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# Robust fixed-structure controller synthesis using the implicit small-gain bound[☆]

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**Abstract**

In this paper we explore the applicability of the implicit small-gain guaranteed cost bound for controller synthesis. For flexibility in controller synthesis, we adopt the approach of fixed-structure controller design which allows consideration of arbitrary controller structures, including order, internal structure, and decentralization. A numerical example that has been addressed in the literature by means of alternative guaranteed cost bounds is presented to demonstrate the fixed-structure/implicit small-gain approach to robust controller synthesis. © 2000 The Franklin Institute. Published by Elsevier Science Ltd. All rights reserved.

*Keywords:* Robust stability; Robust performance; Guaranteed cost bounds; Fixed-structure controllers; Implicit small gain theorem

## 1. Introduction

One of the principal objectives of robust control theory is to synthesize feedback controllers with a priori guarantees of robust stability and performance. In $\mu$ synthesis

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[1] these guarantees are achieved by means of bounds involving frequency-dependent scales and multipliers which account for the structure of the uncertainty as well as its real or complex nature. An alternative robustness approach involves bounding the effect of real or complex uncertain parameters on the $\mathscr{H}_2$ performance of the closed-loop system. These guaranteed cost bounds take the form of modifications to the usual Lyapunov equation to provide bounds for robust stability and $\mathscr{H}_2$ performance.

A diverse collection of guaranteed cost bounds have been developed. An overview of many of the early guaranteed cost bounds can be found in [2], while positive-real-type guaranteed cost bounds are discussed in [3]. More recently, Popov-type guaranteed cost bounds have provided links with frequency-dependent scales and multipliers while providing reliable bounds for the peak real structured singular value [4–6]. Finally, the introduction of shift terms has been shown to reduce the conservatism of guaranteed cost bounds [7,8] for structured real uncertainty without requiring frequency-dependent scales and multipliers.

The goal of this paper is to explore the applicability of the implicit small-gain guaranteed cost bound of Haddad et al. [7] to controller synthesis. As shown in [7], unlike the quadratic stability bounded-real-type bound of Petersen and Hollot [9] and Khargonekar et al. [10], the implicit small gain bound can distinguish between real and complex uncertainty and is particularly effective in capturing internal uncertainty structure. For flexibility in controller synthesis, we adopt the approach of fixed-structure controller synthesis [11] which allows consideration of arbitrary controller structures, including order, internal structure, and decentralization [12]. Finally, to demonstrate the fixed-structure/implicit small gain approach to robust controller synthesis, we consider a flexible structure example that has been addressed in the literature by means of alternative guaranteed cost bounds.

In this paper we use the following standard notation. Let $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ denote real numbers, $n\times 1$ real column vectors, and $n\times m$ real matrices, respectively. Let $A^{\mathrm{T}}$ denote the transpose of $A$ and let $M \geqslant 0$ (resp., $M > 0$) denote the fact that the Hermitian matrix $M$ is nonnegative (resp., positive) definite. Furthermore, let $\mathbb{S}^n$ (resp., $\mathbb{N}^n$) denote the set of $n\times n$ symmetric (resp., nonnegative definite) matrices. Finally, $\mathbb{E}$ denotes the expectation operator and $\mathscr{R}(A)$ denotes the range space of the matrix $A$.

## 2. Robust stability and performance

In this section we state the robust stability and performance problem. This problem involves a set $\mathscr{U} \subset \mathbb{R}^{n\times n}$ of uncertain perturbations $\Delta A$ of the nominal system matrix $A$. The objective of this problem is to determine a fixed-order, strictly proper dynamic compensator $(A_c, B_c, C_c)$ that stabilizes the plant for all variations in $\mathscr{U}$ and minimizes the worst-case $\mathscr{H}_2$ norm of the closed-loop system.

