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Example of Indeterminacy in Classical Dynamics

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The case of a particle moving along a nonsmooth constraint under the action of uniform gravity is presented as an example of indeterminacy in a classical situation. The indeterminacy arises from certain initial conditions having nonunique solutions and is due to the failure of the Lipschitz condition at the corresponding points in the phase space of the equation of motion.

1. INTRODUCTION

An often unstated assumption of classical mechanics is that the laws of dynamics yield deterministic models. This assumption is formally captured in Newton's principle of determinacy (Arnold, 1984, p. 4):

- *The initial positions and velocities of all the particles of a mechanical system uniquely determine all of its motion.*

The developments in physics since the early decades of this century have shown that our physical world is not completely empirically deterministic, that is, the motion of a mechanical system cannot be fully determined from physical measurements of the initial positions and velocities of its points. In particular, chaos theory has shown that infinite precision is required in the measurements of initial conditions for the motion to be fully predicted even qualitatively. On the other hand, Heisenberg's uncertainty principle holds that simultaneous measurements of positions and velocities can be made only with limited precision. The presence of noise further limits the accuracy of measurement. In spite of these fundamental limitations on our ability to make predictions from empirical observations, it is generally believed that models obtained from classical mechanics are completely deterministic and, if obser-

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vations could be made with infinite precision, then predictions could be made with unlimited accuracy. In this paper, we present a counterexample to this widely held notion.

The counterexample, given in Section 2, consists of a particle moving along a nonsmooth (C^1 but not twice differentiable) constraint in a uniform gravitational field. It is shown that, for certain initial conditions, the equation of motion possesses multiple solutions. The motion of the particle starting from these initial conditions cannot, therefore, be uniquely determined based on physical laws. Thus this example provides an instance of indeterminacy in classical dynamics as a direct counterexample to the principle of determinacy stated above.

In Section 3, we present a modification of this counterexample. The modification consists in replacing the original constraint by a spatially periodic nonsmooth constraint that divides the configuration space of the particle into "potential wells." The equation of motion in this case possesses multiple solutions for initial conditions that correspond to zero total mechanical energy. For a smooth (C^∞) constraint, the particle is forever confined to remain in the potential well in which it is initially located if the total mechanical energy is zero (or less). In the case we consider, if the particle is initially located in one of these potential wells with zero total mechanical energy, then there exist solutions of the equation of motion which correspond to the particle leaving the potential well after a finite amount of time. At any given instant, the only prediction that can be made about the particle is that it is located somewhere in any one of a certain number of potential wells and, furthermore, this number increases with the passage of time.

Both of the examples mentioned above possess equilibria that are finite-time repellers; solutions starting infinitesimally close to such points escape every given neighborhood in finite time. Mechanical systems can exhibit similar behavior in the presence of non-Lipschitzian dissipation (Zak, 1993) or controls (Bhat and Bernstein, 1996). However, the examples presented here are completely classical and involve neither dissipation nor controls.

2. AN EXAMPLE OF INDETERMINACY

Consider a particle of unit mass constrained to move without friction in a vertical plane along the curve $y = h(x)$ under the action of uniform gravity. For convenience, assume the gravitational acceleration to be unity.

The total mechanical energy of the particle is given by

$$E(x, \dot{x}) = \tfrac{1}{2}\dot{x}^2[1 + h'(x)^2] + h(x) \tag{1}$$

while the Lagrangian for the particle is given by

$$L(x, \dot{x}) = \tfrac{1}{2}\dot{x}^2[1 + h'(x)^2] - h(x) \tag{2}$$

The Lagrangian yields the equation of motion

$$\ddot{x}[1 + h'(x)^2] + \dot{x}^2 h'(x)h''(x) + h'(x) = 0 \tag{3}$$

Now, consider

$$h(x) = -|x|^\alpha, \qquad x \in \mathrm{R} \tag{4}$$

where $\alpha \in (3/2, 2)$. Figure 1 shows a plot of this constraint for $\alpha = 9/5$. We claim that with $h(\cdot)$ given by (4), equation (3) admits nonunique solutions for the initial conditions

$$x(0) = 0, \qquad \dot{x}(0) = 0 \tag{5}$$

To show this, consider the differential equation

$$\dot{q}(t) = [(2 - \alpha)/\sqrt{2}][1 + \alpha^2(q(t))^{4(\alpha-1)/(2-\alpha)}]^{-1/2} \tag{6}$$

