Automatica 44 (2008) 2258–2265

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Stabilization of a 3D axially symmetric pendulum☆

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ARTICLE INFO

Article history:
Received 1 March 2007
Received in revised form
25 September 2007
Accepted 20 January 2008
Available online 7 March 2008

Keywords:
3D pendulum
Spherical pendulum
Lagrange top
Inverted equilibrium
Swing-up

ABSTRACT

Stabilizing controllers are developed for a 3D pendulum assuming that the pendulum has a single axis of symmetry and that the center of mass lies on the axis of symmetry. This assumption allows development of a reduced model that forms the basis for controller design and global closed-loop analysis; this reduced model is parameterized by the constant angular velocity component of the 3D pendulum about its axis of symmetry. Several different controllers are proposed. Controllers based on angular velocity feedback only, asymptotically stabilize the hanging equilibrium. Then controllers are introduced, based on angular velocity and reduced attitude feedback, that asymptotically stabilize either the hanging equilibrium or the inverted equilibrium. These problems can be viewed as stabilization of a Lagrange top. Finally, if the angular velocity about the axis of symmetry is assumed to be zero, controllers are introduced, based on angular velocity and reduced attitude feedback, that asymptotically stabilize either the hanging equilibrium or the inverted equilibrium. This problem can be viewed as stabilization of a spherical pendulum.

1. Introduction

Pendulum models have provided a rich source of examples that have motivated and illustrated many recent developments in nonlinear dynamics and control. Much of the published research treats 1D planar pendulum models or 2D spherical pendulum models or some multi-body version of these. In Shen, Sanyal, Chaturvedi, Bernstein, and McClamroch (2004), a large part of this published research is summarized, emphasizing both control design and dynamical system results. In the closely related papers Chaturvedi, Bacconi, Sanyal, Bernstein, and McClamroch (2005), Chaturvedi, McClamroch, and Bernstein (2007) and Chaturvedi and McClamroch (2007), based on the developments in Shen et al. (2004), controllers for stabilization of equilibrium manifolds of a 3D pendulum are obtained. Controllers are introduced that provide asymptotic stabilization of a reduced attitude equilibrium. The reduced attitude of the 3D rigid pendulum is defined as the attitude or orientation of the 3D rigid pendulum, modulo rotation about a vertical axis. Stabilization results presented in these papers correspond to the stabilization of the 3D pendulum to a *rest* position. Thus at equilibrium, the 3D pendulum is completely at rest and does not spin.

The present paper considers control of a 3D pendulum for zero or nonzero spin motions, assuming that the pendulum has a single axis of symmetry and is supported at a pivot that is assumed to be frictionless and inertially fixed. The rigid body is axially symmetric. The location of its center of mass is distinct from the location of the pivot; the center of mass and the pivot are assumed to lie on the axis of symmetry of the pendulum. Forces that arise from uniform and constant gravity act on the pendulum.

It can be shown that if the center of mass and the pivot lie on a principal axis, then there exist invariant solutions of the 3D pendulum that correspond to spins about the axis of symmetry. In this paper we stabilize these constant (zero/nonzero) spin motions corresponding to the hanging and the inverted attitudes. Two independent control moments are assumed to act about the two principal axes of the pendulum that are not the axis of symmetry; in other words, there is no control moment about the axis of symmetry of the pendulum.

The formulation of the models depends on construction of a Euclidean frame fixed to the pendulum with origin at the pivot and an inertial Euclidean frame with origin at the pivot. We assume that the pendulum fixed frame is selected to be coincident with the principal axes of the pendulum, so that the center of mass of the pendulum lies on the axis of symmetry of the pendulum. We also assume that the inertial frame is selected so that the first two axes lie in a horizontal plane and the "positive" third axis points down. These assumptions are shown to guarantee that

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Alessandro Astolfi under the direction of Editor Hassan Khalil. This research is supported by NSF Grant CMS-0555797.

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doi:10.1016/j.automatica.2008.01.013

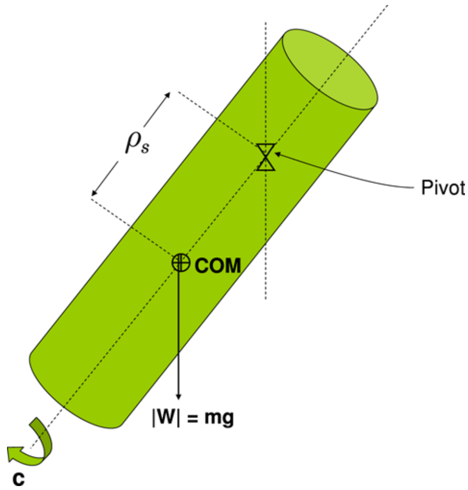

Fig. 1. A schematic of a cylindrically symmetric 3D axisymmetric pendulum.

the angular velocity component about the axis of symmetry of the rigid pendulum is constant. This conservation property allows development of reduced equations of motion for the 3D axially symmetric pendulum. The resulting reduced model is expressed in terms of two components of the angular velocity vector of the pendulum and the reduced attitude vector of the pendulum.

The main contributions of this paper are as follows. Controllers are developed that asymptotically stabilize the hanging relative equilibrium or the inverted relative equilibrium of the pendulum; for the special case that there is zero angular velocity about the axis of symmetry of the pendulum, controllers are developed that asymptotically stabilize the hanging reduced equilibrium or the inverted reduced equilibrium. If the angular velocity component about the axis of symmetry is nonzero, these control results can be compared with results in the literature on stabilization of Lagrange tops, e.g. Wan, Coppola, and Bernstein (1995). If the angular velocity component about the axis of symmetry is zero, our control results can be compared with results in the literature on stabilization of spherical pendula, e.g. Shiriaev, Ludvigsen, and Egeland (2004). In all of these cases, our stabilization results are new in the sense that global models are introduced and used for global analysis of the closed-loop systems.

The results are derived using novel Lyapunov functions that are suited to the geometry of the 3D axially symmetric pendulum. An important feature of the development is that the results are stated in terms of a global representation of the reduced attitude. In particular, we avoid the use of Euler angles and other nonglobal attitude representations.

This work compares with Bullo and Murray (1999), which considers PD control laws for systems evolving over Lie groups. In contrast with the PD-based laws in Bullo and Murray (1999) that generally give a conservative domain of attraction, we provide almost-global asymptotic stabilization results. Finally, we note that results in this paper avoid the artificial need to develop a "swing-up" controller, a locally asymptotically stabilizing controller, and a strategy for switching between the two as in Astrom and Furuta (2000) and Shiriaev et al. (2004).

2. Models of the 3D axially symmetric pendulum

In this section we introduce reduced models for the controlled 3D axially symmetric pendulum, and we summarize stability properties of the uncontrolled 3D axially symmetric pendulum. A schematic of a cylindrical 3D axisymmetric pendulum is shown in Fig. 1.

Since the pendulum is assumed to be axially symmetric, we choose the pendulum fixed coordinate frame so that the inertia matrix is $J = \mathrm{diag}(J_t, J_t, J_a)$. Let ρ denote the vector from the pivot to the center of mass of the pendulum; in the pendulum fixed coordinate frame it is a constant vector given by $\rho = (0, 0, \rho_s)^{\mathrm{T}}$, where ρ_s is a nonzero scalar. The angular velocity vector of the pendulum is denoted by $\omega = (\omega_x, \omega_y, \omega_z)^{\mathrm{T}}$, expressed in the pendulum fixed coordinate frame. As introduced in Shen et al. (2004) the reduced attitude vector $\Gamma = (\Gamma_x, \Gamma_y, \Gamma_z)^{\mathrm{T}}$ of the pendulum is the unit vector pointing in the direction of gravity, expressed in the pendulum fixed coordinate frame.