*Robust Stability and Performance Problem*: Given the $n$th-order stabilizable and detectable plant

$$\dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + D_1 w(t), \qquad t \in [0, \infty), \tag{1}$$

$$y(t) = Cx(t) + D_2 w(t), \tag{2}$$

where $w(\cdot)$ denotes a unit-intensity white noise signal, determine an $n_c$th-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{3}$$

$$u(t) = C_c x_c(t), \tag{4}$$

such that the closed-loop system (1)–(4) is asymptotically stable for all $\Delta A \in \mathscr{U}$ and the performance criterion

$$J(A_c, B_c, C_c) \triangleq \sup_{\Delta A \in \mathscr{U}} \limsup_{t \to \infty} \frac{1}{t} \mathbb{E} \int_0^t [x^T(s) R_1 x(s) + u^T(s) R_2 u(s)] \, ds, \tag{5}$$

is minimized, where $R_1 \geqslant 0$ and $R_2 > 0$.

For each uncertain variation $\Delta A \in \mathscr{U}$, the closed-loop system (1)–(4) can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{D}w(t), \quad t \in [0, \infty), \tag{6}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}, \quad \Delta\tilde{A} \triangleq \begin{bmatrix} \Delta A & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix},$$

and where the closed-loop disturbance $\tilde{D}w(t)$ has intensity

$$\tilde{V} \triangleq \tilde{D}\tilde{D}^T = \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{bmatrix},$$

where $V_1 \triangleq D_1 D_1^T$, $V_2 \triangleq D_2 D_2^T > 0$, and $V_{12} \triangleq D_1 D_2^T = 0$.

## 3. Sufficient conditions for robust stability and performance

In this section we assign explicit structure to the set $\mathscr{U}$ and provide robust stability and performance guarantees in terms of a solution to a modified Riccati equation. Specifically, the uncertainty set $\mathscr{U}$ is defined by

$$\mathscr{U} = \left\{ \Delta A \in \mathbb{R}^{n \times n} \colon \Delta A = \sum_{i=1}^{r} \delta_i A_i, \ |\delta_i| \leqslant \gamma^{-1}, \quad i = 1, \ldots, r \right\}, \tag{7}$$

where $\gamma$ is a positive number and, for $i = 1, \ldots, r$, $A_i \in \mathbb{R}^{n \times n}$ is a fixed matrix denoting the structure of the parametric uncertainty and $\delta_i$ is an uncertain real parameter. Note that $\mathscr{U}$ given by (7) includes repeated parameters without loss of generality. For example, if $\delta_1 = \delta_2$, then discard $\delta_2$ and replace $A_1$ by $A_1 + A_2$. Furthermore,

$\mathscr{U}$ includes real full block uncertainty. For example, if

$$\Delta A = \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{bmatrix},$$

then $\Delta A = \sum_{i=1}^{4} \delta_i A_i$, where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and likewise for $A_2$, $A_3$, and $A_4$.

With the uncertainty set $\mathscr{U}$ given by (7), the closed-loop system (6) has structured uncertainty of the form

$$\Delta\tilde{A} = \sum_{i=1}^{r} \delta_i \tilde{A}_i, \tag{8}$$

where

$$\tilde{A}_i \triangleq \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 1, \ldots, r.$$

We now introduce a modified Riccati equation whose solution guarantees robust stability and robust performance for the closed-loop system (6) with $\mathscr{U}$ given by (7). For $i = 1, \ldots, r$, let $\tilde{S}_i \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ and define $\tilde{Z}_i \triangleq [(\tilde{S}_i + \tilde{S}_i^{\mathrm{T}})^2]^{1/2}$. Note that $-\tilde{Z}_i \leqslant \alpha(\tilde{S}_i + \tilde{S}_i^{\mathrm{T}}) \leqslant \tilde{Z}_i$ for all $\alpha \in [-1, 1]$. If $\tilde{S}_i$ is skew symmetric then $\tilde{Z}_i = 0$. Furthermore, for $i = 1, \ldots, r$, define $\tilde{I}_i \triangleq [\tilde{S}_i \quad \tilde{A}_i^{\mathrm{T}}][\tilde{S}_i \quad \tilde{A}_i^{\mathrm{T}}]^{\dagger}$, where $()^{\dagger}$ denotes the Moore–Penrose generalized inverse. Note that $\tilde{I}_i$ is symmetric and idempotent, that is, $\tilde{I}_i = \tilde{I}_i^{\mathrm{T}} = \tilde{I}_i^2$. Furthermore, since $\tilde{I}_i[\tilde{S}_i \quad \tilde{A}_i^{\mathrm{T}}] = [\tilde{S}_i \quad \tilde{A}_i^{\mathrm{T}}]$, it follows that $\tilde{I}_i\tilde{S}_i = \tilde{S}_i$ and $\tilde{A}_i\tilde{I}_i = \tilde{A}_i$. If $\tilde{S}_i = \tilde{A}_i$ and $\tilde{A}_i$ is an EP matrix [13], that is, $\mathscr{R}(\tilde{A}_i) = \mathscr{R}(\tilde{A}_i^{\mathrm{T}})$, then $\tilde{I}_i = \tilde{A}_i^{\dagger}\tilde{A}_i$. Recall that normal matrices (and thus symmetric and skew-symmetric matrices) are EP.

