Discrete-Time Adaptive Command Following and Disturbance Rejection With Unknown Exogenous Dynamics

Jesse B. Hoagg, Mario A. Santillo, and Dennis S. Bernstein, *Fellow, IEEE*

***Abstract*—We present an adaptive controller that requires limited model information for stabilization, command following, and disturbance rejection for mult-input multi-output minimum-phase discrete-time systems. Specifically, the controller requires knowledge of the open-loop system's relative degree as well as a bound on the first nonzero Markov parameter. Notably, the controller does not require knowledge of the command or the disturbance spectrum as long as the command and disturbance signals are generated by a Lyapunov-stable linear system. Thus, the command and disturbance signals are combinations of discrete-time sinusoids and steps. In addition, the Markov-parameter-based adaptive controller uses feedback action only, and thus does not require a direct measurement of the command or disturbance signals. Using a logarithmic Lyapunov function, we prove global asymptotic convergence for command following and disturbance rejection as well as Lyapunov stability of the adaptive system when the open-loop system is asymptotically stable.**

***Index Terms*—Adaptive control, discrete time, Lyapunov stability.**

I. Introduction

The ADAPTIVE control literature focuses primarily on adaptive stabilization, adaptive tracking, and model reference adaptive control. These adaptive control problems have been approached using parameter-estimation-based adaptive controllers and high-gain adaptive controllers. In addition to stabilization and command following, disturbance rejection is a third common objective, arising in noise control, vibration suppression, and structural control. In the present paper, we consider the combined stabilization, command following, and disturbance rejection problem for uncertain minimum-phase discrete-time systems with command and disturbance signals generated by exogenous dynamics with unknown spectra. Furthermore, unlike adaptive feedforward control, we do not require a direct measurement of the command or disturbance signals.

A discrete-time adaptive feedback disturbance rejection algorithm based on a retrospective performance measure is developed in [1]. The retrospective performance of a system is the performance of the system at the current time assuming that the current controller was used over a past window of time. In [1], the retrospective performance is used in connection with time-series modeling of the plant and the controller to develop an adaptive disturbance rejection algorithm that requires knowledge of only the numerator of the transfer function from the control to the performance, and does not require knowledge of the disturbance spectrum. Extensions of this method and experimental results are given in [2] and [3].

Although the discrete-time adaptive control literature is more limited than the continuous-time literature, there are discrete-time versions of many continuous-time algorithms [4]–[7], as well as adaptive control algorithms unique to discrete time [8], [9]. The authors present in [8] five algorithms for stabilization and command following of single-input single-output (SISO) and multi-input multi-output (MIMO) minimum-phase systems. Although these algorithms require only that the command signal be bounded, they are based on the assumption that an ideal tracking controller exists. Disturbance rejection is not addressed. The authors consider in [10] output regulation with a known plant and an unknown exosystem that generates reference and disturbance signals.

In the present paper, we develop a discrete-time adaptive MIMO output feedback controller for stabilization, command following, and disturbance rejection in minimum-phase systems. This Markov-parameter-based adaptive control algorithm requires knowledge of only the open-loop system's relative degree and a bound on the first nonzero Markov parameter. We assume that the command and disturbance signals are generated by a Lyapunov-stable linear system so that the command and disturbance signals consist of discrete-time sinusoids and steps. However, we do not require any information regarding the spectrum of the command or the disturbance, and we do not require a direct measurement of the command or the disturbance. We prove globally asymptotic command following and disturbance rejection, as well as Lyapunov stability of the closed-loop error system when the open-loop dynamics are asymptotically stable. If there are no command or disturbance signals, then we prove output stabilization, that is, global asymptotic convergence of the output to zero.

The present paper uses three key tools to prove global convergence of the performance variable. First, we use a nonminimal state-space realization of the plant. Similar nonminimal state-space realizations are considered in [9] and [11]. The nonminimal state-space realization has a state that consists entirely of delayed inputs and outputs, which allows us to represent dynamic output feedback as static full-state feedback. More precisely, the dynamic output feedback can be written as the

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J. B. Hoagg was with the Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109-2140 USA (e-mail: jhoagg@umich.edu).

M. A. Santillo and D. S. Bernstein are with the Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109-2140 USA (e-mail: santillo@umich.edu; dsbaero@umich.edu).

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product of a known feedback vector and a matrix of estimated controller parameters. Second, we prove the existence of an ideal fixed-gain controller that incorporates a deadbeat internal model controller. For more information on deadbeat internal model control, see [12]. Lastly, we use a logarithmic Lyapunov-like function to prove asymptotic command following and disturbance rejection. Logarithmic Lyapunov functions, that is, quadratic functions that incorporate a logarithm, are used in [4]–[7] and [13]–[15] to prove Lyapunov stability of discrete-time systems. A quadratic Lyapunov-like function is used in [16] to establish the convergence of discrete-time systems. Using the logarithmic Lyapunov function, we prove global asymptotic convergence for command following and disturbance rejection as well as Lyapunov stability of the adaptive system when the open-loop system is asymptotically stable.

II. Problem Formulation

Consider the MIMO discrete-time system

$$x(k+1) = Ax(k) + Bu(k) + D_1 w(k) \tag{2.1}$$

$$y(k) = Cx(k) + D_2 w(k) \tag{2.2}$$

where $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^{l_y}$, $u(k) \in \mathbb{R}^{l_u}$, $w(k) \in \mathbb{R}^{l_w}$, and $k \geq 0$. Our goal is to design an adaptive output feedback controller under which the performance variable y converges to zero in the presence of the exogenous signal w. Note that w can represent either a command signal to be followed, an external disturbance to be rejected, or both. For example, if $D_1 = 0$ and $D_2 \neq 0$, then the objective is to have the output Cx follow the command signal $-D_2 w$. On the other hand, if $D_1 \neq 0$ and $D_2 = 0$, then the objective is to reject the disturbance w from the performance measurement Cx. The combined command following and disturbance rejection problem is considered when D_1 and D_2 are block matrices. More precisely, if

$$D_1 = [\hat{D}_1 \quad 0], \qquad D_2 = [0 \quad \hat{D}_2]$$

and

$$w(k) = \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix}$$

then the objective is to have Cx follow the command $-\hat{D}_2 w_2$ while rejecting the disturbance w_1. Lastly, if D_1 and D_2 are empty matrices, then the objective is output stabilization, that is, global asymptotic convergence of $y = Cx$ (and thus x) to zero.

In the nonadaptive case, a sufficient condition for command following and disturbance rejection is $l_u \geq l_y$ [12], [17]. Furthermore, we require that $l_y \geq l_u$ because the construction of an ideal fixed-gain controller in Section IV requires that the first nonzero Markov parameter from u to y be left invertible. Thus, we require that $l_y = l_u$. Henceforth, $l \triangleq l_y = l_u$. Weakenings of these conditions and some of the assumptions listed later are discussed in the conclusions.

Next, define the transfer function matrix

$$G_{yu}(z) \triangleq C(zI - A)^{-1}B = \sum_{i=d}^{\infty} z^{-i} H_i \tag{2.3}$$

and define d to be the smallest positive integer i such that the ith Markov parameter $H_i \triangleq CA^{i-1}B$ is nonzero. We make the following assumptions.

A1) The triple (A, B, C) is controllable and observable.

A2) If $\lambda \in \mathbb{C}$ and

$$\operatorname{rank} \begin{bmatrix} A - \lambda I & B \\ C & 0 \end{bmatrix} < \text{normal rank} \begin{bmatrix} A - zI & B \\ C & 0 \end{bmatrix}$$

then $|\lambda| < 1$.

A3) d is known.

A4) H_d is nonsingular.

A5) There exists $\bar{H}_d \in \mathbb{R}^{l \times l}$ such that $2H_d^{\mathrm{T}} H_d \leq H_d^{\mathrm{T}} \bar{H}_d + \bar{H}_d^{\mathrm{T}} H_d$ and $\bar{H}_d$ is known.

A6) There exists an integer $\bar{n}$ such that $n \leq \bar{n}$ and $\bar{n}$ is known.

A7) The performance variable $y(k)$ is measured and is available for feedback.

A8) The exogenous signal $w(k)$ is generated by

$$x_w(k+1) = A_w x_w(k) \tag{2.4}$$

$$w(k) = C_w x_w(k) \tag{2.5}$$

where $x_w \in \mathbb{R}^{n_w}$ and A_w has distinct eigenvalues, all of which are on the unit circle.

A9) There exists an integer $\bar{n}_w$ such that $n_w \leq \bar{n}_w$ and $\bar{n}_w$ is known.

A10) The exogenous signal $w(k)$ is not measured.

A11) $A, B, C, D_1, D_2, A_w, C_w, n, n_w$, and H_d are not known.

Assumption A1) implies that the McMillan degree of $G_{yu}(z)$ is n. In the SISO case, Assumption A1) prevents pole-zero cancellation when forming the transfer function $G_{yu}(z)$, which implies that the order of $G_{yu}(z)$ is n.

Let $G_{yu}(z)$ have a left coprime matrix-fraction description $G_{yu}(z) = \mu(z)^{-1}\nu(z)$, where $\mu(z)$ and $\nu(z)$ are $l \times l$ polynomial matrices. Without loss of generality, we assume that $\mu(z)$ is in column-Hermite form, that is, $\mu(z)$ is upper triangular, where each diagonal entry is a monic polynomial whose degree is higher than the degree of all of the remaining entries in its column [18, Th. 6.3-2]. Thus, we can write

$$\mu(z) = z^m \mu_0 + z^{m-1}\mu_1 + \cdots + z\mu_{m-1} + \mu_m \tag{2.6}$$

where $m \leq n$ and $\mu_0, \ldots, \mu_m \in \mathbb{R}^{l \times l}$ are upper triangular. Note that the leading coefficient matrix μ_0 is not necessarily I_l. However, it can be seen that there exists an $l \times l$ upper-triangular polynomial matrix

$$Q(z) \triangleq \begin{bmatrix} z^{h_{11}} & q_{12} z^{h_{12}} & \cdots & q_{1l} z^{h_{1l}} \\ & z^{h_{22}} & \cdots & q_{2l} z^{h_{2l}} \\ & & \ddots & \vdots \\ & & & z^{h_{ll}} \end{bmatrix} \tag{2.7}$$

such that the leading term of $\alpha(z) \triangleq Q(z)\mu(z)$ is $z^m I_l$. Thus, we can write

$$\alpha(z) = z^m I_l + z^{m-1}\alpha_1 + z^{m-2}\alpha_2 + \cdots + z\alpha_{m-1} + \alpha_m \tag{2.8}$$

where $\alpha_1, \ldots, \alpha_m \in \mathbb{R}^{l \times l}$. Furthermore, $G_{yu}(z)$ has the matrix-fraction description $G_{yu}(z) = \alpha(z)^{-1}\beta(z)$, where $\beta(z) \triangleq$

$Q(z)\nu(z)$, and we can write

$$\beta(z) = z^{m-d}\beta_d + z^{m-d-1}\beta_{d+1} + \cdots + z\beta_{m-1} + \beta_m \quad (2.9)$$

where $\beta_d, \ldots, \beta_m \in \mathbb{R}^{l\times l}$. Note that if the input to G_{yu} is $u = \delta(0)e_i$, where $\delta(0)$ is the unit impulse at $k=0$ and e_i is the ith column of I_l, then the output is

$$y(k) = \begin{cases} 0, & 0 \le k < d \\ \beta_d e_i, & k = d. \end{cases} \quad (2.10)$$

Thus, it follows that $\beta_d = H_d$. Note that $\alpha(z)$ and $\beta(z)$ are not necessarily left coprime. However, since $\mu(z)$ and $\nu(z)$ are left coprime, it follows that $Q(z)$ is the greatest common left divisor of $\alpha(z)$ and $\beta(z)$. Furthermore, since $\det Q(z) = z^{h_{11}+\cdots+h_{ll}}$, the pole-zero cancellation that occurs when forming the transfer function $G_{yu}(z) = \alpha(z)^{-1}\beta(z)$ occurs only at $z=0$.

Define the transfer function matrix

$$G_{yw}(z) \overset{\triangle}{=} C(zI-A)^{-1}D_1 + D_2 \quad (2.11)$$

and assuming that G_{yw} has a matrix-fraction description of the form $G_{yw} = \alpha(z)^{-1}\gamma(z)$, which is not necessarily left coprime, we can write

$$\gamma(z) = z^m\gamma_0 + z^{m-1}\gamma_1 + \cdots + z\gamma_{m-1} + \gamma_m \quad (2.12)$$

where $\gamma_0, \ldots, \gamma_m \in \mathbb{R}^{l\times l_w}$. Therefore, for $k \ge m$, the state-space system (2.1), (2.2) has the time-series representation

$$y(k) = \sum_{i=1}^{m} -\alpha_i y(k-i) + \sum_{i=d}^{m} \beta_i u(k-i) + \sum_{i=0}^{m} \gamma_i w(k-i). \quad (2.13)$$

Definition 2.1: Let G be a strictly proper transfer function matrix. Then the normal rank of G is $\operatorname{rank} G = \operatorname{rank} G(\lambda)$ for almost all $\lambda \in \mathbb{C}$.

Next, note that it follows from (2.3) and Assumption A4) that, for all sufficiently large $\lambda \in \mathbb{C}$, $\operatorname{rank} G_{yu}(\lambda) = l$. Thus, $G_{yu}(z)$ has full normal rank, that is, normal rank $G_{yu} = l$. Consequently, normal rank $\nu = l$.

Definition 2.2: Let G be a strictly proper $s \times t$ transfer function matrix with the Smith–McMillan form

$$G(z) = U_1(z) \begin{bmatrix} \frac{q_1(z)}{p_1(z)} & & & 0 \\ & \ddots & & \\ & & \frac{q_r(z)}{p_r(z)} & \\ 0 & & & 0_{(s-r)\times(t-r)} \end{bmatrix} U_2(z) \quad (2.14)$$

where $r =$ normal rank G, U_1 and U_2 are unimodular matrices, and $q_1, \ldots, q_r, p_1, \ldots, p_r$ are monic polynomials such that, for all $i = 1, \ldots, r$, q_i and p_i are coprime, and for all $i = 1, \ldots, r-1$, p_{i+1} divides p_i and q_i divides q_{i+1}. Then the poles of G, counting multiplicity, are the roots of $p_1 \cdots p_r$, and the transmission zeros of G, counting multiplicity, are the roots of $q_1 \cdots q_r$.

Lemma 2.1: Let G be a strictly proper $s \times t$ transfer function matrix with a left coprime matrix-fraction description $G(z) = P(z)^{-1}Z(z)$. Then, $\lambda \in \mathbb{C}$ is a transmission zero of G if and only if $\operatorname{rank} Z(\lambda) <$ normal rank Z. Furthermore, $p \in \mathbb{C}$ is a pole of G if and only if $\det P(p) = 0$.