Euler's equations in scalar form for the rotational dynamics of the 3D axially symmetric pendulum, taking into account the moment due to gravity and the control moments, are

$$J_t\dot{\omega}_x = (J_t - J_a)\omega_z\omega_y - mg\rho_s\Gamma_y + \tau_x, \tag{1}$$

$$J_t\dot{\omega}_y = (J_a - J_t)\omega_z\omega_x + mg\rho_s\Gamma_x + \tau_y, \tag{2}$$

$$J_a\dot{\omega}_z = 0. \tag{3}$$

Here τ_x and τ_y denote the control moments. As shown in Shen et al. (2004) the rotational kinematics of the 3D pendulum can be expressed in terms of the reduced attitude vector according to the three scalar differential equations

$$\dot{\Gamma}_x = \Gamma_y\omega_z - \Gamma_z\omega_y, \tag{4}$$

$$\dot{\Gamma}_y = -\Gamma_x\omega_z + \Gamma_z\omega_x, \tag{5}$$

$$\dot{\Gamma}_z = \Gamma_x\omega_y - \Gamma_y\omega_x. \tag{6}$$

This model can be viewed as defining the motion of the 3D pendulum on the quotient space $TSO(3)/S^1 \cong \mathbb{R}^3 \times S^2$. Hence, we can view the motion of the 3D pendulum as evolving on $\mathbb{R}^3 \times S^2$ according to Eqs. (1)–(6).

Eq. (3) implies that the angular velocity component ω_z about the pendulum axis of symmetry satisfies

$$\omega_z = c, \tag{7}$$

where c is a constant. Ignoring (3) and substituting (7) into (1), (2), (4) and (5) lead to the reduced dynamics equations

$$J_t\dot{\omega}_x = c(J_t - J_a)\omega_y - mg\rho_s\Gamma_y + \tau_x, \tag{8}$$

$$J_t\dot{\omega}_y = c(J_a - J_t)\omega_x + mg\rho_s\Gamma_x + \tau_y, \tag{9}$$

and the reduced kinematics equations

$$\dot{\Gamma}_x = c\Gamma_y - \Gamma_z\omega_y, \tag{10}$$

$$\dot{\Gamma}_y = -c\Gamma_x + \Gamma_z\omega_x, \tag{11}$$

$$\dot{\Gamma}_z = \Gamma_x\omega_y - \Gamma_y\omega_x. \tag{12}$$

The motion of the 3D pendulum can be viewed as evolving on $\mathbb{R}^2 \times S^2$ according to (8)–(12).

In the remainder of this paper, we develop controllers that asymptotically stabilize an equilibrium of (8)–(12). Note that an equilibrium of (8)–(12) corresponds to a relative equilibrium of (1)–(6) which represents a pure spin of the 3D pendulum about its axis of symmetry. For the case where $c = 0$, a relative equilibrium solution of Eqs. (1)–(6) is an ordinary equilibrium solution.

The uncontrolled equations (8)–(12) have two distinct equilibrium solutions, namely

$$\omega_x = \omega_y = 0, \qquad \Gamma = \Gamma_h = (0, 0, 1) \tag{13}$$

and

$$\omega_x = \omega_y = 0, \qquad \Gamma = \Gamma_i = (0, 0, -1). \tag{14}$$

The first equilibrium is referred to as the hanging equilibrium, since the center of mass of the pendulum is directly below the pivot. The second equilibrium is referred to as the inverted equilibrium, since the center of mass of the pendulum is directly above the pivot. Note that these are relative equilibria of the uncontrolled equations (1)–(6) corresponding to a pure spin of the pendulum about its axis of symmetry. As shown in Shen et al. (2004), the hanging equilibrium of (1)–(6) is stable in the sense of Lyapunov, and the inverted equilibrium of (1)–(6) is unstable. We next present a result for the stability of the hanging and inverted equilibrium of (8)–(12).

Theorem 1. *Consider the 3D axially symmetric pendulum given by the Eqs. (8)–(12). Then the linearized dynamics about the hanging equilibrium is Lyapunov stable for all* $c \in \mathbb{R}$ *and the linearized dynamics about the inverted equilibrium is Lyapunov stable if and only if* $J_a^2c^2 > 4mg\rho_sJ_t$.

Proof. To linearize the dynamics in Eqs. (8)–(12) about the equilibrium $(0, 0, \Gamma_h)$ or $(0, 0, \Gamma_i)$, we consider a perturbation of the variables $(\omega_x, \omega_y, \Gamma) \in \mathbb{R}^2 \times S^2$ in the tangent plane $T_{(0,0,\Gamma_h)}\{\mathbb{R}^2 \times S^2\}$ or $T_{(0,0,\Gamma_i)}\{\mathbb{R}^2 \times S^2\}$. For all perturbations that lie in the tangent plane $T_{(0,0,\Gamma_h)}\{\mathbb{R}^2 \times S^2\}$ or $T_{(0,0,\Gamma_i)}\{\mathbb{R}^2 \times S^2\}$, the Z-component of the perturbation in Γ does not vary. Hence, we express the linearization using perturbations of $x = [\omega_x\ \omega_y\ \Gamma_x\ \Gamma_y]^{\mathrm{T}}$.

Linearizing the dynamics in (8)–(12), we obtain $\Delta\dot{x} = A\Delta x$, where Δx represents a perturbation vector of x from its equilibrium value and

$$A = \begin{bmatrix} 0 & ck_2 & 0 & -k_1 \\ -ck_2 & 0 & k_1 & 0 \\ 0 & -\gamma & 0 & c \\ \gamma & 0 & -c & 0 \end{bmatrix}, \tag{15}$$

where $k_2 = \frac{J_a - J_t}{J_t} \in \mathbb{R}$, $k_1 = \frac{mg\rho_s}{J_t} > 0$, $c \in \mathbb{R}$ and $\gamma = 1$ for the hanging equilibrium and $\gamma = -1$ for the inverted equilibrium. Computing the eigenvalues of (15), we obtain

$$\lambda = \pm\frac{1}{2}\sqrt{\pm 2\sqrt{\mathcal{D}} - 2c^2k_2^2 - 2c^2 - 4\gamma k_1},$$

where $\mathcal{D} = c^4k_2^4 - 2c^4k_2^2 + 4c^2k_2^2\gamma k_1 + c^4 + 4c^2\gamma k_1 + 8c^2k_2\gamma k_1$ and $\sqrt{\cdot}$ represents the square root with positive real part. Since every complex number with a nonzero imaginary part has one square root with positive real part, all eigenvalues of the matrix A lie in the CLHP iff they are purely imaginary. Hence, A in (15) is Lyapunov stable iff $\mathcal{D} > 0$ and $2\sqrt{\mathcal{D}} - 2c^2k_2^2 - 2c^2 - 4\gamma k_1 < 0$.