For convenience in stating the next result, define the shifted dynamics matrix $\tilde{A}_{s\gamma} \triangleq \tilde{A} + \gamma^{-2}\sum_{i=1}^{r}\alpha_i\beta_i\tilde{A}_i\tilde{S}_i$ and define

$$\tilde{R} \triangleq \begin{bmatrix} R_1 & 0 \\ 0 & C_{\mathrm{c}}^{\mathrm{T}} R_2 C_{\mathrm{c}} \end{bmatrix}.$$

**Theorem 3.1** (Haddad et al. [7]). *For $i = 1, \ldots, r$, let $\alpha_i \in \mathbb{R}$, $\beta_i > 0$, and let $\tilde{S}_i \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$. Furthermore, suppose there exists an $\tilde{n} \times \tilde{n}$ nonnegative-definite matrix $\tilde{P}$ satisfying*

$$0 = \tilde{A}_{s\gamma}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A}_{s\gamma} + \sum_{i=1}^{r} [\gamma^{-2}(\alpha_i^2\tilde{S}_i^{\mathrm{T}}\tilde{S}_i + \beta_i^2\tilde{P}\tilde{A}_i\tilde{A}_i^{\mathrm{T}}\tilde{P}) + \gamma^{-1}\beta_i^{-1}|\alpha_i|\tilde{Z}_i + \beta_i^{-2}\tilde{I}_i] + \tilde{R}. \tag{9}$$

*Then* $(\tilde{A} + \Delta\tilde{A}, \tilde{E})$ *is detectable for all* $\Delta A \in \mathscr{U}$ *if and only if* $\tilde{A} + \Delta\tilde{A}$ *is asymptotically stable for all* $\Delta A \in \mathscr{U}$. *In this case,*

$$J(A_c, B_c, C_c) \leqslant \operatorname{tr} \tilde{P}\tilde{V}. \tag{10}$$

**Remark 3.1.** If $\tilde{A}_{s\gamma}$ is asymptotically stable then the existence of a nonnegative-definite matrix $\tilde{P}$ satisfying (9) is equivalent to the existence of a frequency-domain condition guaranteeing robust stability of $\tilde{A} + \Delta\tilde{A}$, $\Delta A \in \mathscr{U}$, in terms of an *implicit* small gain condition involving the shifted dynamics matrix $\tilde{A}_{s\gamma}$ which is a function of the uncertainty set bound $\gamma$. For details see [7].

To apply Theorem 3.1 to controller synthesis, we use the Riccati Eq. (9) to guarantee that the closed-loop system is robustly stable. This leads to the following optimization problem.

*Optimization Problem.* Determine $(A_c, B_c, C_c)$ that minimizes $\mathscr{J}(A_c, B_c, C_c) \triangleq \operatorname{tr} \tilde{P}\tilde{V}$, where $\tilde{P} \in \mathbb{N}^{\tilde{n}}$ satisfies (9) and such that $(A_c, B_c, C_c)$ is controllable and observable.

The relationship between the Optimization Problem and the Robust Stability and Performance Problem is straightforward, as shown by the following proposition.