Assumption A2) states that the invariant zeros of (A,B,C) are contained in the open unit circle. Since, by Assumption A1), (A,B,C) is minimal, it follows that the invariant zeros of (A,B,C) are exactly the transmission zeros of $G_{yu}(z)$. Therefore, Assumption A2) is equivalent to the assumption that the transmission zeros of $G_{yu}(z)$ are contained in the open unit circle. Since $\mu(z)$ and $\nu(z)$ are left coprime, it follows from Lemma 2.1 that Assumption A2) is equivalent to the assumption that, if $\lambda \in \mathbb{C}$ and $\operatorname{rank} \nu(\lambda) <$ normal rank ν, then $|\lambda| < 1$. Furthermore, since normal rank $\nu = l$ by Assumption A4), it follows that Assumption A2) implies that, if $\lambda \in \mathbb{C}$ and $\det \nu(\lambda) = 0$, then $|\lambda| < 1$. Consequently, since $\det \beta(\lambda) = \det Q(\lambda) \det \nu(\lambda) = z^{h_{11}+\cdots+h_{ll}} \det \nu(\lambda)$, it follows that, if $\lambda \in \mathbb{C}$ and $\det \beta(\lambda) = 0$, then $|\lambda| < 1$.

For SISO systems, Assumption A5) specializes to the assumption that $\operatorname{sgn} H_d$ is known and an upper bound on the magnitude $|H_d|$ is known. For MIMO systems, Assumption A5) is a generalization of this SISO assumption. In particular, if H_d is positive definite, then Assumption A5) specializes to the assumption that an upper bound on the magnitude of $\lambda_{\max}(H_d)$ is known. Similarly, if H_d is negative definite, then Assumption A5) specializes to the assumption that an upper bound on the magnitude of $|\lambda_{\min}(H_d)|$ is known. More precisely, if H_d is positive definite, then Assumption A5) is satisfied with $\bar{H}_d > \lambda_{\max}(H_d)I_l$, while, if H_d is negative definite, then Assumption A5) is satisfied with $\bar{H}_d > |\lambda_{\min}(H_d)|I_l$. Note that Assumptions A4) and A5) imply that $\bar{H}_d$ is nonsingular.

Assumption A8) restricts our consideration to command and disturbance signals that consist of discrete-time sinusoids and steps. The assumption that the eigenvalues of A_w are distinct entails no loss in generality compared to the assumption that the eigenvalues of A_w are semisimple, that is, appear only in Jordan blocks of order 1. For example, consider the system

$$x_w(k+1) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} x_w(k), \qquad w(k) = x_w(k) \quad (2.15)$$

where $x_w(k) \overset{\triangle}{=} [x_{w1}(k) \quad x_{w2}(k)]^{\mathrm{T}}$. We consider two cases. First, suppose that $x_{w1}(0) \neq 0$ and construct the system

$$x_{w\mathrm{r}}(k+1) = \lambda x_{w\mathrm{r}}(k), \quad w_{\mathrm{r}}(k) = \begin{bmatrix} 1 \\ \frac{x_{w2}(0)}{x_{w1}(0)} \end{bmatrix} x_{w\mathrm{r}}(k). \quad (2.16)$$

Then, with $x_{wr}(0) = x_{w1}(0)$, it follows that

$$w_{\mathrm{r}}(k) = \begin{bmatrix} 1 \\ \frac{x_{w2}(0)}{x_{w1}(0)} \end{bmatrix} \lambda^k x_{w\mathrm{r}}(0) = \begin{bmatrix} \lambda^k x_{w1}(0) \\ \lambda^k x_{w2}(0) \end{bmatrix} = w(k). \quad (2.17)$$

A similar argument applies to the case $x_{w2}(0) \neq 0$. Therefore, it follows that there exists a system with distinct eigenvalues whose output is identical to the output of (2.4), (2.5). Of course, Jordan blocks of order greater than 1 give rise to unbounded disturbances, which are not considered.

Assumption A10) implies that a direct measurement of the command and disturbance is not required, while Assumption A11) implies that the spectrum of the command and disturbance signals is unknown. We stress that $y(k)$ is the only signal available for feedback.

III. Nonminimal State-Space Realization

We use a nonminimal state-space realization of the time-series system (2.13) whose state consists entirely of measured information. More specifically, the state consists of past values of the performance variable $y(k)$ and the control $u(k)$. To construct the nonminimal state-space realization of the time-series system (2.13), we introduce the following notation. For a positive integer p, define the nilpotent matrix

$$\mathcal{N}_p \triangleq \begin{bmatrix} 0_{l\times l} & \cdots & 0_{l\times l} & 0_{l\times l} \\ I_l & \cdots & 0_{l\times l} & 0_{l\times l} \\ \vdots & \ddots & \vdots & \vdots \\ 0_{l\times l} & \cdots & I_l & 0_{l\times l} \end{bmatrix} \in \mathbb{R}^{lp\times lp} \tag{3.1}$$

and define

$$E_1 \triangleq \begin{bmatrix} I_l \\ 0_{l(p-1)\times l} \end{bmatrix} \in \mathbb{R}^{lp\times l} \tag{3.2}$$

where the dimension p is given by context.

Now, let $n_c \geq m$ and consider the $2ln_c$-order nonminimal state-space realization of (2.13)

$$\phi(k+1) = \mathcal{A}\phi(k) + \mathcal{B}u(k) + \mathcal{D}_1 W(k) \tag{3.3}$$

$$y(k) = \mathcal{C}\phi(k) + \mathcal{D}_2 W(k) \tag{3.4}$$

where

$$\mathcal{A} \triangleq \mathcal{A}_{\rm nil} + \begin{bmatrix} E_1\mathcal{C} \\ 0_{ln_c\times 2ln_c} \end{bmatrix}, \quad \mathcal{B} \triangleq \begin{bmatrix} 0_{ln_c\times l} \\ E_1 \end{bmatrix} \tag{3.5}$$

$$\mathcal{C} \triangleq [-\alpha_1 \quad \cdots \quad -\alpha_m \quad 0_{l\times l(n_c-m)} \quad 0_{l\times l(d-1)} \quad \beta_d \quad \cdots \quad \beta_m \quad 0_{l\times l(n_c-m)}] \tag{3.6}$$

$$\mathcal{D}_1 \triangleq \begin{bmatrix} E_1\mathcal{D}_2 \\ 0_{ln_c\times(m+1)l_w} \end{bmatrix}, \quad \mathcal{D}_2 \triangleq [\gamma_0 \quad \cdots \quad \gamma_m] \tag{3.7}$$

where

$$\mathcal{A}_{\rm nil} \triangleq \begin{bmatrix} \mathcal{N}_{n_c} & 0_{ln_c\times ln_c} \\ 0_{ln_c\times ln_c} & \mathcal{N}_{n_c} \end{bmatrix} \tag{3.8}$$

is nilpotent; and

$$\phi(k) \triangleq \begin{bmatrix} y(k-1) \\ \vdots \\ y(k-n_c) \\ u(k-1) \\ \vdots \\ u(k-n_c) \end{bmatrix}, \quad W(k) \triangleq \begin{bmatrix} w(k) \\ \vdots \\ w(k-m) \end{bmatrix}. \tag{3.9}$$

Note that the definition of $\mathcal{C}$ in (3.6) requires $n_c \geq m$. The triple $(\mathcal{A},\mathcal{B},\mathcal{C})$ is stabilizable and detectable. However, $(\mathcal{A},\mathcal{B},\mathcal{C})$ is neither controllable nor observable. In particular, $(\mathcal{A},\mathcal{B},\mathcal{C})$ has n controllable and observable eigenvalues, while the remaining $2ln_c - n$ eigenvalues are located at 0. Moreover, $(\mathcal{A},\mathcal{B})$ has $ln_c - n$ uncontrollable eigenvalues at 0, while $(\mathcal{A},\mathcal{C})$ has ln_c unobservable eigenvalues at 0. Note that in this basis, the state $\phi(k)$ contains only past values of the performance variable y and the control u.

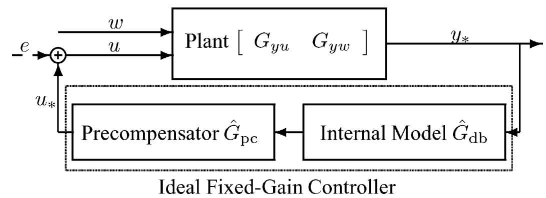

Fig. 1. Closed-loop system with the ideal fixed-gain controller. The pseudo-input e facilitates the proof of Theorem 4.1, but is otherwise set to zero.

Now, we consider the time-series controller

$$u(k) = \sum_{i=1}^{n_c} M_i u(k-i) + \sum_{i=1}^{n_c} N_i y(k-i) \tag{3.10}$$

where, for all $i = 1,\ldots,n_c$, $M_i \in \mathbb{R}^{l\times l}$ and $N_i \in \mathbb{R}^{l\times l}$. The control can be written as

$$u(k) = \theta\phi(k) \tag{3.11}$$

where

$$\theta \triangleq [N_1 \quad \cdots \quad N_{n_c} \quad M_1 \quad \cdots \quad M_{n_c}] \in \mathbb{R}^{l\times 2ln_c}. \tag{3.12}$$

The control (3.11), which is a dynamic output feedback in terms of y, can be computed by recording and using n_c past values of the performance variable y and the control u. However, (3.11) is a full-state-feedback control law for the nonminimal state-space system (3.3)–(3.8). The closed-loop system consisting of (3.3)–(3.8) with the linear time-invariant feedback (3.11) is

$$\phi(k+1) = \tilde{\mathcal{A}}\phi(k) + \mathcal{D}_1 W(k) \tag{3.13}$$

$$y(k) = \mathcal{C}\phi(k) + \mathcal{D}_2 W(k) \tag{3.14}$$

where

$$\tilde{\mathcal{A}} \triangleq \mathcal{A} + \mathcal{B}\theta = \mathcal{A}_{\rm nil} + \begin{bmatrix} E_1\mathcal{C} \\ E_1\theta \end{bmatrix}. \tag{3.15}$$

IV. Ideal Fixed-Gain Controller

In this section, we prove the existence and derive properties of an ideal fixed-gain controller of the form (3.10) for the open-loop system (2.1) and (2.2). This controller, whose structure is illustrated in Fig. 1, is used in subsequent sections to construct an error system for analyzing the adaptive closed-loop system. We stress that the ideal controller is not intended for implementation. An ideal fixed-gain controller consists of two distinct parts, specifically, a precompensator, which cancels the transmission zeros of the open-loop system, and a deadbeat internal model controller, which operates in feedback on the observable states of the precompensator cascaded with the open-loop system.

First, we demonstrate how to construct the ideal fixed-gain controller. Using Assumption A4), consider the $l \times l$ exactly proper precompensator

$$u_*(k) = -H_d^{-1}\sum_{i=1}^{m-d}\beta_{d+i}u_*(k-i) + u_{\rm db}(k) \tag{4.1}$$

which has a minimal state-space realization of the form

$$\hat{x}_{\rm pc}(k+1) = \hat{A}_{\rm pc}\hat{x}_{\rm pc}(k) + \hat{B}_{\rm pc}u_{\rm db}(k) \tag{4.2}$$

$$u_*(k) = \hat{C}_{\rm pc}\hat{x}_{\rm pc}(k) + u_{\rm db}(k) \tag{4.3}$$

where $\hat{x}_{\rm pc} \in \mathbb{R}^{\hat{n}_{\rm pc}}$ and $\hat{n}_{\rm pc}$ is the McMillan degree of $\hat{G}_{\rm pc}(z) \stackrel{\triangle}{=} \beta(z)^{-1}z^{m-d}H_d$, which is the transfer function from $u_{\rm db}$ to u_*. Note that $\hat{n}_{\rm pc} \le l(m-d)$. The poles of the precompensator $\hat{G}_{\rm pc}(z)$ are exactly the transmission zeros of the open-loop transfer function $G_{yu}(z)$. Furthermore, Assumption A2) implies that the transmission zeros of $G_{yu}(z)$, and thus the poles of $\hat{G}_{\rm pc}(z)$, are asymptotically stable. Therefore, the cascade

$$\begin{aligned} G_{yu}(z)\hat{G}_{\rm pc}(z) &= \alpha(z)^{-1}\beta(z)\beta(z)^{-1}z^{m-d}H_d \\ &= \alpha(z)^{-1}z^{m-d}H_d \end{aligned} \tag{4.4}$$

has asymptotically stable pole-zero cancellation. Let $n_{\rm o}$ be the McMillan degree of $G_{yu}(z)\hat{G}_{\rm pc}(z)$, and note that $n_{\rm o} \le lm$.

Define the pseudo-input

$$e(k) \stackrel{\triangle}{=} u(k) - u_*(k) \tag{4.5}$$

and cascade the precompensator (4.2), (4.3) with the open-loop system (2.1), (2.2) to obtain

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ \hat{x}_{\rm pc}(k+1) \end{bmatrix} &= \begin{bmatrix} A & B\hat{C}_{\rm pc} \\ 0 & \hat{A}_{\rm pc} \end{bmatrix}\begin{bmatrix} x(k) \\ \hat{x}_{\rm pc}(k) \end{bmatrix} \\ &\quad + \begin{bmatrix} B \\ \hat{B}_{\rm pc} \end{bmatrix}u_{\rm db}(k) + \begin{bmatrix} B \\ 0 \end{bmatrix}e(k) + \begin{bmatrix} D_1 \\ 0 \end{bmatrix}w(k) \end{aligned} \tag{4.6}$$

$$y_*(k) = [\,C \quad 0\,]\begin{bmatrix} x(k) \\ \hat{x}_{\rm pc}(k) \end{bmatrix} + D_2w(k) \tag{4.7}$$

where y_* is the ideal system output. Since the poles of $\hat{G}_{\rm pc}(z)$ cancel the transmission zeros of $G_{yu}(z)$,

$$\left(\begin{bmatrix} A & B\hat{C}_{\rm pc} \\ 0 & \hat{A}_{\rm pc} \end{bmatrix}, \begin{bmatrix} B \\ \hat{B}_{\rm pc} \end{bmatrix}, [\,C \quad 0\,]\right) \tag{4.8}$$

is not minimal. However, since (A,B) and $(\hat{A}_{\rm pc},\hat{B}_{\rm pc})$ are controllable, it follows that (4.8) is controllable. Thus

$$\left(\begin{bmatrix} A & B\hat{C}_{\rm pc} \\ 0 & \hat{A}_{\rm pc} \end{bmatrix}, [\,C \quad 0\,]\right) \tag{4.9}$$

is not observable. In fact, it follows from the pole-zero cancellations between $\hat{G}_{\rm pc}(z)$ and $G_{yu}(z)$ that the unobservable modes of (4.9) are exactly the poles of $\hat{G}_{\rm pc}(z)$, all of which are asymptotically stable.