It can be shown that if $\mathcal{D} > 0$, then $2\sqrt{\mathcal{D}} - 2c^2k_2^2 - 2c^2 - 4\gamma k_1 < 0$ iff $c^4k_2^2 + k_1^2 - 2c^2k_2k_1\gamma > 0$.

Since $c^4k_2^2 + k_1^2 - 2c^2k_2k_1\gamma \geq c^4k_2^2 + k_1^2 - 2|c^2k_2k_1\gamma|$ and $\gamma = \pm 1$, therefore $c^4k_2^2 + k_1^2 - 2c^2k_2k_1\gamma \geq (c^2|k_2| - |k_1|)^2$. Thus, all eigenvalues of the matrix A lie on the imaginary axis iff $\mathcal{D} > 0$. It can also be shown that $\mathcal{D} = c^4(k_2^2 - 1)^2 + 4c^2\gamma k_1(k_2 + 1)^2$. If $\gamma = 1$, it is clear that $\mathcal{D} > 0$. Thus, the linearized dynamics about the hanging equilibrium is Lyapunov stable for all $c \in \mathbb{R}$. For the case $\gamma = -1$, $\mathcal{D} > 0$ if and only if $c^4(k_2^2 - 1)^2 > 4c^2k_1(k_2 + 1)^2$. Substituting for k_1 and k_2 yields

$$c^2\left[\left(\frac{J_a - J_t}{J_t}\right)^2 - 1\right]^2 > 4\left(\frac{J_a - J_t}{J_t} - 1\right)^2\frac{mg\rho_s}{J_t}.$$

Simplifying the above, we obtain $J_a^2c^2 > 4mg\rho_sJ_t$. Thus, if $J_a^2c^2 > 4mg\rho_sJ_t$ then all eigenvalues lie on the imaginary axis and are nonrepeated, or else at least two lie in the ORHP. Therefore, the linearized dynamics about the inverted equilibrium is Lyapunov stable iff $J_a^2c^2 > 4mg\rho_sJ_t$. ∎

As shown in Theorem 1, the equilibrium of the uncontrolled system (8)–(12) is at best, Lyapunov stable. This background provides motivation for the study of controllers that asymptotically stabilize either the hanging equilibrium or the inverted equilibrium.

3. Stabilization of the hanging equilibrium of the Lagrange top

In this section we assume that the constant angular velocity $c \neq 0$. For this case, the 3D axially symmetric pendulum described by Eqs. (8)–(12) is effectively a Lagrange top; hence that terminology is used in this section. We propose two classes of feedback controllers that asymptotically stabilize the hanging equilibrium of the reduced model described by Eqs. (8)–(12). In each case, we obtain almost-global asymptotic stability.

We begin by considering controllers based on the feedback of the angular velocity of the form

$$\tau_x = -\psi_x(\omega_x), \tag{16}$$

$$\tau_y = -\psi_y(\omega_y), \tag{17}$$

where $\psi_x : \mathbb{R} \to \mathbb{R}$ and $\psi_y : \mathbb{R} \to \mathbb{R}$ are smooth functions satisfying the sector inequalities

$$\begin{cases} \varepsilon_1|x|^2 \leq x\psi_x(x) \leq \varepsilon_2|x|^2, \\ \varepsilon_1|x|^2 \leq x\psi_y(x) \leq \varepsilon_2|x|^2, \end{cases} \tag{18}$$

for every $x \in \mathbb{R}$ where $\varepsilon_2 \geq \varepsilon_1 > 0$.

Lemma 1. *Consider the 3D axially symmetric pendulum given by Eqs. (8)–(12). Let* (ψ_x, ψ_y) *be smooth functions satisfying (18) and choose* τ_x *and* τ_y *as in (16) and (17). Then the hanging equilibrium of (8)–(12) is asymptotically stable. Furthermore, let* $\varepsilon \in (0, 2mg\rho_s)$ *and define*

$$\mathcal{H}_\varepsilon \triangleq \left\{(\omega_x, \omega_y, \Gamma) \in (\mathbb{R}^2 \times S^2) : \frac{1}{2}\left[J_t(\omega_x^2 + \omega_y^2) + mg\rho_s\|\Gamma - \Gamma_h\|^2\right] \leq 2mg\rho_s - \varepsilon\right\}. \tag{19}$$

Then, all solutions of the closed-loop system given by (8)–(12) and (16) and (17), such that $(\omega_x(0), \omega_y(0), \Gamma(0)) \in \mathcal{H}_\varepsilon$, *satisfy* $(\omega_x(t), \omega_y(t), \Gamma(t)) \in \mathcal{H}_\varepsilon$ *for all* $t \geq 0$, *and* $\lim_{t\to\infty}\omega_x(t) = 0$, $\lim_{t\to\infty}\omega_y(t) = 0$ *and* $\lim_{t\to\infty}\Gamma(t) = \Gamma_h$.

Proof. Consider the closed-loop system given by (8)–(12) and (16) and (17). We propose the following candidate Lyapunov function

$$V(\omega_x, \omega_y, \Gamma) = \frac{1}{2}\left[J_t(\omega_x^2 + \omega_y^2) + mg\rho_s\|\Gamma - \Gamma_h\|^2\right]. \tag{20}$$

Note that the Lyapunov function is positive definite on $\mathbb{R}^2 \times S^2$ and $V(0, 0, \Gamma_h) = 0$. Furthermore, the derivative $\dot{V}$ along a solution of the closed loop is given by

$$\begin{aligned} \dot{V}(\omega_x, \omega_y, \Gamma) &= -\omega_x\psi_x(\omega_x, \omega_y) - \omega_y\psi_y(\omega_x, \omega_y), \\ &\leq -\varepsilon_1(\omega_x^2 + \omega_y^2) \leq 0, \end{aligned}$$

where the last inequality follows from (18). Thus V is positive definite and $\dot{V}$ is negative semidefinite on $\mathbb{R}^2 \times S^2$. Next, note that the set $\mathcal{H}_\varepsilon$ can be expressed as the sub-level set

$$\mathcal{H}_\varepsilon = \{(\omega_x, \omega_y, \Gamma) \in \mathbb{R}^2 \times S^2 : V(\omega_x, \omega_y, \Gamma) \leq 2mg\rho_s - \varepsilon\}.$$

Since $\dot{V}(\omega_x, \omega_y, \Gamma) \leq 0$ on $\mathcal{H}_\varepsilon$, all solutions such that $(\omega_x(0), \omega_y(0), \Gamma(0)) \in \mathcal{H}_\varepsilon$ satisfy $(\omega_x(t), \omega_y(t), \Gamma(t)) \in \mathcal{H}_\varepsilon$ for all $t \geq 0$. Thus, $\mathcal{H}_\varepsilon$ is an invariant set of the closed loop.

Furthermore, from the invariant set theorem, we obtain that the solutions satisfying $(\omega_x(0), \omega_y(0), \Gamma(0)) \in \mathcal{H}_\varepsilon$ converge to the largest invariant set in $\{(\omega_x, \omega_y, \Gamma) \in \mathcal{H}_\varepsilon : (\omega_x, \omega_y) = (0, 0)\}$. Thus, $\omega_x \equiv \omega_y \equiv 0$ implies that $\Gamma_x = \Gamma_y = 0$ and $\dot{\Gamma}_z = 0$ and hence, $\Gamma_z = \pm 1$. Thus, as $t \to \infty$, either $\Gamma \to \Gamma_h$ or $\Gamma \to \Gamma_i$. However, since $(0, 0, \Gamma_i) \notin \mathcal{H}_\varepsilon$, it follows that $\Gamma \to \Gamma_h$ as $t \to \infty$. Thus, $(0, 0, \Gamma_h)$ is an asymptotically stable equilibrium of the closed loop given by (8)–(12) and (16) and (17), with $\mathcal{H}_\varepsilon$ as a domain of attraction. ∎

The conclusions of Lemma 1 can be strengthened to show that the domain of attraction is nearly global. This is presented in the following theorem.

