrm{rk}(\mathbf{A}) \leqslant \mathrm{tr}(\mathbf{A})$ yields $r \leqslant \mathrm{tr}(\mathbf{\Sigma K})$, whereas $\mathrm{rk}(\mathbf{I}_n - \mathbf{A}) \leqslant \mathrm{tr}(\mathbf{I}_n - \mathbf{A})$ implies $\mathrm{rk}(\mathbf{I}_r - \mathbf{\Sigma K}) + n - r \leqslant n - \mathrm{tr}(\mathbf{\Sigma K})$. Hence, $\mathrm{rk}(\mathbf{I}_r - \mathbf{\Sigma K}) \leqslant r - \mathrm{tr}(\mathbf{\Sigma K})$, and, since the right-hand side of this inequality is either negative or zero, we have $\mathrm{rk}(\mathbf{I}_r - \mathbf{\Sigma K}) = 0$. In consequence, $\mathbf{\Sigma K} = \mathbf{I}_r$, which means that $\mathbf{A}$ is idempotent. □

Two further relevant characterizations, both of which appear to be new, are given in the proposition below. Statement (ii) therein generalizes Note 5 in [12], according to which Hermitian $\mathbf{A} \in \mathbb{C}_{n,n}$ satisfies $\mathbf{A}^2\mathbf{A}^\dagger\mathbf{A}^* = \mathbf{A}^2$, and if $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$ then $\mathrm{rk}(\mathbf{A}) \leqslant \mathrm{tr}(\mathbf{A}^2)$, and the equality holds if and only if $\mathbf{A}^2 = \mathbf{A}$.

Proposition 2. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$. *Then:*

(i) $\mathbf{A}$ *is idempotent if and only if* $\mathrm{rk}(\mathbf{A}) + \mathrm{tr}(\mathbf{A}^2\mathbf{A}^\dagger\mathbf{A}^*) = 2\mathrm{Re}[\mathrm{tr}(\mathbf{A})]$,
(ii) $\mathbf{A}$ *is an orthogonal projector if and only if* $\mathbf{A}$ *is EP and* $\mathrm{rk}(\mathbf{A}) + \mathrm{tr}(\mathbf{A}^*\mathbf{A}) = 2\,\mathrm{Re}[\mathrm{tr}(\mathbf{A})]$,

where $\mathrm{Re}[\mathrm{tr}(\mathbf{A})]$ *denotes the real part of* $\mathrm{tr}(\mathbf{A})$.

Proof. The proof is based on the representation established in Lemma. As can be verified, it follows from (1.1) that

$$\mathbf{A}^* = \mathbf{U}\begin{pmatrix} \mathbf{K}^*\mathbf{\Sigma} & \mathbf{0} \\ \mathbf{L}^*\mathbf{\Sigma} & \mathbf{0} \end{pmatrix}\mathbf{U}^* \quad \text{and} \quad \mathbf{A}^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{K}^*\mathbf{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{L}^*\mathbf{\Sigma}^{-1} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

whence

$$\mathbf{A}^*\mathbf{A} = \mathbf{U}\begin{pmatrix} \mathbf{K}^*\mathbf{\Sigma}^2\mathbf{K} & \mathbf{K}^*\mathbf{\Sigma}^2\mathbf{L} \\ \mathbf{L}^*\mathbf{\Sigma}^2\mathbf{K} & \mathbf{L}^*\mathbf{\Sigma}^2\mathbf{L} \end{pmatrix}\mathbf{U}^* \quad \text{and} \quad \mathbf{A}\mathbf{A}^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*.$$

Utilizing the properties of the trace, we arrive at $\mathrm{tr}(\mathbf{A}^2\mathbf{A}^\dagger\mathbf{A}^*) = \mathrm{tr}(\mathbf{A}\mathbf{A}^\dagger\mathbf{A}^*\mathbf{A}) = \mathrm{tr}(\mathbf{K}^*\mathbf{\Sigma}^2\mathbf{K})$. Thus, the equality on the right-hand side of the equivalence (i) is satisfied if and only if

$$r + \mathrm{tr}(\mathbf{K}^*\mathbf{\Sigma}^2\mathbf{K}) = 2\mathrm{Re}[\mathrm{tr}(\mathbf{\Sigma K})]. \tag{3.1}$$

By observing that (3.1) can be equivalently expressed as:

$$\mathrm{tr}[(\mathbf{I}_r - \mathbf{\Sigma K})(\mathbf{I}_r - \mathbf{\Sigma K})^*] = 0,$$

which holds if and only if $\mathbf{\Sigma K} = \mathbf{I}_r$, we conclude that (3.1) is equivalent to $\mathbf{A}^2 = \mathbf{A}$.

In view of the characterization of EP matrices given in the proof of Proposition 1, it can be easily verified that the conjunction on the right-hand side of the equivalence (ii) is satisfied if and only if $\mathbf{L} = \mathbf{0}$ holds along with (3.1). By virtue of point (i) of the proposition, this means that $\mathbf{A}$ is both EP and idempotent, or, in other words, that $\mathbf{A}$ is an orthogonal projector. □

Note that $2\mathrm{Re}[\mathrm{tr}(\mathbf{A})]$, which appears in both statements of Proposition 2, satisfies $2\mathrm{Re}[\mathrm{tr}(\mathbf{A})] = \mathrm{tr}(\mathbf{A} + \mathbf{A}^*)$.