**Proposition 3.1.** *Let* $(A_c, B_c, C_c)$ *be given. If* $\tilde{P} \in \mathbb{N}^{\tilde{n}}$ *satisfies* (9) *and* $(\tilde{A} + \Delta\tilde{A}, \tilde{E})$ *is detectable for all* $\Delta A \in \mathscr{U}$, then $\tilde{A} + \Delta\tilde{A}$ *is asymptotically stable for all* $\Delta A \in \mathscr{U}$, *and* $J(A_c, B_c, C_c) \leqslant \mathscr{J}(A_c, B_c, C_c)$.

**Proof.** Since (9) has a solution $\tilde{P} \in \mathbb{N}^{\tilde{n}}$ and $(\tilde{A} + \Delta\tilde{A}, \tilde{E})$ is detectable for all $\Delta A \in \mathscr{U}$, the hypotheses of Theorem 3.1 are satisfied so that robust stability and robust performance are guaranteed. Now, $J(A_c, B_c, C_c) \leqslant \mathscr{J}(A_c, B_c, C_c)$ is merely a restatement of (10). □

It follows from Proposition 3.1 that the satisfaction of (9) along with the generic detectability condition leads to robust stability along with an upper bound for the $\mathscr{H}_2$ performance. Hence, by deriving necessary conditions for the Optimization Problem we obtain sufficient conditions for characterizing dynamic output feedback controllers guaranteeing robust stability and performance.

## 4. Robust controller synthesis via the implicit small-gain guaranteed cost bound

In this section we state constructive sufficient conditions for characterizing fixed-order (i.e., full- and reduced-order) robust controllers. These results are obtained by minimizing the worst-case $\mathscr{H}_2$ cost bound (10) subject to (9). To apply Theorem 3.1 to robust controller synthesis, let $\tilde{S}_i$, $i = 1, \ldots, r$, have the form

$$\tilde{S}_i = \begin{bmatrix} S_i & 0_{n \times n_c} \\ 0_{n_c \times n} & 0_{n_c \times n_c} \end{bmatrix}, \tag{11}$$

where $S_i \in \mathbb{R}^{n\times n}$. With $\tilde{S}_i$, $i = 1,\dots,r$, given by (11) it can be shown that

$$\tilde{I}_i = \begin{bmatrix} \hat{I}_i & 0_{n\times n_c} \\ 0_{n_c\times n} & 0_{n_c\times n_c} \end{bmatrix}, \quad \tilde{Z}_i = \begin{bmatrix} Z_i & 0_{n\times n_c} \\ 0_{n_c\times n} & 0_{n_c\times n_c} \end{bmatrix},$$

where $\hat{I}_i = [S_i \quad A_i^{\mathrm{T}}][S_i \quad A_i^{\mathrm{T}}]^{\dagger}$ and $Z_i = [(S_i + S_i^{\mathrm{T}})^2]^{1/2}$. Furthermore, for convenience in stating the main theorem, define the notation $A_{s\gamma} \triangleq A + \gamma^{-2}\sum_{i=1}^{r}\alpha_i\beta_i A_i S_i$, $\Sigma \triangleq BR_2^{-1}B^{\mathrm{T}}$, and $\bar{\Sigma} \triangleq C^{\mathrm{T}}V_2^{-1}C$.

**Theorem 4.1.** *Let $n_c \leqslant n$ and suppose there exist $n\times n$ nonnegative-definite matrices $P, Q, \hat{P}, \hat{Q}$ satisfying*

$$0 = A_{s\gamma}^{\mathrm{T}}P + PA_{s\gamma} + R_1 + \sum_{i=1}^{r}[\gamma^{-2}(\alpha_i^2 S_i^{\mathrm{T}}S_i + \beta_i^2 PA_iA_i^{\mathrm{T}}P) + \gamma^{-1}\beta_i^{-1}|\alpha_i|Z_i + \beta_i^{-2}\hat{I}_i] - P\Sigma P + \tau_\perp^{\mathrm{T}}P\Sigma P\tau_\perp, \tag{12}$$

$$0 = \left[A_{s\gamma} + \sum_{i=1}^{r}\gamma^{-2}\beta_i^2 A_iA_i^{\mathrm{T}}(P + \hat{P})\right]Q + Q\left[A_{s\gamma} + \sum_{i=1}^{r}\gamma^{-2}\beta_i^2 A_iA_i^{\mathrm{T}}(P + \hat{P})\right]^{\mathrm{T}} + V_1 - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}}, \tag{13}$$