Note that $\alpha \in (3/2, 2)$ implies that $4(\alpha - 1)/(2 - \alpha) > 4$. Hence the right-hand side of (6) is C^4 in q and bounded on R. It thus follows that there exists a unique function $\tau(\cdot)$ on $[0, \infty)$ that satisfies (6) and the initial condition $\tau(0) = 0$. Moreover, $\tau(\cdot)$ is twice continuously differentiable.

It follows by direct substitution that the function $[\tau(\cdot)]^{2/(2-\alpha)}$ satisfies (3) and (5). In fact, this same function delayed in time by an arbitrary positive constant T also satisfies (3) and (5). To make this precise, define

$$x_T(t) = 0, \qquad t \leq T \tag{7}$$

$$= [\tau(t - T)]^{2/(2-\alpha)}, \qquad t > T \tag{8}$$

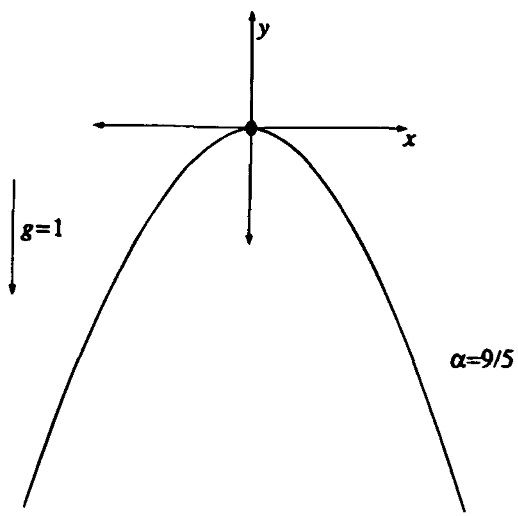

Fig. 1. Constrained particle in uniform gravity.

Then it follows by direct substitution that, for every $T \geq 0$, the functions $\pm x_T(\cdot)$ satisfy (3) and (5). The functions x_T and $-x_T$ correspond to the particle remaining at rest at $x = 0$ for time T and then moving off to the right and left, respectively.

Figure 2 shows the phase portrait for (3) with $\alpha = 9/5$. The origin is a saddle-point equilibrium and the sets $\mathcal{S} = \{(x, \dot{x})\colon E(x, \dot{x}) = 0, x\dot{x} \leq 0\}$ and $\mathcal{U} = \{(x, \dot{x})\colon E(x, \dot{x}) = 0, x\dot{x} \geq 0\}$, which are shown in Fig. 2, are the corresponding stable and unstable manifolds, respectively. Solutions to initial conditions contained in $\mathcal{S}$ converge to the origin in finite time, while solutions to initial conditions contained in $\mathcal{U}$ converge to the origin in backward time. For the solutions x_T described above, $(x_T(t), \dot{x}_T(t))$ lies in $\mathcal{U}$ for all $t \geq 0$. It is easy to see that for every initial condition in $\mathcal{S}$, (3) possesses multiple solutions. For such initial conditions, the motion of the particle cannot be uniquely determined. This phenomenon represents indeterminacy in a classical situation and is a counterexample to Newton's principle of determinacy stated above.