Theorem 2 in [12] states that when $\mathbf{A} \in \mathbb{C}_{n,n}$ is such that $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$, then $\mathrm{rk}(\mathbf{A}) \leqslant \mathrm{tr}(\mathbf{A}^2\mathbf{A}^\dagger\mathbf{A}^*)$, and equality holds if and only if $\mathbf{A}^2 = \mathbf{A}$. This result is in [12] accompanied by a relatively complicated proof. From point (i) of Proposition 2 we straightforwardly obtain the following more general result.

Corollary 3. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$. *Then* $\mathbf{A}$ *is idempotent if and only if* $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$ *and* $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^2\mathbf{A}^\dagger\mathbf{A}^*)$.

The paper is concluded with some relevant remarks concerning Hermitian idempotent matrices, or, in other words, orthogonal projectors. The first observation is that when $\mathbf{A}$ is idempotent, then each of the equalities $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^\dagger)$ and $\mathrm{tr}(\mathbf{A}\mathbf{A}^*) = \mathrm{tr}(\mathbf{A})$, which can be looked at as modified versions of the condition $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{A})$, is equivalent to the requirement that $\mathbf{A}$ is an orthogonal projector; see e.g., [9, Theorem 3.5].

Let now $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ be orthogonal projectors. It is known that $\mathrm{tr}(\mathbf{PQ}) \leqslant \mathrm{rk}(\mathbf{PQ})$, with equality if and only if $\mathbf{PQ}$ is an orthogonal projector, or, equivalently, $\mathbf{PQ} = \mathbf{QP}$; see e.g., [13, Corollary 1]. Whence, if one of the projectors, say $\mathbf{Q}$, is nonsingular (what means that $\mathbf{Q} = \mathbf{I}_n$), it follows that $\mathrm{rk}(\mathbf{P}) = \mathrm{tr}(\mathbf{P})$. Another upper bound for $\mathrm{tr}(\mathbf{PQ})$ was given in [14, Theorem 1.24] and reads

$$\mathrm{tr}(\mathbf{PQ}) \leqslant \min\{\mathrm{tr}(\mathbf{P}), \mathrm{tr}(\mathbf{Q})\}. \tag{3.2}$$

Equality holds in (3.2) if and only if either $\mathbf{P} - \mathbf{Q}$ or $\mathbf{Q} - \mathbf{P}$ is an orthogonal projector; see [15, Section 3].

Further useful characterizations of a similar type are possible. For example, by exploiting representation (1.1) we obtain the two characterizations given in the proposition below, which, to the best of our knowledge, are not available in the literature. The condition $\mathrm{rk}(\mathbf{A}) = \mathrm{tr}(\mathbf{PQ})$, given in the equivalence (i) therein, was inspired by statement 16 in [16, Theorem 8.1], which reads $\mathrm{tr}[(\mathbf{A}^*\mathbf{A})^2] = \mathrm{tr}[(\mathbf{A}^*)^2\mathbf{A}^2]$ and proves to be equivalent to the requirement that $\mathbf{A}$ is normal. The equality given in the equivalence (i) was obtained from statement 16 in [16, Theorem 8.1] by replacing $\mathbf{A}^*$ with $\mathbf{A}^\dagger$.

Proposition 3. *Let* $\mathbf{A} \in \mathbb{C}_{n,n}$*. Then:*

(i) $\mathbf{A}$ *is EP if and only if* $\text{rk}(\mathbf{A}) = \text{tr}(\mathbf{PQ})$,
(ii) $\mathbf{A}$ *is nilpotent if and only if* $\text{tr}(\mathbf{PQ}) = 0$,

where $\mathbf{P} = \mathbf{AA}^\dagger$ *and* $\mathbf{Q} = \mathbf{A}^\dagger\mathbf{A}$.

The final comments provide further characterizations of known classes of matrices. Let $\mathbf{A} \in \mathbb{C}_{n,n}$. From $\text{tr}[(\mathbf{A}^* - \mathbf{A}^\dagger)(\mathbf{A}^* - \mathbf{A}^\dagger)^*] \geqslant 0$, we obtain

$$2\text{rk}(\mathbf{A}) \leqslant \text{tr}(\mathbf{A}^*\mathbf{A}) + \text{tr}[(\mathbf{A}^*\mathbf{A})^\dagger], \tag{3.3}$$

which provides a new upper bound for the rank of $\mathbf{A}$. Equality holds in (3.3) if and only if $\mathbf{A}$ is a partial isometry, i.e., $\mathbf{A}^* = \mathbf{A}^\dagger$. Similarly, by exploiting $\text{tr}[(\mathbf{PQ} - \mathbf{QP})(\mathbf{PQ} - \mathbf{QP})^*] \geqslant 0$, where $\mathbf{Q}$ and $\mathbf{P}$ are orthogonal projectors, we arrive at $\text{tr}(\mathbf{PQPQ}) \leqslant \text{tr}(\mathbf{PQ})$, with equality holding if and only if $\mathbf{PQ} = \mathbf{QP}$. Finally, we note that when $\mathbf{P} = \mathbf{AA}^\dagger$ and $\mathbf{Q} = \mathbf{A}^\dagger\mathbf{A}$, then $\mathbf{PQ} = \mathbf{QP}$ corresponds to the property of $\mathbf{A}$ known as bi-EPness or weak-EPness [5, p. 2799].

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