Next, let $\hat{x}_{\rm db} \in \mathbb{R}^{\hat{n}_{\rm db}}$, and let

$$\hat{x}_{\rm db}(k+1) = \hat{A}_{\rm db}\hat{x}_{\rm db}(k) + \hat{B}_{\rm db}y_*(k) \tag{4.10}$$

$$u_{\rm db}(k) = \hat{C}_{\rm db}\hat{x}_{\rm db}(k) \tag{4.11}$$

be an internal model controller (whose existence is shown later) for the observable states of (4.6) and (4.7) that guarantees exact command following and disturbance rejection in finite time, that is, (4.10), (4.11) is a deadbeat internal model controller. Thus, the ideal fixed-gain controller consists of the precompensator (4.2), (4.3) and the deadbeat internal model controller (4.10), (4.11). Define the transfer function matrix of the deadbeat internal model controller (4.10), (4.11) by

$$\hat{G}_{\rm db}(z) \stackrel{\triangle}{=} \hat{C}_{\rm db}(zI - \hat{A}_{\rm db})^{-1}\hat{B}_{\rm db}.$$

The following theorem constructs the ideal fixed-gain controller

$$u_*(k) = \sum_{i=1}^{n_{\rm c}} M_{*i}u_*(k-i) + \sum_{i=1}^{n_{\rm c}} N_{*i}y_*(k-i) \tag{4.12}$$

which can be expressed as

$$u_*(k) = \theta_*\phi_{**}(k) \tag{4.13}$$

where

$$\theta_* \stackrel{\triangle}{=} [\,N_{*1} \quad \cdots \quad N_{*n_{\rm c}} \quad M_{*1} \quad \cdots \quad M_{*n_{\rm c}}\,] \tag{4.14}$$

and

$$\phi_{**}(k) \stackrel{\triangle}{=} \begin{bmatrix} y_*(k-1) \\ \vdots \\ y_*(k-n_{\rm c}) \\ u_*(k-1) \\ \vdots \\ u_*(k-n_{\rm c}) \end{bmatrix}. \tag{4.15}$$

The closed-loop system with the ideal fixed-gain controller is shown in Fig. 1 and is given by

$$\phi(k+1) = \tilde{\mathcal{A}}_*\phi(k) + \mathcal{D}_1W(k) \tag{4.16}$$

$$y(k) = \mathcal{C}\phi(k) + \mathcal{D}_2W(k) \tag{4.17}$$

where

$$\tilde{\mathcal{A}}_* \stackrel{\triangle}{=} \mathcal{A} + \mathcal{B}\theta_* = \mathcal{A}_{\rm nil} + \begin{bmatrix} E_1\mathcal{C} \\ E_1\theta_* \end{bmatrix}. \tag{4.18}$$

Theorem 4.1: Consider the ideal closed-loop system consisting of (4.16), (4.17), where $\tilde{\mathcal{A}}_*$, $\mathcal{B}$, and $\mathcal{C}$ are given by (4.18), (3.5), and (3.6), respectively. Furthermore, let

$$n_{\rm c} \ge n_{\rm o} + 2ln_w + m - d. \tag{4.19}$$

Then there exists an ideal linear output-feedback controller (4.12) of order $n_{\rm c}$ such that the following statements hold.

1) For all initial conditions $\phi_{**}(0)$ and $x_w(0)$ and all integers $k \ge k_0$, where

$$k_0 \stackrel{\triangle}{=} n_{\rm o} + n_{\rm c} + d - m \tag{4.20}$$

it follows that $y_*(k) = 0$.

2) $\tilde{\mathcal{A}}_*$ is asymptotically stable.

3) For $i = 1, 2, 3, \ldots$

$$\mathcal{C}\tilde{\mathcal{A}}_*^{i-1}\mathcal{B} = \begin{cases} H_d, & i = d \\ 0, & i \neq d. \end{cases} \tag{4.21}$$

Proof: We show that a time-series representation of the fixed-gain controller (4.2), (4.3), (4.10), and (4.11) depicted in Fig. 1 exists and satisfies 1)–3).

First, consider the cascade (4.6), (4.7), and recall that (4.8) is controllable but not observable. Furthermore, the unobservable modes of (4.9) are precisely the poles of $\hat{G}_{\rm pc}(z)$, all of which are asymptotically stable because of Assumption A2). Therefore, it follows from the Kalman decomposition that there exists a nonsingular matrix $T \in \mathbb{R}^{(n+\hat{n}_{\rm pc})\times(n+\hat{n}_{\rm pc})}$ such that

$$\begin{bmatrix} A_{\rm o} & 0 \\ A_{21} & A_{\bar{\rm o}} \end{bmatrix} = T \begin{bmatrix} A & B\hat{C}_{\rm pc} \\ 0 & \hat{A}_{\rm pc} \end{bmatrix} T^{-1} \tag{4.22}$$

$$[\,C_{\rm o} \quad 0\,] = [\,C \quad 0\,]\,T^{-1} \tag{4.23}$$

where $A_{\rm o} \in \mathbb{R}^{n_{\rm o}\times n_{\rm o}}$, $(A_{\rm o}, C_{\rm o})$ is observable, and $A_{\bar{\rm o}}$ is asymptotically stable.

Now, defining $\begin{bmatrix} x_{\rm o}(k) \\ x_{\bar{\rm o}}(k) \end{bmatrix} \triangleq T \begin{bmatrix} x(k) \\ \hat{x}_{\rm pc}(k) \end{bmatrix}$, where $x_{\rm o}(k) \in \mathbb{R}^{n_{\rm o}}$, and applying this change of basis to the cascade (4.6) and (4.7) yields

$$\begin{bmatrix} x_{\rm o}(k+1) \\ x_{\bar{\rm o}}(k+1) \end{bmatrix} = \begin{bmatrix} A_{\rm o} & 0 \\ A_{21} & A_{\bar{\rm o}} \end{bmatrix} \begin{bmatrix} x_{\rm o}(k) \\ x_{\bar{\rm o}}(k) \end{bmatrix} + \begin{bmatrix} B_{\rm o} \\ B_{\bar{\rm o}} \end{bmatrix} u_{\rm db}(k) + \begin{bmatrix} B_{e,{\rm o}} \\ B_{e,\bar{\rm o}} \end{bmatrix} e(k) + \begin{bmatrix} D_{1,{\rm o}} \\ D_{1,\bar{\rm o}} \end{bmatrix} w(k) \tag{4.24}$$

$$y_*(k) = [\,C_{\rm o} \quad 0\,] \begin{bmatrix} x_{\rm o}(k) \\ x_{\bar{\rm o}}(k) \end{bmatrix} + D_2 w(k) \tag{4.25}$$

where $x_{\rm o} \in \mathbb{R}^{n_{\rm o}}$ and

$$\begin{bmatrix} B_{\rm o} \\ B_{\bar{\rm o}} \end{bmatrix} = T \begin{bmatrix} B \\ \hat{B}_{\rm pc} \end{bmatrix}, \quad \begin{bmatrix} B_{e,{\rm o}} \\ B_{e,\bar{\rm o}} \end{bmatrix} = T \begin{bmatrix} B \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} D_{1,{\rm o}} \\ D_{1,\bar{\rm o}} \end{bmatrix} = T \begin{bmatrix} D_1 \\ 0 \end{bmatrix}. \tag{4.26}$$

Note that $(A_{\rm o}, B_{\rm o}, C_{\rm o})$ is a minimal realization of the transfer function matrix

$$\begin{aligned} G_{\rm o}(z) &\triangleq C_{\rm o}[zI - A_{\rm o}]^{-1}B_{\rm o} = G_{yu}(z)\hat{G}_{\rm pc}(z) \\ &= \alpha(z)^{-1} z^{m-d} H_d. \end{aligned} \tag{4.27}$$

Next, we consider a deadbeat internal model controller of the form (4.10), (4.11) designed for the observable subsystem of (4.24), (4.25) given by

$$x_{\rm o}(k+1) = A_{\rm o}x_{\rm o}(k) + B_{\rm o}u_{\rm db}(k) + B_{e,{\rm o}}e(k) + D_{1,{\rm o}}w(k) \tag{4.28}$$

$$y_*(k) = C_{\rm o}x_{\rm o}(k) + D_2 w(k). \tag{4.29}$$

The invariant zeros of $(A_{\rm o}, B_{\rm o}, C_{\rm o})$ are located at the origin, and thus do not coincide with the eigenvalues of A_w by Assumption A8). Since, in addition, $(A_{\rm o}, B_{\rm o}, C_{\rm o})$ is minimal, the dimension of y equals the dimension of u, and normal rank $G_{\rm o} = l$, it follows from Theorem A.1 with $\hat{n} = n_{\rm o}$, $\hat{n}_w = n_w$, and $\hat{l}_y = l$ that, for all $\hat{n}_{\rm db}$ satisfying

$$\hat{n}_{\rm db} \geq n_{\rm o} + 2ln_w \tag{4.30}$$

there exists a discrete-time controller (4.10), (4.11) such that the dynamics matrix

$$\tilde{\mathcal{A}}_{\rm dbo} \triangleq \begin{bmatrix} A_{\rm o} & B_{\rm o}\hat{C}_{\rm db} \\ \hat{B}_{\rm db}C_{\rm o} & \hat{A}_{\rm db} \end{bmatrix} \tag{4.31}$$

of the closed-loop system (4.10), (4.11), (4.28), and (4.29), which represents the feedback interconnection of $G_{\rm o}$ and $\hat{G}_{\rm db}$, is nilpotent. Furthermore, with $e(k) \equiv 0$, for all initial conditions $(x_{\rm o}(0), x_{\bar{\rm o}}(0), \hat{x}_{\rm db}(0), x_w(0))$ and all integers $k \geq n_{\rm o} + \hat{n}_{\rm db}$, it follows that $y_*(k) = 0$.

The closed-loop system (4.10), (4.11), (4.24), and (4.25) is

$$\begin{bmatrix} x_{\rm o}(k+1) \\ \hat{x}_{\rm db}(k+1) \\ x_{\bar{\rm o}}(k+1) \end{bmatrix} = \begin{bmatrix} A_{\rm o} & B_{\rm o}\hat{C}_{\rm db} & 0 \\ \hat{B}_{\rm db}C_{\rm o} & \hat{A}_{\rm db} & 0 \\ A_{21} & B_{\bar{\rm o}}\hat{C}_{\rm db} & A_{\bar{\rm o}} \end{bmatrix} \begin{bmatrix} x_{\rm o}(k) \\ \hat{x}_{\rm db}(k) \\ x_{\bar{\rm o}}(k) \end{bmatrix} + \begin{bmatrix} B_{e,{\rm o}} \\ 0 \\ B_{e,\bar{\rm o}} \end{bmatrix} e(k) + \begin{bmatrix} D_{1,{\rm o}} \\ \hat{B}_{\rm db}D_2 \\ D_{1,\bar{\rm o}} \end{bmatrix} w(k) \tag{4.32}$$

$$y_*(k) = [\,C_{\rm o} \quad 0 \quad 0\,] \begin{bmatrix} x_{\rm o}(k) \\ \hat{x}_{\rm db}(k) \\ x_{\bar{\rm o}}(k) \end{bmatrix} + D_2 w(k). \tag{4.33}$$

Since $\tilde{\mathcal{A}}_{\rm dbo}$ is nilpotent and $A_{\bar{\rm o}}$ is asymptotically stable, it follows that

$$\begin{bmatrix} A_{\rm o} & B_{\rm o}\hat{C}_{\rm db} & 0 \\ \hat{B}_{\rm db}C_{\rm o} & \hat{A}_{\rm db} & 0 \\ A_{21} & B_{\bar{\rm o}}\hat{C}_{\rm db} & A_{\bar{\rm o}} \end{bmatrix} \tag{4.34}$$

is asymptotically stable.

To construct the ideal fixed-gain controller, we first write the transfer function matrix of (4.10), (4.11) as

$$\hat{G}_{\rm db}(z) = \hat{M}(z)^{-1}\hat{N}(z) \tag{4.35}$$

where

$$\hat{M}(z) = z^{\hat{n}_{\rm db}} I_l + z^{\hat{n}_{\rm db}-1}\hat{M}_1 + \cdots + z\hat{M}_{\hat{n}_{\rm db}-1} + \hat{M}_{\hat{n}_{\rm db}} \tag{4.36}$$

$$\hat{N}(z) = z^{\hat{n}_{\rm db}-1}\hat{N}_1 + z^{\hat{n}_{\rm db}-2}\hat{N}_2 + \cdots + z\hat{N}_{\hat{n}_{\rm db}-1} + \hat{N}_{\hat{n}_{\rm db}} \tag{4.37}$$

where, for $i = 1, \ldots, \hat{n}_{\rm db}$, $\hat{M}_i \in \mathbb{R}^{l\times l}$ and $\hat{N}_i \in \mathbb{R}^{l\times l}$. Therefore, (4.10), (4.11) has the time-series representation

$$u_{\rm db}(k) = -\sum_{i=1}^{\hat{n}_{\rm db}} \hat{M}_i u_{\rm db}(k-i) + \sum_{i=1}^{\hat{n}_{\rm db}} \hat{N}_i y_*(k-i). \tag{4.38}$$

Now, let $\hat{n}_{\rm db} = n_{\rm c} + d - m$, and note that, since (4.19) holds, $\hat{n}_{\rm db} = n_{\rm c} + d - m \geq n_{\rm o} + 2ln_w$, as required by (4.30). With

$e(k) \equiv 0$, and thus $u(k) = u_*(k)$ for all $k \geq k_0$, the ideal fixed-gain controller, which consists of the precompensator (4.1) and the deadbeat internal model controller (4.38), is given by (4.12), where, for $i = 1, 2, \ldots, n_c$

$$M_{*i} \triangleq -H_d^{-1}\beta_{d+i} - \sum_{j=1}^{i} \hat{M}_j H_d^{-1}\beta_{d+i-j} \quad (4.39)$$

$$N_{*i} \triangleq \hat{N}_i \quad (4.40)$$

where, for all $i > m$, $\beta_i = 0$, and for all $i > \hat{n}_{db}$, $\hat{M}_i = \hat{N}_i = 0$.

To show 1), consider the $2ln_c$-order nonminimal state-space realization of the controller (4.13), (4.39), and (4.40) given by

$$\phi_{**}(k+1) = \mathcal{A}_c\phi_{**}(k) + \mathcal{B}_c y_*(k) \quad (4.41)$$

$$u_*(k) = \mathcal{C}_c\phi_{**}(k) \quad (4.42)$$

where

$$\mathcal{A}_c \triangleq \mathcal{A}_{nil} + \begin{bmatrix} 0_{ln_c \times 2ln_c} \\ E_1\theta_* \end{bmatrix}, \quad \mathcal{B}_c \triangleq \begin{bmatrix} E_1 \\ 0_{ln_c \times l} \end{bmatrix}, \quad \mathcal{C}_c \triangleq \theta_*. \quad (4.43)$$

Note that $\mathcal{A}_c = \mathcal{A} + \mathcal{B}\mathcal{C}_c - \mathcal{B}_c\mathcal{C}$. Therefore, the ideal closed-loop system (3.3)–(3.8) and (4.41)–(4.43) is

$$\begin{bmatrix} \phi_*(k+1) \\ \phi_{**}(k+1) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B}\mathcal{C}_c \\ \mathcal{B}_c\mathcal{C} & \mathcal{A}_c \end{bmatrix} \begin{bmatrix} \phi_*(k) \\ \phi_{**}(k) \end{bmatrix} + \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix} e(k) + \begin{bmatrix} \mathcal{D}_1 \\ \mathcal{B}_c\mathcal{D}_2 \end{bmatrix} W(k) \quad (4.44)$$

$$y_*(k) = [\mathcal{C} \quad 0] \begin{bmatrix} \phi_*(k) \\ \phi_{**}(k) \end{bmatrix} + \mathcal{D}_2 W(k) \quad (4.45)$$

where

$$\phi_*(k) \triangleq \begin{bmatrix} y_*(k-1) \\ \vdots \\ y_*(k-n_c) \\ u(k-1) \\ \vdots \\ u(k-n_c) \end{bmatrix}. \quad (4.46)$$

The closed-loop system (4.44) and (4.45) is a nonminimal representation of the closed-loop system (4.32) and (4.33). Furthermore, every unobservable or uncontrollable mode of (4.44) and (4.45) is located at zero. Thus, the spectrum of

$$\tilde{\mathcal{A}}_{cl} \triangleq \begin{bmatrix} \mathcal{A} & \mathcal{B}\mathcal{C}_c \\ \mathcal{B}_c\mathcal{C} & \mathcal{A}_c \end{bmatrix} \quad (4.47)$$

consists of the eigenvalues of (4.34) as well as $4ln_c - n - \hat{n}_{pc} - \hat{n}_{db}$ eigenvalues located at zero. Therefore, since (4.34) is asymptotically stable, it follows that (4.47) is asymptotically stable. Furthermore, since (4.44), (4.45) is a nonminimal representation of (4.32), (4.33), it follows that, with $e(k) \equiv 0$, for all initial conditions $\phi_{**}(0)$ and $x_w(0)$ and all $k \geq n_o + \hat{n}_{db} = k_0$, $y_*(k) = 0$. Thus, we have verified 1).