$$0 = \left(A_{s\gamma} - Q\bar{\Sigma} + \sum_{i=1}^{r}\gamma^{-2}\beta_i^2 A_iA_i^{\mathrm{T}}P\right)^{\mathrm{T}}\hat{P} + \hat{P}\left(A_{s\gamma} - Q\bar{\Sigma} + \sum_{i=1}^{r}\gamma^{-2}\beta_i^2 A_iA_i^{\mathrm{T}}P\right) + \sum_{i=1}^{r}\gamma^{-2}\beta_i^2\hat{P}A_iA_i^{\mathrm{T}}\hat{P} + P\Sigma P - \tau_\perp^{\mathrm{T}}P\Sigma P\tau_\perp, \tag{14}$$

$$0 = \left(A_{s\gamma} - \Sigma P + \sum_{i=1}^{r}\gamma^{-2}\beta_i^2 A_iA_i^{\mathrm{T}}P\right)\hat{Q} + \hat{Q}\left(A_{s\gamma} - \Sigma P + \sum_{i=1}^{r}\gamma^{-2}\beta_i^2 A_iA_i^{\mathrm{T}}P\right)^{\mathrm{T}} + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}}, \tag{15}$$

$$\operatorname{rank}\,\hat{Q} = \operatorname{rank}\,\hat{P} = \operatorname{rank}\,\hat{Q}\hat{P} = n_c, \tag{16}$$

$$\hat{Q}\hat{P} = G^{\mathrm{T}}M\Gamma, \qquad \Gamma G^{\mathrm{T}} = I_{n_c}, \quad M \in \mathbb{R}^{n_c\times n_c}, \qquad \tau \triangleq G^{\mathrm{T}}\Gamma, \qquad \tau_\perp \triangleq I_n - \tau, \tag{17}$$

*and let $A_c$, $B_c$, and $C_c$ be given by*

$$A_c = \Gamma(A_{s\gamma} - Q\bar{\Sigma} - \Sigma P + \gamma^{-2}\sum_{i=1}^{r}\beta_i^2 A_iA_i^{\mathrm{T}}P)G^{\mathrm{T}}, \tag{18}$$

$$B_c = \Gamma QC^{\mathrm{T}}V_2^{-1}, \tag{19}$$

$$C_c = -R_2^{-1}B^{\mathrm{T}}PG^{\mathrm{T}}. \tag{20}$$

*Then* $(\tilde{A}+\Delta\tilde{A},\tilde{E})$ *is detectable for all* $\Delta A\in\mathscr{U}$ *if and only if* $\tilde{A}+\Delta\tilde{A}$ *is asymptotically stable for all* $\Delta A\in\mathscr{U}$. *In this case, the worst-case* $\mathscr{H}_2$ *performance criterion satisfies the bound*

$$J(A_c,B_c,C_c)\leqslant \operatorname{tr}\left[PV_1+Q\left(P\Sigma P-\tau_\perp^{\mathrm{T}}P\Sigma P\tau_\perp-\gamma^{-2}\sum_{i=1}^{r}\hat{P}A_iA_i^{\mathrm{T}}\hat{P}\right)\right]$$

$$=\operatorname{tr}\Big[QR_1+P(Q\bar{\Sigma}Q-\tau_\perp Q\bar{\Sigma}Q\tau_\perp^{\mathrm{T}})-\hat{Q}\left(\sum_{i=1}^{r}\gamma^{-2}\alpha_i^2S_i^{\mathrm{T}}S_i+\gamma^{-1}\beta_i^{-1}|\alpha_i|Z_i+\beta_i^{-2}\hat{I}_i\right)\Big]. \tag{21}$$

**Proof.** The proof is constructive in nature. We first obtain necessary conditions for the Optimization Problem and show by construction that these conditions serve as sufficient conditions for closed-loop stability. Specifically, it can be shown (see [14] for a similar construction) that the existence of $P$, $Q$, $\hat{P}$, $\hat{Q}\in\mathbb{N}^n$ satisfying Eqs. (12)–(15) implies the existence of $\tilde{P}\in\mathbb{N}^{\tilde{n}}$ satisfying (9) where $\tilde{P}$ is given by

$$\tilde{P}=\begin{bmatrix}P+\hat{P} & -\hat{P}G^{\mathrm{T}}\\ -G\hat{P} & G\hat{P}G^{\mathrm{T}}\end{bmatrix}.$$

Now, the proof of robust stability and the upper bound on $\mathscr{H}_2$ performance (5) for all uncertain perturbations $\Delta A\in\mathscr{U}$ follows from Theorem 3.1.