Theorem 2. *Consider the 3D axially symmetric pendulum given by Eqs. (8)–(12). Let (ψ_x, ψ_y) be smooth functions satisfying (18). Choose τ_x and τ_y as in (16) and (17). Then, all solutions of the closed-loop system given by (8)–(12) and (16) and (17), such that $(\omega_x(0), \omega_y(0), \Gamma(0)) \in (\mathbb{R}^2 \times S^2) \setminus \mathcal{M}$ satisfy $\lim_{t\to\infty} \omega_x(t) = 0$, $\lim_{t\to\infty} \omega_y(t) = 0$ and $\lim_{t\to\infty} \Gamma(t) = \Gamma_h$. Here, $\mathcal{M}$ is the stable manifold of the closed-loop equilibrium $(0, 0, \Gamma_i)$, and it is a closed nowhere dense set of Lebesgue measure zero.*

Proof. We present an outline of the proof. Denote

$$\mathcal{N} \triangleq \left\{(\omega_x, \omega_y, \Gamma) \in (\mathbb{R}^2 \times S^2) : \frac{1}{2}\left[J_t(\omega_x^2 + \omega_y^2) + mg\rho_s\|\Gamma - \Gamma_h\|^2\right] \le 2mg\rho_s\right\}. \tag{21}$$

Then as in Chaturvedi et al. (2005) and Chaturvedi and McClamroch (2007), it can be shown that all solutions of the closed loop (8)–(12) and (16) and (17), satisfying $(\omega_x(0), \omega_y(0), \Gamma(0)) \in \partial\mathcal{N} \setminus \{(0, 0, \Gamma_i)\}$ enter the set $\mathcal{H}_\varepsilon$ in Lemma 1, for some $\varepsilon > 0$, in finite time. From Lemma 1 and the definition of $\mathcal{N}$, we note that for every $\varepsilon \in (0, 2mg\rho_s)$ and $(\omega_x(0), \omega_y(0), \Gamma(0)) \in \mathcal{H}_\varepsilon \bigcup (\partial\mathcal{N} \setminus \{(0, 0, \Gamma_i)\})$, $\omega(t) \to 0$ and $\Gamma(t) \to \Gamma_h$ as $t \to \infty$. Since

$$\mathcal{N} = \bigcup_{\varepsilon\in(0,2mg\rho_s)} \left(\mathcal{H}_\varepsilon \bigcup \partial\mathcal{N}\right),$$

it follows that all solutions satisfying $(\omega_x(0), \omega_y(0), \Gamma(0)) \in \mathcal{N} \setminus \{(0, 0, \Gamma_i)\}$ converge to the hanging equilibrium.

Next, it can be shown that all solutions of the closed loop (8)–(12) and (16) and (17), enter the set $\mathcal{N}$ in finite time. Thus all solutions either converge to the inverted equilibrium, or the hanging equilibrium. It is sufficient to show that the stable manifold of the inverted equilibrium $(0, 0, \Gamma_i)$, has dimension less than the dimension of $\mathbb{R}^2 \times S^2$ i.e. four, since all other solutions converge to the hanging equilibrium.

Using linearization, it can be shown that the equilibrium $(0, 0, \Gamma_i)$ of the closed loop is unstable and hyperbolic with nontrivial stable and unstable manifolds. Denoting the stable manifold by $\mathcal{M}$, it follows from Theorem 3.2.1 in Guckenheimer and Holmes (1983) that the dimension of the $\mathcal{M}$ is less than four and hence, the Lebesgue measure of this global invariant stable sub-manifold is zero (Krstic & Deng, 1998). Since, the domain of attraction of an asymptotically stable equilibrium is open, $\mathcal{M}$ is closed and hence, nowhere dense (Chaturvedi, Bloch, & McClamroch, 2006). ■

Theorem 2 provides conditions under which the hanging equilibrium of the Lagrange top is made asymptotically stable by feedback of the angular velocity. Since the hanging equilibrium of the uncontrolled Lagrange top is stable in the sense of Lyapunov, any controller of the form (16) and (17) can be viewed as providing damping. Note that such a controller does not require knowledge of the moment of inertia, location of the center of mass, or spin rate of the Lagrange top. In Lemma 1, the hanging equilibrium of the closed loop has a domain of stability that is easily computed. In Theorem 2, the domain of attraction is almost global.

Next we consider controllers based on feedback of the angular velocity and the reduced attitude. These controllers provide more design flexibility than the controllers that depend on angular velocity only; hence they can provide improved closed-loop performance.

Theorem 3. *Consider the 3D axially symmetric pendulum given by Eqs. (8)–(12) with $c \ne 0$. Let $\Phi : [0, 1) \to \mathbb{R}$ be a C^1 monotonically increasing function such that $\Phi(0) = 0$, $\Phi'(x) > 0$ if $x \ne 0$, and $\Phi(x) \to \infty$ as $x \to 1$. Furthermore, let (ψ_x, ψ_y) be smooth functions satisfying the inequality given in (18). Choose*

$$\begin{aligned}\tau_x = {} & -\omega_x + \psi_x\left((\Gamma_z - 1)\Gamma_y\right) - c(J_t - J_a)\omega_y \\ & + J_t(\Gamma_z - 1)(-c\Gamma_x + \Gamma_z\omega_x)\psi_x'\left((\Gamma_z - 1)\Gamma_y\right) \\ & + (\Gamma_z - 1)\Gamma_y\Phi'\left(\frac{1}{4}(\Gamma_z - 1)^2\right) + mg\rho_s\Gamma_y,\end{aligned} \tag{22}$$

$$\begin{aligned}\tau_y = {} & -\omega_y + \psi_y\left((1 - \Gamma_z)\Gamma_x\right) - c(J_a - J_t)\omega_x \\ & + J_t(\Gamma_z - 1)(c\Gamma_y - \Gamma_z\omega_y)\psi_y'\left((1 - \Gamma_z)\Gamma_x\right) \\ & - (\Gamma_z - 1)\Gamma_x\Phi'\left(\frac{1}{4}(\Gamma_z - 1)^2\right) - mg\rho_s\Gamma_x.\end{aligned} \tag{23}$$

Then $(\omega_x, \omega_y, \Gamma) = (0, 0, \Gamma_h)$ is an equilibrium of the closed loop given by (8)–(12) and (22) and (23) that is asymptotically stable with $\mathbb{R}^2 \times (S^2 \setminus \{\Gamma_i\})$ as a domain of attraction.