To show 2), consider the change of basis

$$\begin{bmatrix} \tilde{\mathcal{A}}_* & \mathcal{B}\mathcal{C}_c \\ 0 & \mathcal{A}_{nil} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B}\mathcal{C}_c \\ \mathcal{B}_c\mathcal{C} & \mathcal{A}_c \end{bmatrix} \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \quad (4.48)$$

$$\begin{bmatrix} \mathcal{B} \\ -\mathcal{B} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix} \quad (4.49)$$

$$[\mathcal{C} \quad 0] = [\mathcal{C} \quad 0] \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}. \quad (4.50)$$

Since (4.47) is asymptotically stable and $\mathcal{A}_{nil}$ is nilpotent, it follows from (4.48) that $\tilde{\mathcal{A}}_*$ is asymptotically stable, verifying 2).

To show 3), we compute the closed-loop Markov parameters $\tilde{H}_{y_*e,i}$ from the pseudo-input e to the performance variable y_* using a state-space realization of the closed-loop system and a transfer function matrix representation of the closed-loop system. First, consider the nonminimal state-space realization (4.44) and (4.45). For $i = 1, 2, \ldots$, define the Markov parameters

$$\begin{aligned} \tilde{H}_{y_*e,i} &\triangleq [\mathcal{C} \quad 0] \begin{bmatrix} \mathcal{A} & \mathcal{B}\mathcal{C}_c \\ \mathcal{B}_c\mathcal{C} & \mathcal{A}_c \end{bmatrix}^{i-1} \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix} \\ &= [\mathcal{C} \quad 0] \begin{bmatrix} \tilde{\mathcal{A}}_* & \mathcal{B}\mathcal{C}_c \\ 0 & \mathcal{A}_{nil} \end{bmatrix}^{i-1} \begin{bmatrix} \mathcal{B} \\ -\mathcal{B} \end{bmatrix} \\ &= \mathcal{C}\tilde{\mathcal{A}}_*^{i-1}\mathcal{B} + \sum_{j=1}^{i-1} -\mathcal{C}\tilde{\mathcal{A}}_*^{j-1}\mathcal{B}M_{*i-j} \end{aligned} \quad (4.51)$$

where $M_{*i} = \mathcal{C}_c\mathcal{A}_{nil}^{i-1}\mathcal{B}$ for $i = 1, 2, \ldots, n_c$ and $M_{*i} = 0$ for all $i > n_c$.

Next, consider the transfer function matrix representation of the open-loop system

$$\begin{aligned} y_* &= G_{yu}(z)u + G_{yw}(z)w \\ &= G_{yu}(z)u_* + G_{yu}(z)e + G_{yw}(z)w \\ &= G_{yu}(z)\hat{G}_{pc}(z)\hat{G}_{db}(z)y_* + G_{yu}(z)e + G_{yw}(z)w \end{aligned} \quad (4.52)$$

which implies that the closed-loop system is

$$y_* = \tilde{G}_{ye}e + \tilde{G}_{yw}w \quad (4.53)$$

where

$$\begin{aligned} \tilde{G}_{ye} &\triangleq [I_l - G_{yu}(z)\hat{G}_{pc}(z)\hat{G}_{db}(z)]^{-1}G_{yu}(z) \\ &= [I_l - \alpha(z)^{-1}z^{m-d}H_d\hat{M}(z)^{-1}\hat{N}(z)]^{-1}\alpha(z)^{-1}\beta(z) \\ &= [\alpha(z) - z^{m-d}H_d\hat{M}(z)^{-1}\hat{N}(z)]^{-1}\beta(z) \\ &= \tilde{D}(z)^{-1}\hat{M}(z)H_d^{-1}\beta(z) \end{aligned} \quad (4.54)$$

$$\begin{aligned} \tilde{G}_{yw} &\triangleq [I_l - G_{yu}(z)\hat{G}_{pc}(z)\hat{G}_{db}(z)]^{-1}G_{yw}(z) \\ &= \tilde{D}(z)^{-1}\hat{M}(z)H_d^{-1}\gamma(z), \end{aligned} \quad (4.55)$$

and $\tilde{D}(z) \overset{\triangle}{=} \hat{M}(z)H_d^{-1}\alpha(z) - z^{m-d}\hat{N}(z)$. Notice that $\tilde{D}(z)$ can be written as

$$\tilde{D}(z) = z^{m+\hat{n}_{\rm db}}H_d^{-1} + z^{m+\hat{n}_{\rm db}-1}\tilde{D}_1 + \cdots + \tilde{D}_{m+\hat{n}_{\rm db}} \quad (4.56)$$

where, for $i = 1, 2, \ldots, m + \hat{n}_{\rm db}$, $\tilde{D}_i \in \mathbb{R}^{l\times l}$. Since (4.31) is nilpotent, it follows that the poles of $\tilde{G}_{ye}$ and $\tilde{G}_{yw}$ are located at zero; in particular, $\det \tilde{D}(z) = z^{l(m+\hat{n}_{\rm db})}\det H_d^{-1}$. In fact, it follows from (4.56) that the coefficients of the deadbeat controller $\hat{M}(z)^{-1}\hat{N}(z)$ can be chosen so that $\tilde{D}_1 = \cdots = \tilde{D}_{m+\hat{n}_{\rm db}} = 0$, and thus

$$\tilde{G}_{ye}(z) = \left[z^{m+\hat{n}_{\rm db}}H_d^{-1}\right]^{-1}\tilde{N}(z) = z^{-m-\hat{n}_{\rm db}}H_d\tilde{N}(z) \quad (4.57)$$

where

$$\tilde{N}(z) \overset{\triangle}{=} \hat{M}(z)H_d^{-1}\beta(z) = z^{m+\hat{n}_{\rm db}}\tilde{N}_0 + \cdots + \tilde{N}_{m+\hat{n}_{\rm db}} \quad (4.58)$$

and

$$\tilde{N}_i = \begin{cases} 0, & 0 \le i < d \\ I_l, & i = d \\ H_d^{-1}\beta_i + \sum_{j=1}^{i-d}\hat{M}_j H_d^{-1}\beta_{i-j}, & d < i \le m + \hat{n}_{\rm db}. \end{cases} \quad (4.59)$$

Therefore, it follows from (4.39) that

$$\tilde{N}_i = \begin{cases} 0, & 0 \le i < d \\ I_l, & i = d \\ -M_{*i-d} & d < i \le m + \hat{n}_{\rm db}. \end{cases} \quad (4.60)$$

It follows from (4.57) that the closed-loop Markov parameters $\tilde{H}_{y_*e,i}$ from the pseudo-input e to the performance variable y_* are $\tilde{H}_{y_*e,i} = H_d\tilde{N}_i$ for $i = 1, 2, \ldots, m + \hat{n}_{\rm db}$ and $\tilde{H}_{y_*e,i} = 0$ for $i > m + \hat{n}_{\rm db}$, which implies

$$\tilde{H}_{y_*e,i} = \begin{cases} 0, & 0 \le i < d \\ H_d, & i = d \\ -H_d M_{*i-d}, & d < i \le m + \hat{n}_{\rm db} \\ 0, & i > m + \hat{n}_{\rm db}. \end{cases} \quad (4.61)$$

Then, property 3) follows from comparing the expressions for $\tilde{H}_{y_*e,i}$ given by (4.51) and (4.61). More specifically, since (4.61) implies that $\tilde{H}_{y_*e,1} = \cdots = \tilde{H}_{y_*e,d-1} = 0$, it follows from (4.51) that $\mathcal{C}\mathcal{B} = \mathcal{C}\tilde{\mathcal{A}}_*\mathcal{B} = \cdots = \mathcal{C}\tilde{\mathcal{A}}_*^{d-2}\mathcal{B} = 0$. Next, since $\mathcal{C}\mathcal{B} = \mathcal{C}\tilde{\mathcal{A}}_*\mathcal{B} = \cdots = \mathcal{C}\tilde{\mathcal{A}}_*^{d-2}\mathcal{B} = 0$ and $\tilde{H}_{y_*e,d} = H_d$ [using (4.61)], it follows from (4.51) that $\mathcal{C}\tilde{\mathcal{A}}_*^{d-1}\mathcal{B} = H_d$. Now, since $\mathcal{C}\mathcal{B} = \mathcal{C}\tilde{\mathcal{A}}_*\mathcal{B} = \cdots = \mathcal{C}\tilde{\mathcal{A}}_*^{d-2}\mathcal{B} = 0$, $\mathcal{C}\tilde{\mathcal{A}}_*^{d-1}\mathcal{B} = H_d$, and $\tilde{H}_{y_*e,d+1} = -H_dM_{*1}$ [using (4.61)], it follows from (4.51) that $\mathcal{C}\tilde{\mathcal{A}}_*^{d}\mathcal{B} = 0$. Lastly, since $\mathcal{C}\mathcal{B} = \mathcal{C}\tilde{\mathcal{A}}_*\mathcal{B} = \cdots = \mathcal{C}\tilde{\mathcal{A}}_*^{d-2}\mathcal{B} = 0$, $\mathcal{C}\tilde{\mathcal{A}}_*^{d-1}\mathcal{B} = H_d$, $\mathcal{C}\tilde{\mathcal{A}}_*^{d}\mathcal{B} = 0$, and $\tilde{H}_{y_*e,d+2} = -H_dM_{*2}$ [using (4.61)], it follows from (4.51) that $\mathcal{C}\tilde{\mathcal{A}}_*^{d+1}\mathcal{B} = 0$. Continuing this analysis yields $\mathcal{C}\mathcal{B} = \mathcal{C}\tilde{\mathcal{A}}_*\mathcal{B} = \cdots = \mathcal{C}\tilde{\mathcal{A}}_*^{d-2}\mathcal{B} = 0$, $\mathcal{C}\tilde{\mathcal{A}}_*^{d-1}\mathcal{B} = H_d$, and $\mathcal{C}\tilde{\mathcal{A}}_*^{d}\mathcal{B} = \mathcal{C}\tilde{\mathcal{A}}_*^{d+1} = \cdots = 0$. ■

V. Error System

We now construct an error system using the ideal fixed-gain controller and a controller whose gains are updated by an adaptive law. By Assumption A11), the controller order $n_{\rm c}$ given by (4.19) is unknown. However, since $m \le n$ and $n_{\rm o} \le lm$, it follows that $n_{\rm o} + m + 2ln_w - d \le (l+1)\bar{n} + 2l\bar{n}_w - d$. Therefore, if

$$n_{\rm c} \ge (l+1)\bar{n} + 2l\bar{n}_w - d \quad (5.1)$$

then $n_{\rm c}$ satisfies (4.19). Assumptions A3), A6), and A9) imply that the lower bound on $n_{\rm c}$ given by (5.1) is known.

The closed-loop system consisting of (3.3)–(3.8) with the ideal feedback (4.13) is

$$\phi_{**}(k+1) = \tilde{\mathcal{A}}_*\phi_{**}(k) + \mathcal{D}_1W(k) \quad (5.2)$$

$$y_*(k) = \mathcal{C}\phi_{**}(k) + \mathcal{D}_2W(k) \quad (5.3)$$

where, by 2) of Theorem 4.1, $\tilde{\mathcal{A}}_*$ is asymptotically stable.

Next, consider the controller

$$u(k) = \sum_{i=1}^{n_{\rm c}} M_i(k)u(k-i) + \sum_{i=1}^{n_{\rm c}} N_i(k)y(k-i) \quad (5.4)$$

where, for all $i = 1, \ldots, n_{\rm c}$, $M_i : \mathbb{N} \to \mathbb{R}^{l\times l}$ and $N_i : \mathbb{N} \to \mathbb{R}^{l\times l}$ are given by the adaptive law presented in the following section. The control can be expressed as

$$u(k) = \theta(k)\phi(k) \quad (5.5)$$

where

$$\theta(k) \overset{\triangle}{=} [\, N_1(k) \quad \cdots \quad N_{n_{\rm c}}(k) \quad M_1(k) \quad \cdots \quad M_{n_{\rm c}}(k)\,]. \quad (5.6)$$

Inserting (5.5) into (3.3) yields

$$\phi(k+1) = \mathcal{A}\phi(k) + \mathcal{B}\theta(k)\phi(k) + \mathcal{D}_1W(k). \quad (5.7)$$

Next, defining

$$\tilde{\theta}(k) \overset{\triangle}{=} \theta(k) - \theta_* \quad (5.8)$$

and substituting $\theta(k) = \tilde{\theta}(k) + \theta_*$ into (5.7), the closed-loop system consisting of (3.3), (3.4) with the time-varying feedback (5.5) becomes

$$\phi(k+1) = \tilde{\mathcal{A}}_*\phi(k) + \mathcal{B}\tilde{\theta}(k)\phi(k) + \mathcal{D}_1W(k) \quad (5.9)$$

$$y(k) = \mathcal{C}\phi(k) + \mathcal{D}_2W(k). \quad (5.10)$$

Now, we construct an error system by combining the ideal closed-loop system (5.2), (5.3) with the closed-loop system (5.9), (5.10). Define the error state

$$\tilde{\phi}(k) \overset{\triangle}{=} \phi(k) - \phi_{**}(k) \quad (5.11)$$

and subtract (5.2), (5.3) from (5.9), (5.10) to obtain

$$\tilde{\phi}(k+1) = \tilde{\mathcal{A}}_*\tilde{\phi}(k) + \mathcal{B}\tilde{\theta}(k)\phi(k) \quad (5.12)$$

$$\tilde{y}(k) = \mathcal{C}\tilde{\phi}(k) \quad (5.13)$$

where

$$\tilde{y}(k) \overset{\triangle}{=} y(k) - y_*(k). \quad (5.14)$$

Note that the Markov parameters of the error system (5.12), (5.13) are given by 3) of Theorem 4.1.

The following proposition shows that $y(k)$ is linear in the estimation error $\tilde{\theta}(k)$. This proposition is essential for developing the adaptive law and analyzing the stability of the error system.