3. A FURTHER EXAMPLE OF INDETERMINACY

The indeterminacy seen above can be made even more striking by replacing (4) by

$$h(x) = -|\cos(x)|^{\alpha}, \qquad x \in \mathrm{R} \tag{9}$$

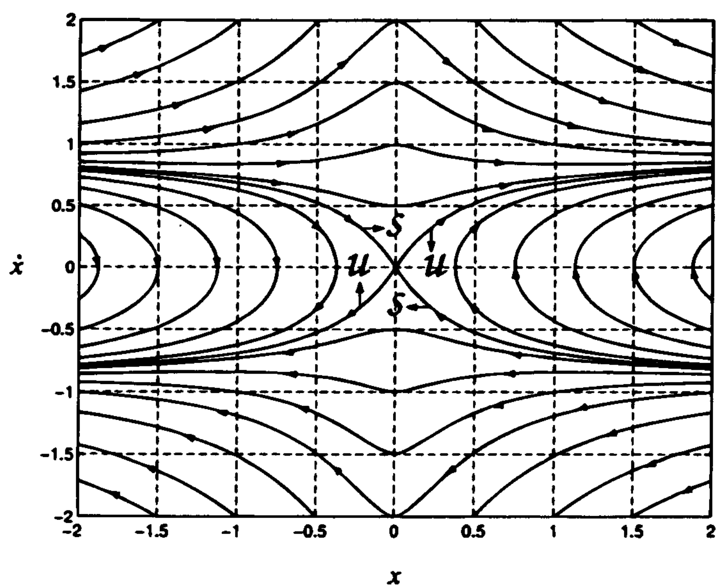

Fig. 2. Phase portrait for (3).

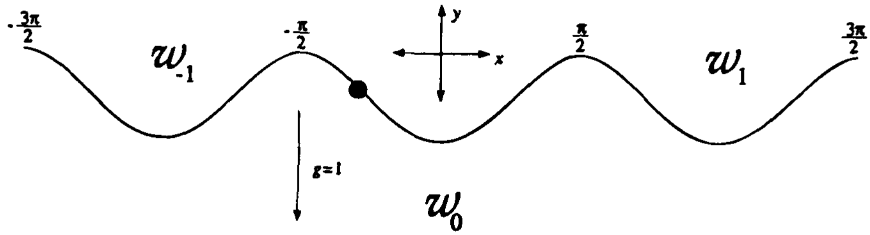

Fig. 3. Particle under periodic constraint.

where $\alpha \in (3/2, 2)$ as before. As Fig. 3, which corresponds to $\alpha = 9/5$, shows, the constraint divides the configuration space into the potential wells $\mathcal{W}_n = \{x: \frac{1}{2}(2n - 1)\pi \leq x \leq \frac{1}{2}(2n + 1)\pi\}$, $n = \ldots, -1, 0, 1, \ldots$.

The points $(\frac{1}{2}(2n + 1)\pi, 0)$, $n = \ldots, -1, 0, 1, \ldots$, are saddle-point equilibria that are connected by heteroclinic orbits. The set $\{(x, \dot{x}): E(x, \dot{x}) = 0\}$ is the union of these heteroclinic orbits. Because of the non-Lipschitzian nature of the equation (3), solutions starting in the stable manifold of any one of these points converge to that point in finite time, while solutions starting in the unstable manifold leave every neighborhood of that point in a finite time.

Suppose the particle is initially located in $\mathcal{W}_0$ with zero total mechanical energy. Then, depending on the direction of its initial velocity, the heteroclinic structure will bring the particle to rest at one of the crests $\pm\pi/2$ in a finite amount of time. As in the previous example, there exist solutions which correspond to the particle staying at rest at $x = \pm\pi/2$ for an arbitrary amount of time before sliding off to the right or the left. Every solution that corresponds to the particle moving off brings the particle to rest at some other crest in a finite time. This argument can be used repeatedly to show that given an initial condition in $\mathcal{W}_0$ with zero total mechanical energy, for every integer n there exists a solution $x(\cdot)$ with $x(t) \in \mathcal{W}_n$ for all $t \geq T$ for some T. In other words, it is not possible to predict in which potential well the particle will be found after a certain finite amount of time from the initial instant. The only prediction that can be made at any instant is that the particle is located somewhere in any one of a certain number of potential wells. Moreover, this number increases with time. This means that we can predict less and less about the particle as time passes.

4. CONCLUSIONS

The examples given in the previous sections show that classical mechanical situations can exhibit a lack of determinacy even in the absence of disturbances and noise. This lack of determinacy is distinct from the empirical indeterminacy that arises from sensitive dependence on initial conditions, the uncertainty principle, and random noise. Our examples provide situations whose outcome cannot be predicted theoretically. Finally, this effect would be difficult to demonstrate empirically due to noise and dissipation.

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