Proof. Consider the system represented by (8)–(12) and (22) and (23). We propose the following candidate Lyapunov function.

$$V(\omega_x, \omega_y, \Gamma) = \frac{J_t}{2}\left[\omega_x - \psi_x\left((\Gamma_z - 1)\Gamma_y\right)\right]^2 + \frac{J_t}{2}\left[\omega_y - \psi_y\left((1 - \Gamma_z)\Gamma_x\right)\right]^2 + 2\Phi\left(\frac{1}{4}(\Gamma_z - 1)^2\right).$$

Note that the above Lyapunov function is positive definite and proper on $\mathbb{R}^2 \times S^2$ with $V(0, 0, \Gamma_h) = 0$.

Suppose that $(\omega_x(0), \omega_y(0), \Gamma(0)) \ne (0, 0, \Gamma_i)$. Computing the derivative of the Lyapunov function along a solution of the closed loop, we obtain

$$\begin{aligned}\dot V(\omega_x, \omega_y, \Gamma) = {} & -\left[\omega_x - \psi_x\left((\Gamma_z - 1)\Gamma_y\right)\right]^2 \\ & - \left[\omega_y - \psi_y\left((1 - \Gamma_z)\Gamma_x\right)\right]^2 \\ & - \Phi'\left(\frac{1}{4}(\Gamma_z - 1)^2\right)\left[(\Gamma_z - 1)\Gamma_y\right]\psi_x\left((\Gamma_z - 1)\Gamma_y\right) \\ & - \Phi'\left(\frac{1}{4}(\Gamma_z - 1)^2\right)\left[(1 - \Gamma_z)\Gamma_x\right]\psi_y\left((1 - \Gamma_z)\Gamma_x\right), \\ \le {} & -\left[\omega_x - \psi_x\left((\Gamma_z - 1)\Gamma_y\right)\right]^2 - \left[\omega_y - \psi_y\left((1 - \Gamma_z)\Gamma_x\right)\right]^2 \\ & - \varepsilon_1\Phi'\left(\frac{1}{4}(\Gamma_z - 1)^2\right)(\Gamma_z - 1)^2(\Gamma_x^2 + \Gamma_y^2) \le 0.\end{aligned} \tag{24}$$

Thus, $\dot V$ is negative semidefinite and hence, each solution remains in the compact invariant set $\mathcal{K} = \{(\omega_x, \omega_y, \Gamma) \in \mathbb{R}^2 \times S^2 : V(\omega_x, \omega_y, \Gamma) \le C\}$, where $C = V(\omega_x(0), \omega_y(0), \Gamma(0))$.

Since $\dot V$ is negative semidefinite and Φ is monotonic with $\Phi'(x) \ne 0$ if $x \ne 0$, we obtain that, $(\Gamma_z - 1)\Gamma_y \to 0$, $(\Gamma_z - 1)\Gamma_x \to 0$, $\omega_x \to \psi_x(0) = 0$ and $\omega_y \to \psi_y(0) = 0$ as $t \to \infty$. Furthermore, by LaSalle's invariant set theorem, each solution converges to the largest invariant set in $\mathcal{S} \triangleq \{(\omega_x, \omega_y, \Gamma) \in \mathcal{K} : \omega_x = \omega_y = 0,\ (\Gamma_z - 1)\Gamma_y = 0,\ (\Gamma_z - 1)\Gamma_x = 0\}$. Since, any closed-loop solution of (8)–(12) in $\mathcal{S}$ satisfies $\omega_x \equiv \omega_y \equiv 0$, we obtain that the solution also satisfies $\Gamma_z =$ constant.

Next, $(\Gamma_z - 1)\Gamma_y \equiv (\Gamma_z - 1)\Gamma_x \equiv 0$ yields either $\Gamma_z = 1$, in which case $\Gamma = \Gamma_h$, or it yields $\Gamma_x = 0$ and $\Gamma_y = 0$, and hence, $\Gamma = \Gamma_h$ or $\Gamma = \Gamma_i$. However, since $V(\omega_x(t), \omega_y(t), \Gamma(t)) \le V(\omega_x(0), \omega_y(0), \Gamma(0))$, therefore $\Gamma(t) \ne \Gamma_i$ for all $t > 0$. Thus, $\Gamma_i \notin \mathcal{S}$. Hence, $\Gamma = \Gamma_h$. Thus, the only invariant solution of the closed loop contained in the set $\mathcal{S}$ is $\omega_x = \omega_y = 0$ and $\Gamma = \Gamma_h$. ■

Theorem 3 provides conditions under which the hanging equilibrium of the Lagrange top is made asymptotically stable by feedback of the angular velocity and feedback of the reduced attitude of the top. Any controller of the form (22) and (23) requires knowledge of the axial and transverse principal moments of inertia, mass, location of the center of mass, and spin rate of the Lagrange top. The controller (22), (23) is globally defined and smooth except at the inverted attitude. The hanging equilibrium of the top is guaranteed to have an almost-global domain of attraction.

These results on stabilization of the hanging equilibrium of a Lagrange top are apparently new. We find no references to this case in the published literature. We include this case for its independent interest and also because it naturally leads to the more familiar problem of stabilization of the inverted equilibrium of a Lagrange top.

4. Stabilization of the inverted equilibrium of the Lagrange top

As in the previous section, we assume that the constant $c \neq 0$, so that the 3D axially symmetric pendulum described by Eqs. (8)–(12) is effectively a Lagrange top; hence that terminology is also used in this section. We now propose the feedback controllers that asymptotically stabilize the inverted equilibrium of the reduced equations (8)–(12). The domain of attraction of the inverted equilibrium is shown to be almost global.

Theorem 4. *Consider the 3D axially symmetric pendulum given by Eqs. (8)–(12) with $c \neq 0$. Let $\Phi : [0, 1) \to \mathbb{R}$ be a C^1 monotonically increasing function such that $\Phi(0) = 0$, $\Phi'(x) > 0$ if $x \neq 0$, and $\Phi(x) \to \infty$ as $x \to 1$. Furthermore, let (ψ_x, ψ_y) be smooth functions satisfying the inequality given in (18). Choose*

$$\begin{aligned}\tau_x = {} & -\omega_x + \psi_x\left((1-\Gamma_i^{\mathrm{T}}\Gamma)\Gamma_y\right) - c(J_t - J_a)\omega_y \\ & - J_t(\Gamma_i^{\mathrm{T}}\Gamma - 1)(-c\Gamma_x + \Gamma_z\omega_x)\psi_x'\left((1-\Gamma_i^{\mathrm{T}}\Gamma)\Gamma_y\right) \\ & - (\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_y\Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right) + mg\rho_s\Gamma_y, \end{aligned} \tag{25}$$

$$\begin{aligned}\tau_y = {} & -\omega_y + \psi_y\left((\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_x\right) - c(J_a - J_t)\omega_x \\ & - J_t(\Gamma_i^{\mathrm{T}}\Gamma - 1)(c\Gamma_y - \Gamma_z\omega_y)\psi_y'\left((\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_x\right) \\ & + (\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_x\Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right) - mg\rho_s\Gamma_x. \end{aligned} \tag{26}$$

Then $(\omega_x, \omega_x, \Gamma) = (0, 0, \Gamma_i)$ is an equilibrium of the closed loop given by (8)–(12) and (25) and (26) that is asymptotically stable with $\mathbb{R}^2 \times (S^2 \setminus \{\Gamma_h\})$ as a domain of attraction.