Proposition 5.1: Consider the error system (5.12) and (5.13). For all $k \geq k_0$,

$$\tilde{y}(k) = y(k) = H_d\tilde{\theta}(k-d)\phi(k-d). \tag{5.15}$$

Proof: Substituting (5.12) into (5.13) yields

$$\tilde{y}(k) = \sum_{i=1}^{k} \mathcal{C}\tilde{\mathcal{A}}_*^{i-1}\mathcal{B}\tilde{\theta}(k-i)\phi(k-i). \tag{5.16}$$

It now follows from 3) of Theorem 4.1 and (5.16) that $\tilde{y}(k) = H_d\tilde{\theta}(k-d)\phi(k-d)$. Furthermore, it follows from 1) of Theorem 4.1 that, for all $k \geq k_0$, $y_*(k) = 0$, that is, $\tilde{y}(k) = y(k)$. Hence, for all $k \geq k_0$, (5.15) holds. ■

VI. Adaptive Controller and Stability Analysis

We now present the adaptive law for the controller (5.5), (5.6) and analyze the properties of the closed-loop error system. Consider the cost function

$$\mathcal{J}(k) \triangleq \frac{1}{2}\tilde{y}^{\mathrm{T}}(k)\tilde{y}(k). \tag{6.1}$$

Substituting (5.15) into (6.1), the gradient of $\mathcal{J}(k)$ with respect to $\tilde{\theta}(k-d)$ is given by

$$\frac{\partial \mathcal{J}(k)}{\partial \tilde{\theta}(k-d)} = H_d^{\mathrm{T}} y(k)\phi^{\mathrm{T}}(k-d). \tag{6.2}$$

Since, by Assumption A11), H_d is unknown, we replace H_d in (6.2) with $\bar{H}_d$, and, in place of (6.2), we use the implementable gradient

$$G(k) \triangleq \bar{H}_d^{\mathrm{T}} y(k)\phi^{\mathrm{T}}(k-d). \tag{6.3}$$

Note that the implementable gradient (6.3) can be used in practice due to Assumptions A3), A5), and A7).

Now, consider the adaptive law

$$\theta(k+1) = \theta(k-d) - \eta(k)G(k) \tag{6.4}$$

where $\eta : \mathbb{N} \to [0,\infty)$ is a step-size function. Note that, if $G(k) = 0$, then $\eta(k)$ is irrelevant. In accordance with Assumptions A10) and A11), the adaptive control law (6.4) does not require a measurement of the exogenous signal $w(k)$ and does not use knowledge of the exogenous dynamics (2.4), (2.5).

Subtracting θ_* from both sides of (6.4) yields the estimator-error update equation

$$\tilde{\theta}(k+1) = \tilde{\theta}(k-d) - \eta(k)G(k). \tag{6.5}$$

The closed-loop error system is thus given by

$$Y(k+1) = \mathcal{A}_Y Y(k) + \mathcal{B}_Y y(k) \tag{6.6}$$

$$\tilde{\theta}(k+1) = \tilde{\theta}(k-d) - \eta(k)G(k) \tag{6.7}$$

$$\vdots$$

$$\tilde{\theta}(k-d+1) = \tilde{\theta}(k-2d) - \eta(k-d)G(k-d) \tag{6.8}$$

where

$$\mathcal{A}_Y \triangleq \mathcal{N}_{l(n_c+d)}, \quad \mathcal{B}_Y \triangleq \begin{bmatrix} I_l \\ 0_{l(n_c+d-1)\times l}\end{bmatrix}$$

$$Y(k) \triangleq \begin{bmatrix} y(k-1) \\ \vdots \\ y(k-n_c-d)\end{bmatrix}. \tag{6.9}$$

Theorem 6.1: Consider the open-loop system (2.1), (2.2) satisfying Assumptions A1)–A11) and the adaptive feedback controller (5.1), (5.5), (5.6), (5.15), and (6.4). Furthermore, for all $k \geq k_0$, let $\zeta(k) \in \mathbb{R}$ be such that

$$0 < \zeta_{\mathrm{l}} \triangleq \inf_{j\geq k_0}\zeta(j) \leq \zeta(k) \leq \zeta_{\mathrm{u}} \triangleq \sup_{j\geq k_0}\zeta(j) < 2. \tag{6.10}$$

Finally, for all $k \in \mathbb{N}$ such that $G(k) \neq 0$, let $\eta(k) \in [0,\infty)$ satisfy

$$\eta(k) = 0, \quad \text{if } k < k_0 \tag{6.11}$$

$$\eta(k) = \zeta(k)\eta_{\mathrm{opt}}(k), \quad \text{if } k \geq k_0 \tag{6.12}$$

where

$$\eta_{\mathrm{opt}}(k) \triangleq \frac{\|y(k)\|_2^2}{\|G(k)\|_{\mathrm{F}}^2}. \tag{6.13}$$

Then, for all initial conditions $x(0)$ and $\theta(0)$, $\theta(k)$ is bounded, $u(k)$ is bounded, $\lim_{k\to\infty} y(k) = 0$, and $x(k)$ satisfying (2.1) is bounded. If, in addition, the open-loop dynamics matrix A is asymptotically stable and $u(k) = 0$ for all $k = 0,\ldots,k_0-1$, then, for all $x_w(0)$, the zero solution of the closed-loop error system (6.6)–(6.8) is Lyapunov stable.

Proof: Let $k \geq k_0$ so that, by Proposition 5.1, $\tilde{y}(k) = y(k)$. Consider the quadratic function

$$J(Y) \triangleq Y^{\mathrm{T}}\mathcal{P}Y \tag{6.14}$$

where $\mathcal{P} > 0$ satisfies the discrete-time Lyapunov equation

$$\mathcal{P} = \mathcal{A}_Y^{\mathrm{T}}\mathcal{P}\mathcal{A}_Y + \mathcal{Q} + \alpha I \tag{6.15}$$

where $\mathcal{Q} > 0$ and $\alpha > 0$. Note that $\mathcal{P}$ exists since $\mathcal{A}_Y$ is asymptotically stable. Defining

$$\Delta J(k) \triangleq J(Y(k+1)) - J(Y(k)) \tag{6.16}$$

it follows from (6.6) that

$$\begin{aligned}\Delta J(k) =& Y^{\mathrm{T}}(k+1)\mathcal{P}Y(k+1) - Y^{\mathrm{T}}(k)\mathcal{P}Y(k)\\ =& -Y^{\mathrm{T}}(k)(\mathcal{Q}+\alpha I)Y(k) + Y^{\mathrm{T}}(k)\mathcal{A}_Y^{\mathrm{T}}\mathcal{P}\mathcal{B}_Y y(k)\\ &+ y^{\mathrm{T}}(k)\mathcal{B}_Y^{\mathrm{T}}\mathcal{P}\mathcal{A}_Y Y(k) + y^{\mathrm{T}}(k)\mathcal{B}_Y^{\mathrm{T}}\mathcal{P}\mathcal{B}_Y y(k)\\ \leq& -Y^{\mathrm{T}}(k)(\mathcal{Q}+\alpha I)Y(k) + y^{\mathrm{T}}(k)\mathcal{B}_Y^{\mathrm{T}}\mathcal{P}\mathcal{B}_Y y(k)\\ &+ \alpha Y^{\mathrm{T}}(k)Y(k)\\ &+ \frac{1}{\alpha}y^{\mathrm{T}}(k)\left[\mathcal{B}_Y^{\mathrm{T}}\mathcal{P}\mathcal{A}_Y\mathcal{A}_Y^{\mathrm{T}}\mathcal{P}\mathcal{B}_Y\right]y(k)\\ \leq& -Y^{\mathrm{T}}(k)\mathcal{Q}Y(k) + \sigma_1 y^{\mathrm{T}}(k)y(k)\end{aligned} \tag{6.17}$$

where $\sigma_1 \triangleq \lambda_{\max}(\mathcal{B}_Y^{\mathrm{T}}\mathcal{P}\mathcal{B}_Y + \frac{1}{\alpha}\mathcal{B}_Y^{\mathrm{T}}\mathcal{P}\mathcal{A}_Y\mathcal{A}_Y^{\mathrm{T}}\mathcal{P}\mathcal{B}_Y)$.

Now, consider the positive-definite, radially unbounded Lyapunov-like function

$$V(Y(k),\tilde{\theta}(k),\ldots,\tilde{\theta}(k-d))$$

$$\stackrel{\triangle}{=} \ln(1 + a_1 Y^{\mathrm{T}}(k)\mathcal{P}Y(k)) + a_2 \sum_{i=0}^{d} \|\tilde{\theta}(k-i)\|_{\mathrm{F}}^2$$
$$= \ln(1 + a_1 J(Y(k))) + a_2 \sum_{i=0}^{d} \|\tilde{\theta}(k-i)\|_{\mathrm{F}}^2 \quad (6.18)$$

where $a_1 > 0$ and $a_2 > 0$ are specified later. The Lyapunov-like difference is thus given by

$$\Delta V(k) \stackrel{\triangle}{=} V(Y(k+1), \tilde{\theta}(k+1), \ldots, \tilde{\theta}(k-d+1)) - V(Y(k), \tilde{\theta}(k), \ldots, \tilde{\theta}(k-d)). \quad (6.19)$$

Evaluating $\Delta V(k)$ along the trajectories of the closed-loop error system (6.6)–(6.8) yields

$$\begin{aligned}\Delta V(k) =& \ln\left[1 + a_1 Y^{\mathrm{T}}(k+1)\mathcal{P}Y(k+1)\right] \\ &- \ln\left[1 + a_1 Y^{\mathrm{T}}(k)\mathcal{P}Y(k)\right] + a_2\eta^2(k)\|G(k)\|_{\mathrm{F}}^2 \\ &- 2a_2\eta(k)\left[\mathrm{tr}(\tilde{\theta}(k-d)G^{\mathrm{T}}(k))\right] \\ =& \ln\left[1 + a_1 J(Y(k)) + a_1\Delta J(k)\right] \\ &- \ln\left[1 + a_1 J(Y(k))\right] + a_2\eta^2(k)\|G(k)\|_{\mathrm{F}}^2 \\ &- 2a_2\eta(k)\left[\mathrm{tr}(\tilde{\theta}(k-d)\phi(k-d)y^{\mathrm{T}}(k)\bar{H}_d)\right] \\ =& \ln\left[1 + a_1 J(Y(k)) + a_1\Delta J(k)\right] \\ &- \ln\left[1 + a_1 J(Y(k))\right] + a_2\eta^2(k)\|G(k)\|_{\mathrm{F}}^2 \\ &- 2a_2\eta(k)y^{\mathrm{T}}(k)\bar{H}_d\tilde{\theta}(k-d)\phi(k-d) \\ =& \ln\left[1 + a_1 J(Y(k)) + a_1\Delta J(k)\right] \\ &- \ln\left[1 + a_1 J(Y(k))\right] \\ &+ a_2\big(-2\eta(k)\phi^{\mathrm{T}}(k-d)\tilde{\theta}^{\mathrm{T}}(k-d)H_d^{\mathrm{T}} \\ &\times \bar{H}_d\tilde{\theta}(k-d)\phi(k-d) + \eta^2(k)\|G(k)\|_{\mathrm{F}}^2\big) \\ =& \ln\left[1 + a_1 J(Y(k)) + a_1\Delta J(k)\right] \\ &- \ln\left[1 + a_1 J(Y(k))\right] + a_2\eta^2(k)\|G(k)\|_{\mathrm{F}}^2 \\ &- a_2\eta(k)\phi^{\mathrm{T}}(k-d)\tilde{\theta}^{\mathrm{T}}(k-d) \\ &\times [H_d^{\mathrm{T}}\bar{H}_d + \bar{H}_d^{\mathrm{T}}H_d]\tilde{\theta}(k-d)\phi(k-d).\end{aligned} \quad (6.20)$$

By Assumption A5) and using (5.15), we have

$$\begin{aligned}\Delta V(k) \le& \ln\left(1 + \frac{a_1\Delta J(k)}{1 + a_1 J(Y(k))}\right) \\ &+ a_2\big[-2\eta(k)\phi^{\mathrm{T}}(k-d)\tilde{\theta}^{\mathrm{T}}(k-d) \\ &\times H_d^{\mathrm{T}}H_d\tilde{\theta}(k-d)\phi(k-d) + \eta^2(k)\|G(k)\|_{\mathrm{F}}^2\big] \\ =& \ln\left(1 + \frac{a_1\Delta J(k)}{1 + a_1 J(Y(k))}\right) \\ &+ a_2\left[-2\eta(k)\|y(k)\|_2^2 + \eta^2(k)\|G(k)\|_{\mathrm{F}}^2\right] \\ =& \ln\left(1 + \frac{a_1\Delta J(k)}{1 + a_1 J(Y(k))}\right) \\ &- 2a_2\eta(k)\|y(k)\|_2^2 + a_2\eta^2(k)\frac{\|y(k)\|_2^2}{\eta_{\mathrm{opt}}(k)} \\ =& \ln\left(1 + \frac{a_1\Delta J(k)}{1 + a_1 J(Y(k))}\right) \\ &- 2a_2\eta_{\mathrm{opt}}^2(k)\frac{\eta(k)}{\eta_{\mathrm{opt}}(k)}\frac{\|y(k)\|_2^2}{\eta_{\mathrm{opt}}(k)} \\ &+ a_2\eta_{\mathrm{opt}}^2(k)\left(\frac{\eta(k)}{\eta_{\mathrm{opt}}(k)}\right)^2\frac{\|y(k)\|_2^2}{\eta_{\mathrm{opt}}(k)} \\ =& \ln\left(1 + \frac{a_1\Delta J(k)}{1 + a_1 J(Y(k))}\right) \\ &- a_2\eta_{\mathrm{opt}}^2(k)\left[2\zeta(k) - \zeta^2(k)\right]\|G(k)\|_{\mathrm{F}}^2 \\ \le& \ln\left(1 + \frac{a_1\Delta J(k)}{1 + a_1 J(Y(k))}\right) - a_2\kappa\eta_{\mathrm{opt}}^2(k)\|G(k)\|_{\mathrm{F}}^2 \\ =& \ln\left(1 + \frac{a_1\Delta J(k)}{1 + a_1 J(Y(k))}\right) - a_2\kappa\frac{\|y(k)\|_2^4}{\|G(k)\|_{\mathrm{F}}^2}\end{aligned} \quad (6.21)$$

where κ is defined by

$$\begin{aligned}\kappa &\stackrel{\triangle}{=} \inf_{j \ge k_0}\left[2\zeta(j) - \zeta^2(j)\right] \\ &= \min\{2\zeta_{\mathrm{l}} - \zeta_{\mathrm{l}}^2, 2\zeta_{\mathrm{u}} - \zeta_{\mathrm{u}}^2\}.\end{aligned} \quad (6.22)$$

Since $0 < \zeta_{\mathrm{l}} \le \zeta_{\mathrm{u}} < 2$, it follows that κ is positive.