Next, to optimize Eq. (10) subject to constraint (9) over the open set

$$\mathscr{S}\triangleq\Big\{(\tilde{P},A_c,B_c,C_c)\colon \tilde{P}>0,\ \tilde{A}_{s\gamma}+\gamma^{-2}\sum_{i=1}^{r}A_i\beta_i^{\mathrm{T}}\beta_iA_i^{\mathrm{T}}\tilde{P}\ \text{is asymptotically stable and } (A_c,B_c,C_c) \text{ is minimal}\Big\}$$

form the Lagrangian

$$\mathscr{L}(A_c,B_c,C_c,\tilde{Q},\tilde{P},\lambda)\triangleq\operatorname{tr}\Big[\lambda\tilde{P}\tilde{V}+\tilde{Q}\Big\{\tilde{A}_{s\gamma}^{\mathrm{T}}\tilde{P}+\tilde{P}\tilde{A}_{s\gamma}+\tilde{R}+\sum_{i=1}^{r}[\gamma^{-2}(\alpha_i^2\tilde{S}_i^{\mathrm{T}}\tilde{S}_i+\beta_i^2\tilde{P}\tilde{A}_i\tilde{A}_i^{\mathrm{T}}\tilde{P})+\gamma^{-1}\beta_i^{-1}|\alpha_i|\tilde{Z}_i+\beta_i^{-2}\tilde{I}_i]\Big\}\Big], \tag{22}$$

where the Lagrange multipliers $\lambda\geqslant 0$ and $\tilde{Q}\in\mathbb{R}^{\tilde{n}\times\tilde{n}}$ are not all zero. We thus obtain

$$\frac{\partial\mathscr{L}}{\partial\tilde{P}}=\left(\tilde{A}_{s\gamma}+\gamma^{-2}\sum_{i=1}^{r}\beta_i^2A_iA_i^{\mathrm{T}}\tilde{P}\right)\tilde{Q}+\tilde{Q}\left(\tilde{A}_{s\gamma}+\gamma^{-2}\sum_{i=1}^{r}\beta_i^2A_iA_i^{\mathrm{T}}\tilde{P}\right)^{\mathrm{T}}+\lambda\tilde{V}. \tag{23}$$

Setting $\partial \mathscr{L}/\partial \tilde{P} = 0$ yields

$$0 = \left(\tilde{A}_{s\gamma} + \gamma^{-2} \sum_{i=1}^{r} \beta_i^2 A_i A_i^{\mathrm{T}} \tilde{P}\right)\tilde{Q} + \tilde{Q}\left(\tilde{A}_{s\gamma} + \gamma^{-2} \sum_{i=1}^{r} \beta_i^2 A_i A_i^{\mathrm{T}} \tilde{P}\right)^{\mathrm{T}} + \lambda \tilde{V}. \quad (24)$$

Since $\tilde{A}_{s\gamma} + \gamma^{-2}\sum_{i=1}^{r} \beta_i^2 A_i A_i^{\mathrm{T}} \tilde{P}$ is assumed to be asymptotically stable, setting $\lambda = 0$ implies $\tilde{Q} = 0$. Hence, it can be assumed, without loss of generality, that $\lambda = 1$. Furthermore, $\tilde{Q}$ is nonnegative definite. The remainder of the proof follows as in [14]. Briefly, the principal steps are as follows.

*Step* 1: Compute $\partial \mathscr{L}/\partial A_c$, $\partial \mathscr{L}/\partial B_c$, and $\partial \mathscr{L}/\partial C_c$.