Proof. Consider the system represented by (8)–(12) and (25) and (26). We propose the following candidate Lyapunov function.

$$\begin{aligned}V(\omega_x, \omega_y, \Gamma) = {} & \frac{J_t}{2}\left[\omega_x - \psi_x\left(-(\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_y\right)\right]^2 \\ & + \frac{J_t}{2}\left[\omega_y - \psi_y\left((\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_x\right)\right]^2 \\ & + 2\Phi\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right). \end{aligned} \tag{27}$$

Note that the above Lyapunov function is positive definite and proper on $\mathbb{R}^2 \times S^2$ with $V(0, 0, \Gamma_i) = 0$.

Suppose that $(\omega_x(0), \omega_y(0), \Gamma(0)) \neq (0, 0, \Gamma_h)$. Computing the derivative of the Lyapunov function along a solution of the closed loop, we obtain

$$\begin{aligned}\dot V(\omega_x, \omega_y, \Gamma) = {} & -\left[\omega_x - \psi_x\left(-(\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_y\right)\right]^2 \\ & - \left[\omega_y - \psi_y\left((\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_x\right)\right]^2 - \Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right) \\ & \times \left[(1-\Gamma_i^{\mathrm{T}}\Gamma)\Gamma_y\right]\psi_x\left((1-\Gamma_i^{\mathrm{T}}\Gamma)\Gamma_y\right) \\ & - \Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right)\left[(\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_x\right]\psi_y\left((\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_x\right), \\ \leq {} & -\left[\omega_x - \psi_x\left(-(\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_y\right)\right]^2 \\ & - \left[\omega_y - \psi_y\left((\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_x\right)\right]^2 \\ & - \varepsilon_1\Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right)(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2(\Gamma_x^2 + \Gamma_y^2) \leq 0. \end{aligned} \tag{28}$$

Thus, $\dot V$ is negative semidefinite and hence, each solution remains in the compact invariant set $\mathcal{K} = \{(\omega_x, \omega_y, \Gamma) \in \mathbb{R}^2 \times S^2 : V(\omega_x, \omega_y, \Gamma) \leq C\}$, where $C = V(\omega_x(0), \omega_y(0), \Gamma(0))$.

The remainder of the proof follows exactly the arguments used in Theorem 3. The only solution of the closed-loop system of (8)–(12) and (25) and (26) such that $(\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_y \to 0$, $(\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_x \to 0$, $\omega_x \to \psi_x(0) = 0$ and $\omega_y \to \psi_y(0) = 0$ as $t \to \infty$ is the inverted equilibrium $(\omega_x, \omega_y, \Gamma) = (0, 0, \Gamma_i)$. ■

Theorem 4 provides conditions under which the inverted equilibrium of the Lagrange top is made asymptotically stable by feedback of the angular velocity and feedback of the reduced attitude of the top. Any controller of the form (25) and (26) requires knowledge of the axial and transverse principal moments of inertia, mass, location of the center of mass, and spin rate of the Lagrange top. The controllers (25), (26) are globally defined and smooth except at the hanging attitude. The inverted equilibrium of the top is guaranteed to have an almost-global domain of attraction.

The above stabilization results can be compared with the extensive literature on stabilization of Lagrange tops; see for example Lum, Bernstein, and Coppola (1995), and Wan et al. (1995). The results in Theorem 4 are substantially different from any of these cited results on stabilization of a Lagrange top.

5. Stabilization of the inverted equilibrium of the spherical pendulum

In this section we assume that the angular velocity ω_z is a constant $c = 0$. In this case, the 3D axially symmetric pendulum described by Eqs. (8)–(12) is effectively a spherical pendulum; hence that terminology is used in this section. We propose feedback controllers that asymptotically stabilize the inverted equilibrium of the reduced model described by Eqs. (8)–(12). Since $\omega_z = c = 0$ it corresponds to an equilibrium manifold of the complete model (1)–(6). The domain of attraction of the closed-loop equilibrium is shown to be almost global.

Theorem 5. *Consider the 3D axially symmetric pendulum given by Eqs. (8)–(12) with $c = 0$. Let $\Phi : [0, 1) \to \mathbb{R}$ be a C^1 monotonically increasing function such that $\Phi(0) = 0$, $\Phi'(x) > 0$ if $x \neq 0$, and $\Phi(x) \to \infty$ as $x \to 1$. Furthermore, let (ψ_x, ψ_y) be smooth functions satisfying the inequality given in (18). Assume that $\omega_z(0) = c = 0$, and denote*

$$y_1 \triangleq (1 + \Gamma_z)\Gamma_y, \tag{29}$$

$$y_2 \triangleq (1 + \Gamma_z)\Gamma_x. \tag{30}$$

Choose

$$\begin{aligned}\tau_x = {} & mg\rho_s\Gamma_y + J_t\psi_x'(y_1)\dot y_1 - (\omega_x - \psi_x(y_1)) \\ & + y_1\Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right), \end{aligned} \tag{31}$$

$$\begin{aligned}\tau_y = {} & -mg\rho_s\Gamma_x + J_t\psi_y'(y_2)\dot y_2 - (\omega_y - \psi_y(y_2)) \\ & + y_2\Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right), \end{aligned} \tag{32}$$

where $\dot y_1$ and $\dot y_2$ are obtained by differentiating (29) and (30) and substituting from (10)–(12). Then $(0, 0, \Gamma_i)$ is an equilibrium of the closed loop given by (8)–(12) and (31) and (32) that is asymptotically stable with $\mathbb{R}^2 \times (S^2 \setminus \{\Gamma_h\})$ as a domain of attraction.

Proof. Consider the system given by (8)–(12) and (31) and (32). We propose the following candidate Lyapunov function.

$$V(\omega,\Gamma)=\frac{J_t}{2}[\omega_x-\psi_x(y_1)]^2+\frac{J_t}{2}[\omega_y-\psi_y(y_2)]^2+2\Phi\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma-1)^2\right). \tag{33}$$

Note that the above Lyapunov function is positive definite on $\mathbb{R}^2\times S^2$ and $V(0,\Gamma_i)=0$. Furthermore, $V(\omega_x,\omega_y,\Gamma)$ is a proper function on $\mathbb{R}^2\times S^2$. Next, computing the derivative of the Lyapunov function along a solution of the closed loop, we obtain

$$\begin{aligned}\dot V(\omega,\Gamma)&=-[\omega_x-\psi_x(y_1)]^2-[\omega_y-\psi_y(y_2)]^2\\&\quad-\Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma-1)^2\right)[y_1\psi_x(y_1)+y_2\psi_y(y_2)],\\&\le-(\omega_x-\psi_x(y_1))^2-(\omega_y-\psi_y(y_2))^2\\&\quad-\varepsilon_1\Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma-1)^2\right)(y_1^2+y_2^2)\le 0.\end{aligned} \tag{34}$$

Thus, $\dot V$ is negative semidefinite and hence, each solution remains in the compact invariant set $\mathcal{K}=\{(\omega_x,\omega_y,\Gamma)\in\mathbb{R}^2\times S^2 : V(\omega_x,\omega_y,\Gamma)\le C\}$ where $C=V(\omega_x(0),\omega_y(0),\Gamma(0))$. Next, since $\dot V$ is negative semidefinite and from properties of $\Phi(\cdot)$, we obtain that, $y_1\to 0$, $y_2\to 0$, $\omega_x\to\psi_x(0)=0$ and $\omega_y\to\psi_y(0)=0$ as $t\to\infty$.