Since, for all $x > 0$, $\ln x \le x - 1$, using

$$\|G(k)\|_{\mathrm{F}}^2 \le \sigma_{\max}^2(\bar{H}_d)\|y(k)\|_2^2\|\phi(k-d)\|_2^2 \quad (6.23)$$

and (6.17), we have

$$\begin{aligned}\Delta V(k) \le& a_1\frac{\Delta J(k)}{1 + a_1 J(Y(k))} - a_2\kappa\frac{y^{\mathrm{T}}(k)y(k)}{\sigma_{\max}^2(\bar{H}_d)\|\phi(k-d)\|_2^2} \\ \le& -a_1\frac{Y^{\mathrm{T}}(k)\mathcal{Q}Y(k)}{1 + a_1 Y^{\mathrm{T}}(k)\mathcal{P}Y(k)} \\ &+ a_1\sigma_1\frac{y^{\mathrm{T}}(k)y(k)}{1 + a_1 Y^{\mathrm{T}}(k)\mathcal{P}Y(k)} \\ &- a_2\kappa\frac{y^{\mathrm{T}}(k)y(k)}{\sigma_{\max}^2(\bar{H}_d)\|\phi(k-d)\|_2^2}.\end{aligned} \quad (6.24)$$

Furthermore, defining

$$U_0(k) \stackrel{\triangle}{=} \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-n_c) \end{bmatrix}, \quad Y_0(k) \stackrel{\triangle}{=} \begin{bmatrix} y(k-1) \\ \vdots \\ y(k-n_c) \end{bmatrix} \quad (6.25)$$

it follows from $\|\phi(k-d)\|_2^2 = \|Y_0(k-d)\|_2^2 + \|U_0(k-d)\|_2^2$ that

$$\begin{aligned}\Delta V(k) \le& -a_1\frac{Y^{\mathrm{T}}(k)\mathcal{Q}Y(k)}{1 + a_1 Y^{\mathrm{T}}(k)\mathcal{P}Y(k)} \\ &+ a_1\sigma_1\frac{y^{\mathrm{T}}(k)y(k)}{1 + a_1\lambda_{\min}(\mathcal{P})\|Y(k)\|_2^2}\end{aligned}$$

$$- a_2\kappa \frac{y^{\mathrm{T}}(k)y(k)}{\sigma_{\max}^2(\bar{H}_d)\left[\|Y_0(k-d)\|_2^2 + \|U_0(k-d)\|_2^2\right]}. \tag{6.26}$$

Assumption A2) implies that the invariant zeros of the system (2.1)–(2.5) from u to y are asymptotically stable. Thus, it follows from Theorem B.1 with $p = n_{\mathrm{c}}$ that there exist $b_1 > 0$ and $b_2 > 0$ such that

$$\begin{aligned}\|U_0(k-d)\|_2^2 &\le b_1 + b_2 \left\| \begin{bmatrix} y(k-1) \\ \vdots \\ y(k-n_{\mathrm{c}}-1) \end{bmatrix} \right\|_2^2 \\ &= b_1 + b_2 \left\| \begin{bmatrix} Y_0(k) \\ y(k-n_{\mathrm{c}}-1) \end{bmatrix} \right\|_2^2 \\ &\le b_1 + b_2 \left\| \begin{bmatrix} Y_0(k) \\ y(k-n_{\mathrm{c}}-1) \\ y(k-n_{\mathrm{c}}-2) \\ \vdots \\ y(k-n_{\mathrm{c}}-d) \end{bmatrix} \right\|_2^2 \\ &= b_1 + b_2 \|Y(k)\|_2^2. \end{aligned} \tag{6.27}$$

Therefore, since $\|Y_0(k-d)\|_2^2 \le \|Y(k)\|_2^2$, it follows that

$$\begin{aligned}\Delta V(k) \le& -a_1 \frac{Y^{\mathrm{T}}(k)\mathcal{Q}Y(k)}{1 + a_1 Y^{\mathrm{T}}(k)\mathcal{P}Y(k)} \\ &+ a_1\sigma_1 \frac{y^{\mathrm{T}}(k)y(k)}{1 + a_1\lambda_{\min}(\mathcal{P})\|Y(k)\|_2^2} \\ &- a_2\kappa \frac{y^{\mathrm{T}}(k)y(k)}{\sigma_{\max}^2(\bar{H}_d)\left[b_1 + \|Y_0(k-d)\|_2^2 + b_2\|Y(k)\|_2^2\right]} \\ \le& -a_1 \frac{Y^{\mathrm{T}}(k)\mathcal{Q}Y(k)}{1 + a_1 Y^{\mathrm{T}}(k)\mathcal{P}Y(k)} \\ &+ a_1\sigma_1 \frac{y^{\mathrm{T}}(k)y(k)}{1 + a_1\lambda_{\min}(\mathcal{P})\|Y(k)\|_2^2} \\ &- a_2\kappa \frac{y^{\mathrm{T}}(k)y(k)}{\sigma_{\max}^2(\bar{H}_d)\left[b_1 + (b_2+1)\|Y(k)\|_2^2\right]} \\ =& -a_1 \frac{Y^{\mathrm{T}}(k)\mathcal{Q}Y(k)}{1 + a_1 Y^{\mathrm{T}}(k)\mathcal{P}Y(k)} \\ &+ a_1\sigma_1 \frac{y^{\mathrm{T}}(k)y(k)}{1 + a_1\lambda_{\min}(\mathcal{P})\|Y(k)\|_2^2} \\ &- a_2\kappa \frac{b_3 y^{\mathrm{T}}(k)y(k)}{1 + b_4\|Y(k)\|_2^2} \end{aligned} \tag{6.28}$$

where $b_3 \triangleq 1/(\sigma_{\max}^2(\bar{H}_d)b_1)$ and $b_4 \triangleq (b_2+1)/b_1$.

Next, letting $a_1 \triangleq b_4/\lambda_{\min}(\mathcal{P})$ and $a_2 \triangleq a_1\sigma_1/(b_3\kappa)$, it follows that

$$\Delta V(k) \le -W(Y(k)) \tag{6.29}$$

where

$$W(Y(k)) \triangleq a_1 \frac{Y^{\mathrm{T}}(k)\mathcal{Q}Y(k)}{1 + a_1 Y^{\mathrm{T}}(k)\mathcal{P}Y(k)}. \tag{6.30}$$

To show that $\tilde{\theta}(k)$ and $Y(k)$ are bounded, summing (6.29) from k_0 to $k-1$, where $k_0 \le k-1$, yields

$$\begin{aligned} 0 &\le V(Y(k), \tilde{\theta}(k), \ldots, \tilde{\theta}(k-d)) \\ &\le -\sum_{j=k_0}^{k-1} W(Y(j)) + V(Y(k_0), \tilde{\theta}(k_0), \ldots, \tilde{\theta}(k_0-d)) \\ &\le V(Y(k_0), \tilde{\theta}(k_0), \ldots, \tilde{\theta}(k_0-d)). \end{aligned} \tag{6.31}$$

Thus, $V(Y(k), \tilde{\theta}(k), \ldots, \tilde{\theta}(k-d))$ is bounded. Since $V(Y(k), \tilde{\theta}(k), \ldots, \tilde{\theta}(k-d))$ is positive definite and radially unbounded, it follows that $\tilde{\theta}(k)$ and $Y(k)$ are bounded. Thus, $\theta(k) = \tilde{\theta}(k) + \theta_*$ is bounded.

Now, we show that $\lim_{k\to\infty} Y(k) = 0$. Since V is positive definite, it follows from (6.29) that

$$\begin{aligned} 0 &\le \lim_{k\to\infty} \sum_{j=k_0}^{k} W(Y(j)) \\ &\le -\lim_{k\to\infty} \sum_{j=k_0}^{k} \Delta V(j) \\ &= V(Y(k_0), \tilde{\theta}(k_0), \ldots, \tilde{\theta}(k_0-d)) \\ &\quad - \lim_{k\to\infty} V(Y(k), \tilde{\theta}(k), \ldots, \tilde{\theta}(k-d)) \\ &\le V(Y(k_0), \tilde{\theta}(k_0), \ldots, \tilde{\theta}(k_0-d)) \end{aligned} \tag{6.32}$$

where all three limits exist. Thus, $\lim_{k\to\infty} W(Y(k)) = 0$. Next, note that

$$0 \le \upsilon(\|Y(k)\|) \le W(Y(k)) \tag{6.33}$$

where

$$\upsilon(\|Y(k)\|) \triangleq \frac{a_1\lambda_{\min}(\mathcal{Q})\|Y(k)\|_2^2}{1 + a_1\lambda_{\max}(\mathcal{P})\|Y(k)\|_2^2}. \tag{6.34}$$

Thus, $\lim_{k\to\infty} \upsilon(\|Y(k)\|) = 0$. Rewriting (6.34) as

$$\|Y(k)\| = \sqrt{\frac{\upsilon(\|Y(k)\|)}{a_1(\lambda_{\min}(\mathcal{Q}) - \upsilon(\|Y(k)\|)\lambda_{\max}(\mathcal{P}))}} \tag{6.35}$$

it follows that $\lim_{k\to\infty} Y(k) = 0$, and thus $\lim_{k\to\infty} y(k) = 0$. Finally, it follows from (6.27) that $u(k)$ is bounded. Thus, $\phi(k)$ is bounded. Since $\phi(k)$ is the state of the nonminimal state-space realization (3.3)–(3.8) of the time-series representation (2.13) for the original state-space system (2.1), (2.2), it follows that $x(k)$ is bounded.

To prove the last statement of Theorem 6.1, let $x_w(0)$ be given, and let

$$\mathcal{X}(k) \triangleq \begin{bmatrix} Y(k) \\ \tilde{\theta}(k-d) \\ \vdots \\ \tilde{\theta}(k-2d) \end{bmatrix} \tag{6.36}$$

be the state of the closed-loop error system (6.6)–(6.8). Since V is positive definite, and, by (6.29), ΔV is negative semidefinite, it follows from [19, Lemma A.3.12] that the zero solution of the closed-loop error system is Lyapunov stable starting at k_0. Therefore, given $\varepsilon_0 > 0$, there exists $\delta_0 > 0$ such that, if $\|\mathcal{X}(k_0)\| < \delta_0$, then $\|\mathcal{X}(k)\| < \varepsilon_0$ for all $k \geq k_0$.

Now, assume that the open-loop dynamics matrix A is asymptotically stable and that $u(k) = 0$ for all $k < k_0$. Then, it follows that there exists $\delta_1 > 0$ such that, if $\|\mathcal{X}(0)\| < \delta_1$, then $\|\mathcal{X}(k)\| < \delta_0$ for all $k = 0, \ldots, k_0 - 1$. Consequently, for all $\varepsilon_0 > 0$, there exists $\delta_1 > 0$ such that, if $\|\mathcal{X}(0)\| < \delta_1$, then $\|\mathcal{X}(k)\| < \varepsilon_0$ for all $k \geq 0$. Therefore, the zero solution of the closed-loop error system (6.6)–(6.8) is Lyapunov stable starting at $k = 0$. ■

The step size $\eta_{\rm opt}(k)$ given by (6.13) has the following interpretation. Note that (6.21) can be written as

$$\Delta V(k) \leq \ln\left(1 + \frac{a_1 \Delta J(k)}{1 + a_1 J(Y(k))}\right) + a_2\left[(\eta(k) - \eta_{\rm opt}(k))^2 - \eta_{\rm opt}^2(k)\right] \|G(k)\|_{\rm F}^2. \tag{6.37}$$

Since the quadratic function $(\eta(k) - \eta_{\rm opt}(k))^2 - \eta_{\rm opt}^2(k)$ achieves its minimum at $\eta(k) = \eta_{\rm opt}(k)$, it follows that the upper bound for $\Delta V(k)$ given by (6.37) is minimized by $\eta(k) = \eta_{\rm opt}(k)$.

An analogous optimal step size is constructed in [1], where an ideal (not necessarily deadbeat) controller is assumed to exist. However, in the present paper, an ideal deadbeat internal model controller is proven to exist and have the properties given by Theorem 4.1 and Proposition 5.1. Hence, for all $k \geq k_0$, $\tilde{y}(k) = y(k)$ is known, and thus $\eta_{\rm opt}(k)$ is computable.

In [1], $\tilde{y}(k) = y(k) - y_*(k)$ is unknown since $y_*(k)$ is unknown, and thus the optimal step size is not computable in [1]. To obtain a computable step size in [1], several implementable step sizes are defined. We can construct an analogous step size $\eta_{\rm imp}(k)$. Specifically, $\eta_{\rm imp}(k)$ defined by

$$\eta_{\rm imp}(k) \triangleq \frac{1}{\varepsilon + \sigma_{\max}^2(\bar{H}_d)\|\phi(k-d)\|_2^2} \tag{6.38}$$

where $\varepsilon \geq 0$, satisfies

$$\eta_{\rm imp}(k) \leq \eta_{\rm opt}(k). \tag{6.39}$$

Theorem 6.1 holds with (6.12) replaced by

$$\eta(k) = \zeta(k)\eta_{\rm imp}(k). \tag{6.40}$$

However, (6.38) is not needed in the present paper since $\tilde{y}(k) = y(k)$ is known for all $k \geq k_0$, and thus $\eta_{\rm opt}(k)$ is computable, and thus implementable.

Let $\{\psi(k)\}_{k=k_0}^{\infty}$ satisfy

$$\frac{\zeta_{\rm u}}{2} < \sup\nolimits_{j \geq k_0} \psi(j) < \infty \tag{6.41}$$

and define $\hat{\zeta}(k) \triangleq \frac{\zeta(k)}{\psi(k)}$. Then, if (6.10) holds for $\{\zeta(k)\}_{k=k_0}^{\infty}$, then it also holds with $\{\zeta(k)\}_{k=k_0}^{\infty}$ replaced by $\{\hat{\zeta}(k)\}_{k=k_0}^{\infty}$. The term $\psi(k)$ can be viewed as a tuning variable relating to the magnitude of the bound $\bar{H}_d$ representing the accuracy with which H_d is modeled. In particular, by defining the time-varying bound

$$\bar{H}_{d,k} \triangleq \sqrt{\psi(k)}\bar{H}_d \tag{6.42}$$

$\bar{H}_d$ can be replaced with $\bar{H}_{d,k}$ in Assumption A5) and (6.3). The example in the next section shows that the transient response is directly related to $\psi(k)$, and thus $\zeta(k)$. Therefore, $\psi(k)$ and $\zeta(k)$ are indirectly related to the conservatism of the bound $\bar{H}_d$ on the first nonzero Markov parameter.

VII. Mass-Spring-Dashpot Example

Consider the 3-mass structure with all possible spring and dashpot connections given by

$$M\ddot{q} + C\dot{q} + Kq = \mu \begin{bmatrix} 0 \\ u \\ 0 \end{bmatrix} + \mu \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \tag{7.1}$$

where

$$M \triangleq \operatorname{diag}(m_1, m_2, m_3) \tag{7.2}$$

$$C \triangleq \begin{bmatrix} c_1 + c_{1,2} + c_{1,3} & -c_{1,2} & -c_{1,3} \\ -c_{1,2} & c_{1,2} + c_2 + c_{2,3} & -c_{2,3} \\ -c_{1,3} & -c_{2,3} & c_{1,3} + c_{2,3} + c_3 \end{bmatrix} \tag{7.3}$$

$$K \triangleq \begin{bmatrix} k_1 + k_{1,2} + k_{1,3} & -k_{1,2} & -k_{1,3} \\ -k_{1,2} & k_{1,2} + k_2 + k_{2,3} & -k_{2,3} \\ -k_{1,3} & -k_{2,3} & k_{1,3} + k_{2,3} + k_3 \end{bmatrix} \tag{7.4}$$

$$q \triangleq [\,q_1 \quad q_2 \quad q_3\,]^{\rm T} \tag{7.5}$$

u is the control, and w_1, w_2, and w_3 are disturbances. For this example, the masses are $m_1 = 0.01$ kg, $m_2 = 0.02$ kg, and $m_3 = 0.01$ kg; the damping coefficients are $c_1 = 5$ kg/s, $c_2 = 3$ kg/s, $c_3 = 4$ kg/s, $c_{1,2} = 0.1$ kg/s, $c_{1,3} = 0.2$ kg/s, and $c_{2,3} = 0.3$ kg/s; and the spring constants are $k_1 = 11$ kg/s^2, $k_2 = 12$ kg/s^2, $k_3 = 13$ kg/s^2, $k_{1,2} = 70$ kg/s^2, $k_{1,3} = 60$ kg/s^2, and $k_{2,3} = 30$ kg/s^2. The input gain $\mu = 10^4$ is used for numerical conditioning.