Furthermore, by LaSalle's invariant set theorem, the solution converges to the largest invariant set in $\mathcal{S}\triangleq\{(\omega_x,\omega_y,\Gamma)\in\mathcal{K} : \omega_x=\omega_y=0,\ y_1=0,\ y_2=0\}$. Since, any closed-loop solution in $\mathcal{S}$ satisfies $\omega_x\equiv\omega_y\equiv 0$, we obtain that the solution also satisfies $\Gamma=$ constant. Next, $y_1\equiv y_2\equiv 0$ yields either $\Gamma_z=-1$, in which case $\Gamma=\Gamma_i$, or it yields $\Gamma_x=0$ and $\Gamma_y=0$ which implies that $\Gamma=\Gamma_i$ or $\Gamma=\Gamma_h$. However, since $V(\omega_x(t),\omega_y(t),\Gamma(t))\le V(\omega_x(0),\omega_y(0),\Gamma(0))$, therefore $\Gamma(t)\ne\Gamma_h$ for all $t\ge 0$. Thus, $(0,0,\Gamma_h)\notin\mathcal{S}$. Hence, $\Gamma=\Gamma_i$. Thus, the only invariant solution of the closed loop contained in the set $\mathcal{S}$, is $\omega_x=\omega_y=0$ and $\Gamma=\Gamma_i$. ■

Theorem 5 provides conditions under which the inverted equilibrium of the spherical pendulum is made asymptotically stable by feedback of the angular velocity and feedback of the reduced attitude of the spherical pendulum. Any controller of the form (31) and (32) requires knowledge of the transverse (but not the axial) principal moment of inertia, the mass, and the location of the center of mass of the spherical pendulum. The controllers (31), (32) are globally defined and smooth except at the hanging attitude. Theorem 5 provides a means for stabilizing the inverted equilibrium of the spherical pendulum with an almost-global domain of attraction.

This is a new result for stabilization of the spherical pendulum. The results in Theorem 5 are substantially different from similar results on stabilization of spherical pendulums that have appeared in prior literature (Shirieav, Ludvigsen, & Egeland, 1999; Shiriaev et al., 2004; Shirieav, Pogromsky, Ludvigsen, & Egeland, 2000). Our results provide an almost-globally stabilizing controller that avoids the need to construct a swing-up controller, a locally stabilizing controller, and a switching strategy between the two. In this comparative sense, our results are direct and simple.

6. Simulation results

In this section, we present simulation results for specific controllers that stabilize the inverted equilibrium of the Lagrange top and the spherical pendulum. Consider the model (8)–(12), where $m=140$ kg, $\rho=(0,0,0.5)^{\mathrm{T}}$ m and $J=\mathrm{diag}(40,40,50)$ kg m^2. We choose $\Phi(x)=-k\ln(1-x)$, and $\psi_x(u)=p_xu$,

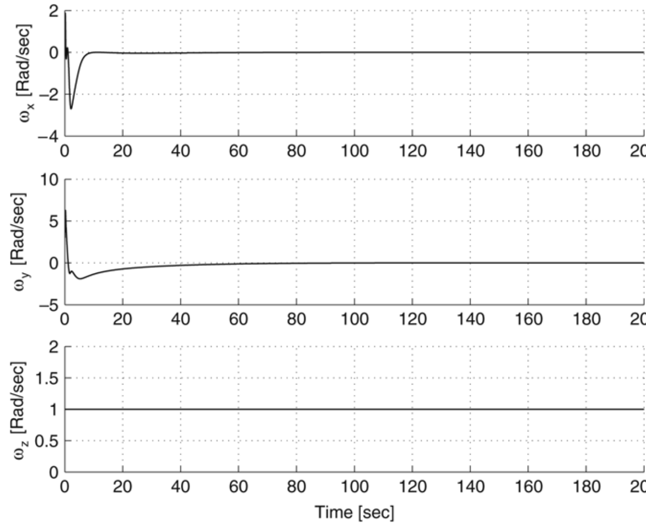

Fig. 2. Evolution of the angular velocity of the Lagrange top in the body frame.

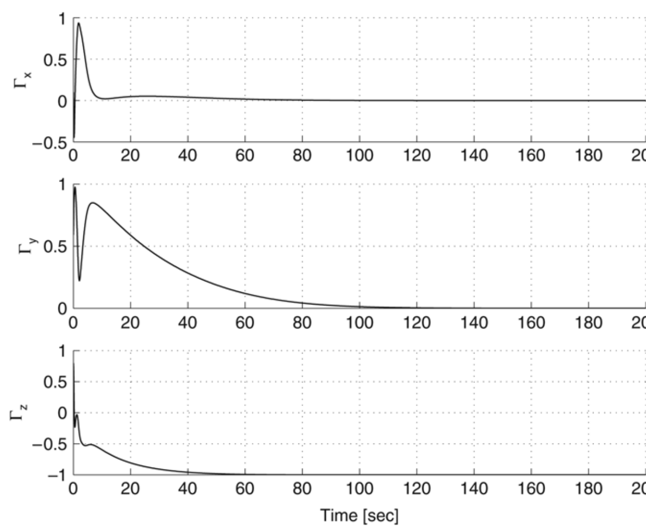

Fig. 3. Evolution of the components of the direction of gravity Γ in the body frame for the Lagrange top.

and $\psi_y(u)=p_yu$, where k, p_x and p_y are positive numbers in the controller (25) and (26).

Consider a Lagrange top with spin rate about its axis of symmetry $c=1$ rad/s. Choose gains as $k=5$ and $p_x=p_y=3$. The following figures describe the evolution of the closed loop. The initial conditions are $\omega(0)=(1,3,1)^{\mathrm{T}}$ rad/s and $\Gamma(0)=(0.1,0.5916,0.8)^{\mathrm{T}}$. Simulation results in Figs. 2 and 3 show that $\omega_x(t)\to 0$, $\omega_y(t)\to 0$ and $\Gamma(t)\to\Gamma_i$ as $t\to\infty$. Figs. 4 and 5 illustrate the motion of the Lagrange top in the inertial frame and the magnitude of the applied control input along each axis, respectively.

Now consider a spherical pendulum with controller given by (31) and (32) with the above specifications, so that it stabilizes the inverted equilibrium. The functions $\Phi(\cdot)$ and (ψ_x,ψ_y) are chosen as before. The following figures describe the evolution of the closed loop. The initial conditions are $\omega(0)=(1,3,0)^{\mathrm{T}}$ rad/s and $\Gamma(0)=(0.1,0.5916,0.8)^{\mathrm{T}}$. Simulation results in Figs. 6 and 7 show that $\omega_x(t)\to 0$, $\omega_y(t)\to 0$ and $\Gamma(t)\to\Gamma_i$ as $t\to\infty$. Figs. 8 and 9 illustrate the motion of the spherical pendulum in the inertial frame and the magnitude of the applied control input along each axis, respectively.