The control objective is to reject the disturbances w_1, w_2, and w_3 while forcing the position of m_2 to follow the command w_4. Thus, the performance variable is given by $y = q_2 - w_4$. We assume that the command and disturbance signals are generated by a Lyapunov-stable discrete-time linear system whose spectrum is unknown.

The continuous-time system (7.1)–(7.5) is sampled at 100 Hz with input provided by a zero-order hold. It follows from [20] that the resulting sampled-data system is minimum phase from u to y. Thus, Assumption A2) is satisfied. Furthermore, the sampled-data system has a delay $d = 1$, and the first nonzero Markov parameter is $H_1 = 0.3$. Let the bound on the first nonzero Markov parameter be $\bar{H}_1 = 1.5H_1 = 0.45$, which satisfies Assumption A5). Thus, the mass-spring-dashpot sampled-data system satisfies Assumptions A1)–A11).

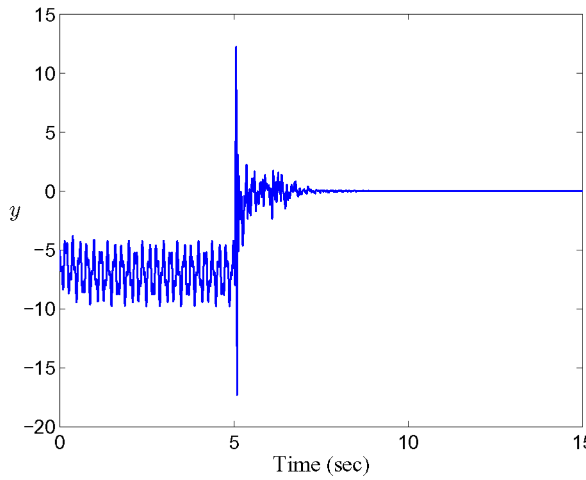

Fig. 2. Adaptive controller with $\eta(k) = \eta_{\rm opt}(k)$ [that is, $\zeta(k) \equiv 1$] is implemented in the feedback loop after 5 s. The performance variable y converges to zero.

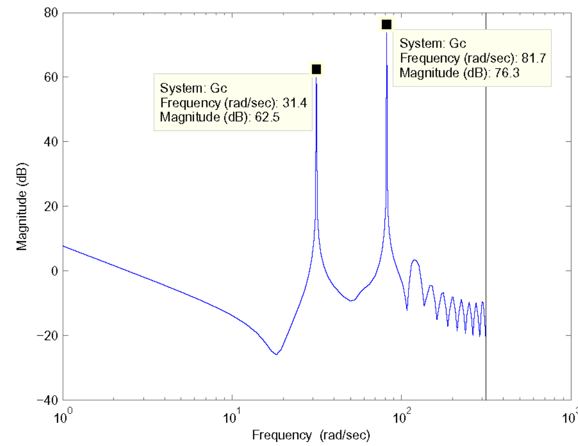

Fig. 3. Bode magnitude plot of the adaptive controller at $t = 15$ s. The adaptive controller places poles at the disturbance frequencies $\omega_1 = 5$ Hz and $\omega_2 = 13$ Hz. The controller magnitude $|G_{\rm c}(e^{\jmath\omega T_{\rm s}})|$ is plotted for ω up to the Nyquist frequency $\omega_{\rm Nyq} = \pi/T_{\rm s} = 314$ rad/s.

The three unknown disturbance signals are discrete-time sinusoids with frequency $\omega_1 = 5$ Hz, while the unknown command signal is a discrete-time sinusoid with frequency $\omega_2 = 13$ Hz plus a constant bias. More specifically, the unknown disturbance and command signals are

$$w_1(k) = \sin 2\pi\omega_1 T_{\rm s} k \tag{7.6}$$

$$w_2(k) = -1.5 \sin 2\pi\omega_1 T_{\rm s} k \tag{7.7}$$

$$w_3(k) = 2 \sin 2\pi\omega_1 T_{\rm s} k \tag{7.8}$$

$$w_4(k) = \sin 2\pi\omega_2 T_{\rm s} k + 7 \tag{7.9}$$

where the sample time is $T_{\rm s} = 0.01$ s. The open-loop system is given the initial conditions $q(0) = [1 \quad 2 \quad 0]^{\rm T}$ m and $\dot{q}(0) = [-1 \quad -2 \quad 0]^{\rm T}$ m/s. Fig. 2 is a time history of the performance variable y. The system is allowed to run open loop for 5 s. Then, the adaptive controller (5.5) and (6.4) with $n_{\rm c} = 20$, $d = 1$, $\bar{H}_1 = 0.45$, and $\eta(k) = \eta_{\rm opt}(k)$ is implemented in feedback with the initial condition $\theta(0) = 0$. The performance variable y converges to zero, which implies that the position q_2 asymptotically follows the command w_4 and rejects the disturbances w_1, w_2, and w_3. In particular, Fig. 3 shows that the controller places poles at the disturbance and command frequencies $\omega_1 = 5$ Hz and $\omega_2 = 13$ Hz. Note that $k_0 = 21$, which corresponds to 0.21 s.

The controller's transient performance has significant peaks, as shown in Fig. 2. This transient behavior is due in part to the bound $\bar{H}_1$ on the first nonzero Markov parameter H_1. However, the speed of adaptation, and thus the transient performance are directly influenced by $\zeta(k)$. Specifically, the controller adapts more slowly when $\zeta(k)$ is small and more quickly when $\zeta(k)$ is large. To demonstrate this effect, consider the adaptive controller (5.5) and (6.4) with $\eta(k) = (1/5)\eta_{\rm opt}(k)$. After the system is allowed to run open loop for 5 s, the adaptive controller (5.5) and (6.4) with $n_{\rm c} = 20$, $d = 1$, $\bar{H}_1 = 0.45$, and $\eta(k) = (1/5)\eta_{\rm opt}(k)$ is implemented in feedback with the initial condition $\theta(0) = 0$. Fig. 4 shows that the performance variable y converges to zero with improved transient performance, but at the expense of convergence time. Equivalently, setting $\zeta(k) \equiv 1$, $\psi(k) \equiv 5$, and replacing $\bar{H}_1$ with $\bar{H}_{1,k} \equiv 0.45\sqrt{5} \cong 1.0$ yields the same result. In this case, the transient performance is viewed as a consequence of how well the bound $\bar{H}_{1,k}$ models the first nonzero Markov parameter H_1.

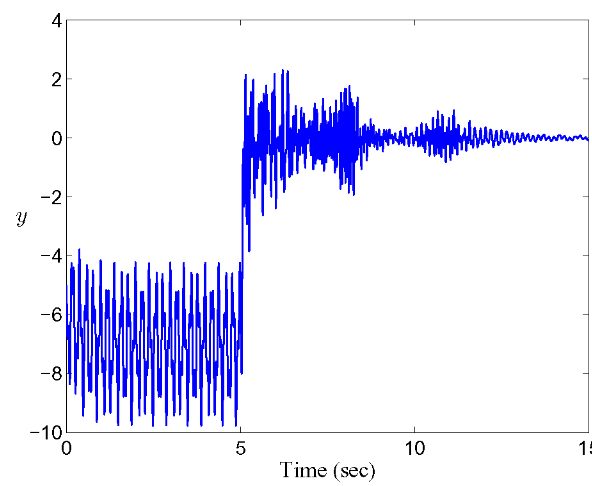

Fig. 4. Adaptive controller with $\eta(k) = (1/5)\eta_{\rm opt}(k)$ [that is, $\zeta(k) \equiv 1/5$] is implemented in the feedback loop after 5 s. The performance variable y converges to zero with improved transient performance, but much slower convergence compared to Fig. 2.

For this mass-spring-dashpot example, slower adaptation can reduce peaks in the transient performance, but faster adaptation causes faster convergence. In fact, these observations hold for many open-loop stable systems; however, if the system is open-loop unstable, then the effects of adaptation speed differ. For the open-loop stable mass-spring-dashpot system, one might consider using slower adaptation when the controller is initially turned on, then increasing the adaptation speed. In particular, let $\zeta(k) = \exp(-3/k)$. Fig. 5 shows a time history of the performance variable y. The system is allowed to run open loop for 5 s. Then, the adaptive controller (5.5) and (6.4) with $n_{\rm c} = 20$, $d = 1$, $\bar{H}_1 = 0.45$, and $\eta(k) = \exp(-3/k)\eta_{\rm opt}(k)$ is

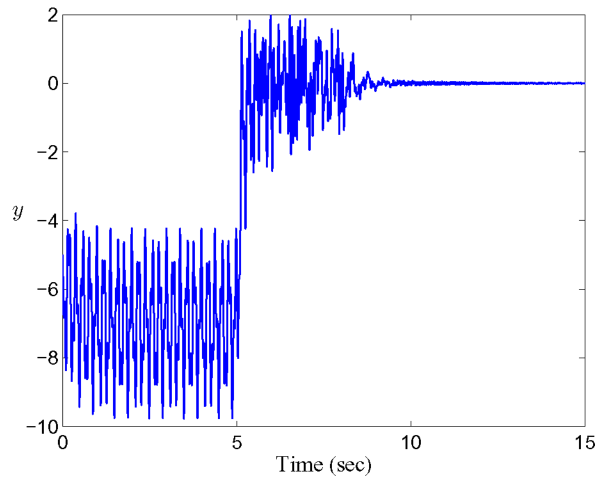

Fig. 5. Adaptive controller with $\eta(k) = \exp(-3/k)\eta_{\rm opt}(k)$ [that is, $\zeta(k) = \exp(-3/k)$] is implemented in the feedback loop after 5 s. The performance variable y converges to zero with improved transient performance compared to Figs. 2 and 4. Furthermore, the performance converges almost as quickly as Fig. 2, and more quickly than Fig. 4.

implemented in feedback with the initial condition $\theta(0) = 0$. The performance variable y converges to zero with improved transient performance and good convergence time. Equivalently, setting $\zeta(k) \equiv 1$, $\psi(k) = \exp(3/k)$, and replacing $\bar{H}_1$ with $\bar{H}_{1,k} = 0.45\sqrt{\exp(3/k)}$ yields the same result.

VIII. Conclusion

We considered adaptive stabilization, command following, and disturbance rejection for MIMO minimum-phase discrete-time systems, where the command and disturbance signals are generated by a linear system with unknown dynamics. Future work includes extending the discrete-time adaptive controller to nonsquare nonminimum-phase plants. Lifting techniques, which transform a high-rate nonminimum-phase system into a low-rate minimum-phase system, can be used in this case [21].

Appendix A

Theorem A.1: Consider the discrete-time system

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}u(k) + \hat{D}_1 w(k) \quad \text{(A.1)}$$

$$y(k) = \hat{C}\hat{x}(k) + \hat{D}_2 w(k) \quad \text{(A.2)}$$

where $\hat{x}(k) \in \mathbb{R}^{\hat{n}}$, $y(k) \in \mathbb{R}^{\hat{l}_y}$, $u(k) \in \mathbb{R}^{\hat{l}_u}$, $w(k) \in \mathbb{R}^{l_w}$, and assume that the following conditions hold.

1) $(\hat{A}, \hat{B}, \hat{C})$ is controllable and observable.
2) $\hat{l}_u \geq \hat{l}_y$.
3) The exogenous signal $w(k)$ is generated from the output of the linear system

$$\hat{x}_w(k+1) = \hat{A}_w \hat{x}_w(k), \quad w(k) = \hat{C}_w \hat{x}_w(k) \quad \text{(A.3)}$$

where $\hat{x}_w(k) \in \mathbb{R}^{\hat{n}_w}$, $(\hat{A}_w, \hat{C}_w)$ is observable, for all $\lambda \in \text{spec}(\hat{A}_w)$, λ is not a transmission zero of $G(z) = \hat{C}(\mathbf{z}I - \hat{A})^{-1}\hat{B}$, and normal rank $G = \min(\hat{l}_u, \hat{l}_y)$.

Furthermore, consider the linear time-invariant controller

$$\hat{x}_c(k+1) = \hat{A}_c\hat{x}_c(k) + \hat{B}_c y(k), \quad u(k) = \hat{C}_c\hat{x}_c(k) \quad \text{(A.4)}$$

where $\hat{x}_c(k) \in \mathbb{R}^{n_{\rm db}}$ so that the closed-loop system is given by

$$x_{\rm cl}(k+1) = A_{\rm cl}x_{\rm cl}(k) + D_{\rm cl}w(k) \quad \text{(A.5)}$$

$$y(k) = C_{\rm cl}x_{\rm cl}(k) + D_2 w(k) \quad \text{(A.6)}$$

where

$$A_{\rm cl} \triangleq \begin{bmatrix} \hat{A} & \hat{B}\hat{C}_c \\ \hat{B}_c\hat{C} & \hat{A}_c \end{bmatrix}, \quad D_{\rm cl} \triangleq \begin{bmatrix} \hat{D}_1 \\ \hat{B}_c\hat{D}_2 \end{bmatrix}$$

$$C_{\rm cl} \triangleq [\hat{C} \quad 0], \quad x_{\rm cl} \triangleq \begin{bmatrix} \hat{x} \\ \hat{x}_c \end{bmatrix}. \quad \text{(A.7)}$$

Then, for all $n_{\rm db} \geq \hat{n} + 2\hat{n}_w\hat{l}_y$, there exists $(\hat{A}_c, \hat{B}_c, \hat{C}_c)$ such that $A_{\rm cl}$ is nilpotent. Consequently, for all initial conditions $x_{\rm cl}(0)$, and $\hat{x}_w(0)$, and, for all $k \geq 2(\hat{n} + \hat{n}_w\hat{l}_y)$, $y(k) = 0$.