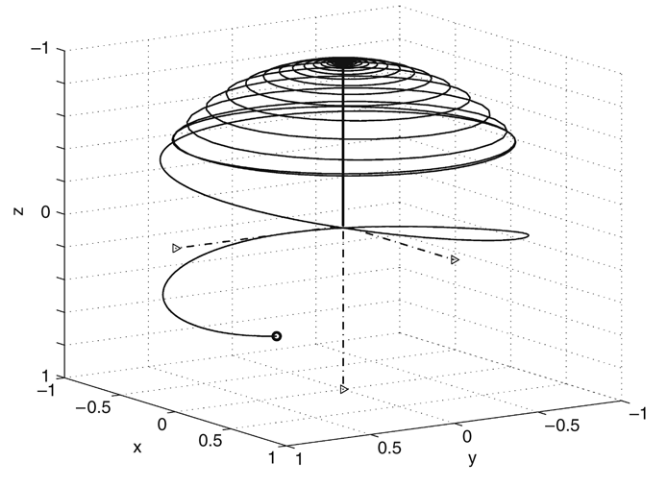

Fig. 4. Motion of the vector between the pivot and the center of mass of the Lagrange top in the inertial frame.

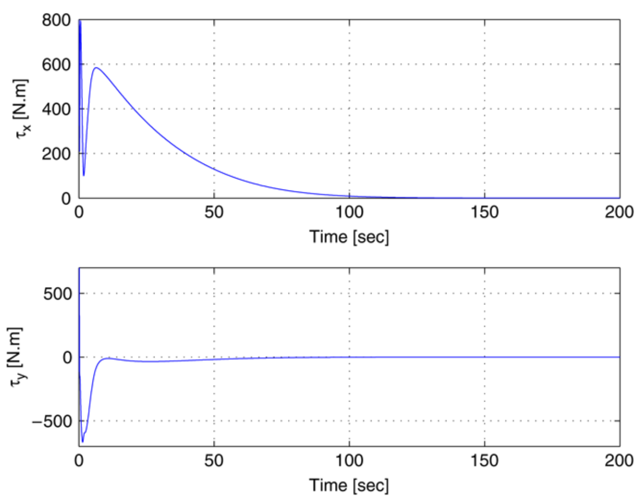

Fig. 5. Magnitude of the applied control moment along each axis.

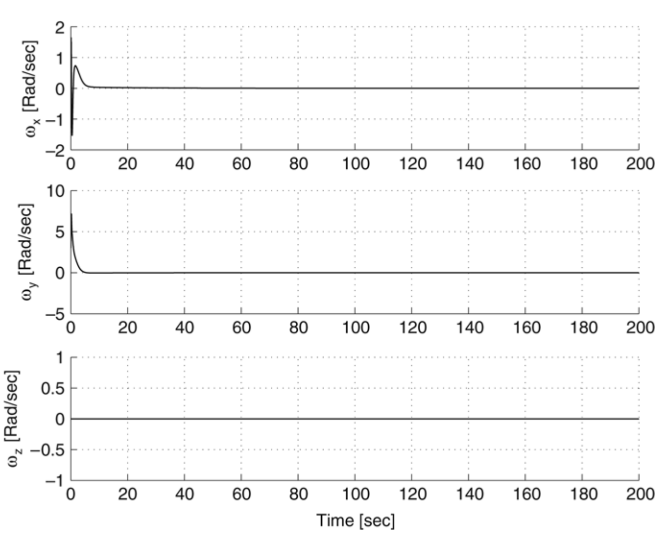

Fig. 6. Evolution of the angular velocity of the spherical pendulum in the body frame.

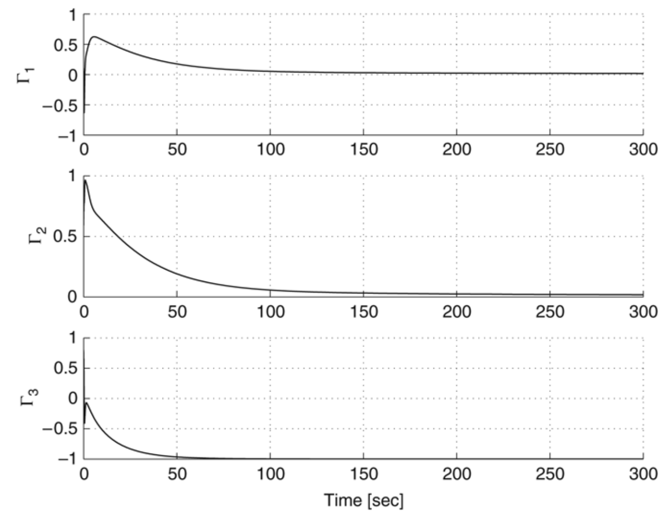

Fig. 7. Evolution of the components of the direction of gravity Γ in the body frame for the spherical pendulum.

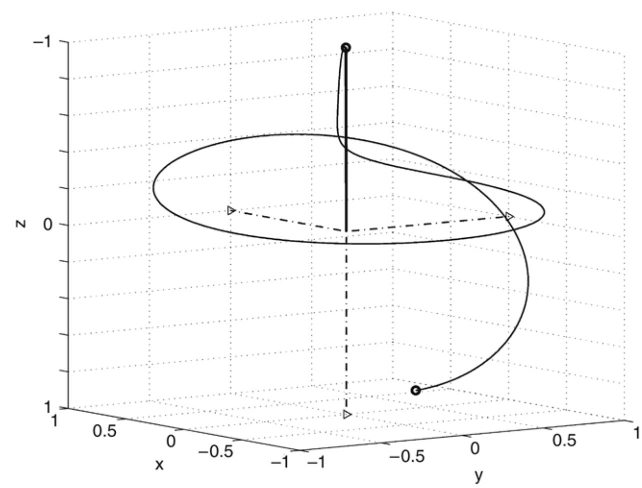

Fig. 8. Motion of the vector between the pivot and the center of mass of the spherical pendulum in the inertial frame.

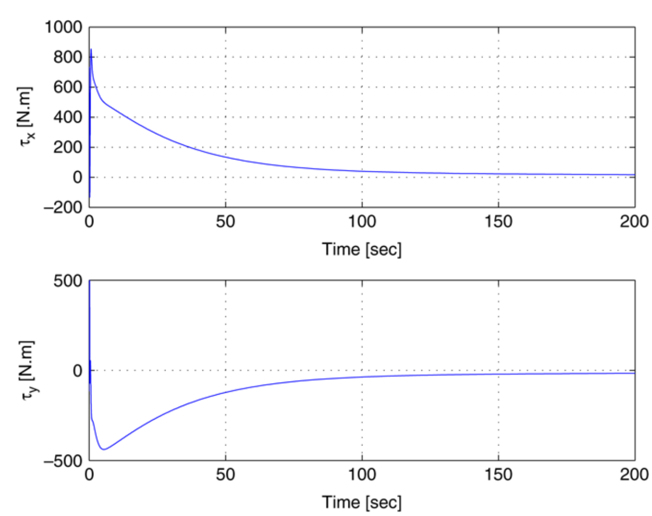

Fig. 9. Magnitude of the applied control moment along each axis.

7. Conclusions

This paper has treated stabilization problems for a 3D pendulum that has a single axis of symmetry. The control action is assumed to provide no external moment about the axis of symmetry. In this case the 3D pendulum has a constant angular velocity about its axis of symmetry. If this angular velocity is nonzero, the 3D pendulum is equivalent to a Lagrange top; if this angular velocity is zero, the 3D pendulum is equivalent to a spherical pendulum. Stabilization results are presented and proved for the Lagrange top and for the spherical pendulum. All of these results are substantially stronger than the results that have been previously presented in the published literature. In addition, the perspective provided by the 3D pendulum provides a unifying framework for all of these developments.

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