Proof: A straightforward extension of the arguments used in Section II to show that A_w can be chosen to have distinct eigenvalues shows that, without loss of generality, $\hat{A}_w$ can be assumed to be cyclic. We consider the open-loop system (A.1)–(A.2) connected in cascade with an internal model of the exogenous dynamics

$$\hat{x}_1(k+1) = A_W\hat{x}_1(k) + B_W y(k) \quad \text{(A.8)}$$

where $A_W \triangleq I_{\hat{l}_y} \otimes \hat{A}_w$, $B_W \triangleq I_{\hat{l}_y} \otimes \hat{B}_w$, and $\hat{B}_w \in \mathbb{R}^{\hat{n}_w}$ is chosen such that $(\hat{A}_w, \hat{B}_w)$ is controllable [22, Fact 5.12.6]. Note that the dynamics (A.8) contains $\hat{l}_y$ copies of the exogenous dynamics $\hat{A}_w$. The cascade (A.1), (A.2), and (A.8) is

$$\begin{bmatrix} \hat{x}(k+1) \\ \hat{x}_1(k+1) \end{bmatrix} = \begin{bmatrix} \hat{A} & 0 \\ B_W\hat{C} & A_W \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{x}_1(k) \end{bmatrix} + \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} \hat{D}_1 \\ B_W\hat{D}_2 \end{bmatrix} w(k) \quad \text{(A.9)}$$

$$\begin{bmatrix} y(k) \\ \hat{x}_1(k) \end{bmatrix} = \begin{bmatrix} \hat{C} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{x}_1(k) \end{bmatrix} + \begin{bmatrix} \hat{D}_2 \\ 0 \end{bmatrix} w(k). \quad \text{(A.10)}$$

Now, we show that the augmented system (A.9), (A.10) is controllable and observable. First, define the stable region

$$\mathcal{S} \triangleq \{\lambda \in \mathbb{C} : |\lambda| < 1\} \quad \text{(A.11)}$$

and the unstable region $\mathcal{U} \triangleq \mathbb{C}\backslash\mathcal{S}$. Let $\mathbf{z} \in \mathcal{U}$ and $\lambda \in \text{spec}(\hat{A}_w) \subset \mathcal{U}$. Since $(\hat{A}, \hat{B})$ is controllable, it follows that

$$\begin{aligned} &\text{rank}\begin{bmatrix} \hat{A} - \mathbf{z}I & \hat{B} & 0 \\ B_W\hat{C} & 0 & A_W - \mathbf{z}I \end{bmatrix} \\ &\geq \text{rank}\begin{bmatrix} \hat{A} - \lambda I & \hat{B} & 0 \\ B_W\hat{C} & 0 & A_W - \lambda I \end{bmatrix} \\ &\geq \text{rank}\left(\begin{bmatrix} I_{\hat{n}} & 0 & 0 \\ 0 & B_W & A_W - \lambda I \end{bmatrix} \times \begin{bmatrix} \hat{A} - \lambda I & \hat{B} & 0 \\ \hat{C} & 0 & 0 \\ 0 & 0 & I_{\hat{l}_y\hat{n}_w} \end{bmatrix}\right). \end{aligned} \quad \text{(A.12)}$$

Conditions 2) and 3) imply that

$$\operatorname{rank}\begin{bmatrix}\hat{A}-\lambda I & \hat{B} & 0\\ \hat{C} & 0 & 0\\ 0 & 0 & I_{\hat{l}_y\hat{n}_w}\end{bmatrix} = \hat{n}+\hat{l}_y+\hat{l}_y\hat{n}_w$$

which is full row rank. Therefore,

$$\begin{aligned}\hat{n}+\hat{l}_y\hat{n}_w &\geq \operatorname{rank}\begin{bmatrix}\hat{A}-\mathbf{z}I & \hat{B} & 0\\ B_W\hat{C} & 0 & A_W-\mathbf{z}I\end{bmatrix}\\ &\geq \operatorname{rank}\begin{bmatrix}I_{\hat{n}} & 0 & 0\\ 0 & B_W & A_W-\lambda I\end{bmatrix}.\end{aligned} \tag{A.13}$$

Since (A_W, B_W) is controllable, it follows that

$$\operatorname{rank}\begin{bmatrix}I_{\hat{n}} & 0 & 0\\ 0 & B_W & A_W-\lambda I\end{bmatrix} = \hat{n}+\hat{l}_y\hat{n}_w$$

and thus

$$\operatorname{rank}\begin{bmatrix}\hat{A}-\mathbf{z}I & \hat{B} & 0\\ B_W\hat{C} & 0 & A_W-\mathbf{z}I\end{bmatrix} = \hat{n}+\hat{l}_y\hat{n}_w. \tag{A.14}$$

Hence, $\left(\begin{bmatrix}\hat{A} & 0\\ B_W\hat{C} & A_W\end{bmatrix}, \begin{bmatrix}\hat{B}\\ 0\end{bmatrix}\right)$ is controllable. Since, in addition, $(\hat{A},\hat{C})$ is observable, it follows that $\left(\begin{bmatrix}\hat{A} & 0\\ B_W\hat{C} & A_W\end{bmatrix}, \begin{bmatrix}\hat{C} & 0\\ 0 & I\end{bmatrix}\right)$ is observable. Thus, there exists an observer-based controller that stabilizes the augmented system (A.9)–(A.10) and yields a closed-loop system with nilpotent dynamics. It follows that, for all $n_{\rm db} \geq \hat{n}+2\hat{n}_w\hat{l}_y$, there exists a linear time-invariant controller (A.4) of order $n_{\rm db}$ such that the equilibrium of the closed-loop system (A.5)–(A.7) is asymptotically stable, where $A_{\rm cl}$ is nilpotent, and, for all initial conditions $x_{\rm cl}(0)$ and $\hat{x}_w(0)$, $\lim_{k\to\infty} y(k) = 0$.

The closed-loop system (A.5)–(A.7) with exogenous input $w(k)$ can be written as

$$x_s(k+1) = A_s x_s(k), \quad y(k) = C_s x_s(k), \tag{A.15}$$

where

$$A_s \triangleq \begin{bmatrix}A_{\rm cl} & D_{\rm cl}\hat{C}_w\\ 0 & \hat{A}_w\end{bmatrix}, \quad C_s \triangleq [\, C_{\rm cl} \quad \hat{D}_2\hat{C}_w \,] \tag{A.16}$$

and $x_s \triangleq \begin{bmatrix}x_{\rm cl}\\ \hat{x}_w\end{bmatrix}$. Since $\lim_{k\to\infty} y(k) = 0$ and $A_{\rm cl}$ is asymptotically stable, it follows from [12] and [17, Lemma 2.1] that there exists $S \in \mathbb{R}^{2(\hat{n}+\hat{n}_w\hat{l}_y)\times\hat{n}_w}$ such that

$$A_{\rm cl}S - S\hat{A}_w = D_{\rm cl}\hat{C}_w \tag{A.17}$$

$$C_{\rm cl}S = \hat{D}_2\hat{C}_w. \tag{A.18}$$

Now, define

$$Q \triangleq \begin{bmatrix}I & -S\\ 0 & I\end{bmatrix}$$

and consider the change of basis

$$\bar{A}_s \triangleq Q^{-1}A_sQ = \begin{bmatrix}A_{\rm cl} & 0\\ 0 & \hat{A}_w\end{bmatrix}, \quad \bar{C}_s \triangleq C_sQ = [\, C_{\rm cl} \quad 0 \,]. \tag{A.19}$$

Then, we have $y(k) = \bar{C}_s\bar{A}_s^kQ^{-1}x_s(0) = C_{\rm cl}A_{\rm cl}^k[x_{\rm cl}(0) + S\hat{x}_w(0)]$. Since, $A_{\rm cl} \in \mathbb{R}^{2(\hat{n}+\hat{n}_w\hat{l}_y)\times 2(\hat{n}+\hat{n}_w\hat{l}_y)}$ is nilpotent, it follows that, for all initial conditions $x_{\rm cl}(0)$ and $\hat{x}_w(0)$, and for all $k \geq 2(\hat{n}+\hat{n}_w\hat{l}_y)$, $y(k) = 0$. ■

APPENDIX B

Consider the discrete-time system (2.1), (2.2), where $y(k) \in \mathbb{R}^l$ and $u(k) \in \mathbb{R}^l$. To derive the inverse system, we increment (2.2) by d steps, yielding

$$y(k+d) = Cx(k+d) + D_2w(k+d) \tag{B.1}$$

$$= CA^dx(k) + H_du(k) + [\, D_2 \quad CD_1 \quad \cdots \quad CA^{d-1}D_1 \,]\begin{bmatrix}w(k+d)\\ \vdots\\ w(k)\end{bmatrix} \tag{B.2}$$

where $H_d \triangleq CA^{d-1}B$ is the first nonzero Markov parameter from u to y. It follows from (B.2) and Assumption A4) that

$$u(k) = -H_d^{-1}CA^dx(k) + H_d^{-1}y(k+d) - H_d^{-1}[\, D_2 \quad CD_1 \quad \cdots \quad CA^{d-1}D_1 \,]\begin{bmatrix}w(k+d)\\ \vdots\\ w(k)\end{bmatrix}.$$

The inverse system is thus given by

$$x(k+1) = A_{\rm R}x(k) + B_{\rm R}y_d(k) + D_{1\rm R}W_d(k) \tag{B.3}$$

$$u(k) = C_{\rm R}x(k) + D_{\rm R}y_d(k) + D_{2\rm R}W_d(k) \tag{B.4}$$

where

$$\begin{aligned}A_{\rm R} &\triangleq A - BH_d^{-1}CA^d, \qquad B_{\rm R} \triangleq BH_d^{-1}\\ C_{\rm R} &\triangleq -H_d^{-1}CA^d, \qquad D_{\rm R} \triangleq H_d^{-1}\\ D_{1\rm R} &\triangleq [\, -BH_d^{-1}D_2 \quad -BH_d^{-1}CD_1 \quad \cdots\\ &\qquad -BH_d^{-1}CA^{d-2}D_1 \quad D_1 - BH_d^{-1}CA^{d-1}D_1 \,]\\ D_{2\rm R} &\triangleq [\, -H_d^{-1}D_2 \quad -H_d^{-1}CD_1 \quad \cdots -H_d^{-1}CA^{d-1}D_1 \,]\\ y_d(k) &\triangleq y(k+d), \qquad W_d(k) \triangleq \begin{bmatrix}w(k+d)\\ \vdots\\ w(k)\end{bmatrix}.\end{aligned} \tag{B.5}$$

Since, by Assumption A1), (A,B,C) is minimal, it follows from [23, Proposition 4.2] that the eigenvalues of $A_{\rm R}$ consist of the invariant zeros of (A,B,C) as well as $n-d$ eigenvalues equal to zero. Therefore, by Assumption A2), $A_{\rm R}$ is asymptotically stable.

Theorem B.1: Consider the system (2.1), (2.2) and its inverse (B.3), (B.4). Let p be a positive integer. Then, subject to Assumptions A1), A2), A4), and A8), there exist $c_1 > 0$ and $c_2 > 0$ such that

$$\|\tilde{U}(k-d)\|_2^2 \leq c_1 + c_2\|\tilde{Y}(k)\|_2^2 \tag{B.6}$$

where

$$\tilde{U}(k) \triangleq \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-p) \end{bmatrix}, \quad \tilde{Y}(k) \triangleq \begin{bmatrix} y(k-1) \\ \vdots \\ y(k-p-1) \end{bmatrix}. \quad \text{(B.7)}$$

Proof: By successive substitution

$$u(k) = C_{\rm R} A_{\rm R}^k x(0) + D_{\rm R} y_d(k) + D_{2{\rm R}} W_d(k) + \sum_{i=1}^{k} C_{\rm R} A_{\rm R}^{i-1} B_{\rm R} y_d(k-i) + \sum_{i=1}^{k} C_{\rm R} A_{\rm R}^{i-1} D_{1{\rm R}} W_d(k-i).$$

Taking the norm of both sides yields

$$\|u(k)\|^2 \le 5\Big\{\|C_{\rm R}\|^2\|A_{\rm R}^k\|^2\|x(0)\|^2 + \|D_{\rm R}\|^2\|y_d(k)\|^2 + \|D_{2{\rm R}}\|^2\|W_d(k)\|^2 + \left[\sum_{i=1}^{k}\|C_{\rm R}\|\|A_{\rm R}^{i-1}\|\|B_{\rm R}\|\|y_d(k-i)\|\right]^2 + \left[\sum_{i=1}^{k}\|C_{\rm R}\|\|A_{\rm R}^{i-1}\|\|D_{1{\rm R}}\|\|W_d(k-i)\|\right]^2\Big\}$$

where $\|\cdot\|$ is the Euclidean norm. Since $A_{\rm R}$ is asymptotically stable, it follows that there exist $\lambda \in [0,1)$ and $c > 0$ such that, for every positive integer k, $\|A_{\rm R}^k\| \le c\lambda^k$. Therefore, there exists $c_3 > 0$ such that

$$\|u(k)\|^2 \le c_3\left[\lambda^{2k} + \|y_d(k)\|^2 + \left(\sum_{i=1}^{k}\lambda^{i-1}\|y_d(k-i)\|\right)^2 + \|W_d(k)\|^2 + \left(\sum_{i=1}^{k}\lambda^{i-1}\|W_d(k-i)\|\right)^2\right].$$

Since, by Assumption A8), $w(k)$ is bounded for all k, it follows that $\|W_d(k)\|^2$ is also bounded, that is, there exists $\rho > 0$ such that $\|W_d(k)\|^2 \le \rho$ for all k. Thus, there exists $c_4 > 0$ such that

$$\|u(k)\|^2 \le c_4\left[\rho + \lambda^{2k} + \|y_d(k)\|^2 + \left(\sum_{i=1}^{\infty}\lambda^{i-1}\right) \times \left(\sum_{i=1}^{k}\lambda^{i-1}\|y_d(k-i)\|^2\right) + \left(\rho\sum_{i=1}^{\infty}\lambda^{i-1}\right)^2\right].$$

Since $|\lambda| < 1$, it follows that $\sum_{i=1}^{\infty}\lambda^{i-1} = \frac{1}{1-\lambda}$, where $0^0 \triangleq 1$. Thus, it follows that there exist $c_5 > 0$ and $c_6 > 0$ such that

$$\|u(k)\|^2 \le c_5\left[c_6 + \|y_d(k)\|^2 + \sum_{i=1}^{k}\lambda^{i-1}\|y_d(k-i)\|^2\right]. \quad \text{(B.8)}$$

Summing both sides of (B.8) from $k-p$ to $k-1$ yields

$$\sum_{j=k-p}^{k-1}\|u(j)\|^2 \le c_5\left[c_7 + \sum_{j=k-p}^{k-1}\|y_d(j)\|^2 + \sum_{j=k-p}^{k-1}\sum_{i=1}^{j}\lambda^{i-1}\|y_d(j-i)\|^2\right] \quad \text{(B.9)}$$

where $c_7 > 0$. Introducing $\tau \triangleq j - i$ yields

$$\begin{aligned}\sum_{j=k-p}^{k-1}\|u(j)\|^2 &\le c_5\left[c_7 + \sum_{j=k-p}^{k-1}\|y_d(j)\|^2 + \sum_{\tau=k-p}^{k-2}\sum_{j=\tau+1}^{k-1}\lambda^{j-\tau-1}\|y_d(\tau)\|^2\right] \\ &\le c_8\left[c_7 + \sum_{j=k-p}^{k-1}\|y_d(j)\|^2 + \sum_{\tau=k-p-1}^{k-2}\|y_d(\tau)\|^2\right] \\ &\le c_1 + c_2\sum_{j=k-p-1}^{k-1}\|y_d(j)\|^2 \end{aligned} \quad \text{(B.10)}$$

where $c_8 > 0$. Decrementing (B.10) by d steps and using the definitions of $y_d(k)$, $\tilde{U}(k)$, and $\tilde{Y}(k)$ from (B.5) and (B.7) yields (B.6). ∎

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Mario A. Santillo received the B.S. degree in aeronautical and mechanical engineering from Rensselaer Polytechnic Institute, Troy, NY, in 2003, and the M.S.E. degree in aerospace engineering from the University of Michigan, Ann Arbor, in 2005.

He is currently a National Defense Science and Engineering Graduate (NDSEG) Fellow, University of Michigan. His current research interests include adaptive control and control of fluids.

Dennis S. Bernstein (M'82–F'00) received the Ph.D. degree from the University of Michigan, Ann Arbor, in 1982. He is currently a Professor in the Department of Aerospace Engineering, University of Michigan. He is the author of *Matrix Mathematics*. His current research interests include system identification and adaptive control for aerospace applications.

He is the Editor-in-Chief of the IEEE CONTROL SYSTEMS MAGAZINE.

Jesse B. Hoagg received the B.S.E. degree in civil and environmental engineering from Duke University, Durham, NC, in 2002, and the Ph.D. degree in aerospace engineering from the University of Michigan, Ann Arbor, in 2006.

His current research interests include adaptive control and control of structures.