*Step* 2: Partition (9) and (24) into six equations (a)–(f) corresponding to the $n \times n$, $n \times n_c$, and $n_c \times n_c$ sub-blocks of $\tilde{P}$ and $\tilde{Q}$, respectively. Next, since the compensator triple $(A_c, B_c, C_c)$ is controllable and observable, using a minor extension of the result from Albert [15] and Lemma 12.2 of Wonham [16], (c) implies that the lower-right $n_c \times n_c$ block of $\tilde{P}$ is positive definite. Using similar arguments we can show that the lower-right $n_c \times n_c$ block of $\tilde{Q}$ is positive definite. See [14] for details.

*Step* 3: Form (b) times the $n \times n_c$ sub-block of $\tilde{Q}$ plus the $n_c \times n_c$ sub-block of $\tilde{Q}$ times (c) to define the projection matrix $\tau$ and the new variables $P$, $Q$, $\hat{P}$, $\hat{Q}$, $G$, and $\Gamma$.

*Step* 4: Use the result of Steps 1 and 3 to solve for the compensator gains (18)– (20).

*Step* 5: Manipulate (a), (b), (d), and (e) to yield (12)–(15).

*Step* 6: Use the results of Step 3 to show that (10) is equivalent to (21).

For a detailed exposition of a similar proof, see [14]. □

**Remark 4.1.** In the full-order case, set $n_c = n$ so that $G = \Gamma = \tau = I_n$ and $\tau_\perp = 0$. In this case the last term in each of (12)– (15) is zero and (15) is superfluous.

Theorem 4.1 provides constructive sufficient conditions that yield dynamic feedback gains $A_c$, $B_c$, and $C_c$ for robust stability and performance. When solving (12)–(15) numerically, the values of $\gamma$, $\alpha_i$, $\beta_i$, and $S_i$, $i = 1, \ldots, r$, can be adjusted to examine tradeoffs between $\mathscr{H}_2$ performance and robustness. As discussed in [7], to further reduce conservatism, one can view the scalars $\alpha_i$, $\beta_i$, and the matrix $S_i$ as free parameters and optimize the worst-case $\mathscr{H}_2$ performance bound $\mathscr{J}$ with respect to $\alpha_i$, $\beta_i$, and $S_i$. Specifically, by using $\partial \mathscr{J}/\partial \alpha_i = 0$, $\partial \mathscr{J}/\partial \beta_i = 0$, and $\partial \mathscr{J}/\partial S_i = 0$ within a numerical optimization algorithm, the optimal robust reduced-order controllers and scaling parameters $\alpha_i$, $\beta_i$, $S_i$, $i = 1, \ldots, r$, can be determined simultaneously. For further details, see [7].

## 5. Flexible structure example

Consider the dynamic system shown in Fig. 1, which represents a flexible structure with uncertain high-frequency dynamics [17]. The equations of motion for this system are

$$m_1 \ddot{x}_1 + c_1 \dot{x}_1 - c_2(\dot{x}_2 - \dot{x}_1) + k_1 x_1 - k_2(x_2 - x_1) = u,$$

$$m_2 \ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0.$$

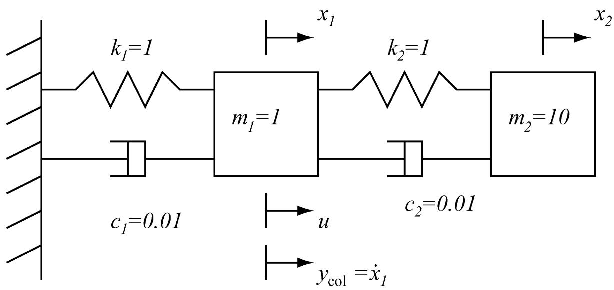


Fig. 1. Two-mass system.

Here we consider the case of a colocated sensor and actuator pair, where the output is given by $y_{\mathrm{col}} = \dot{x}_1$. Letting $m_1 = 1$, $m_2 = 10$, $k_1 = k_2 = 1$, and $c_1 = c_2 = 0.01$ and transforming to real normal coordinates yields the plant state-space realization

$$\dot{x} = \begin{bmatrix} -0.0002 & 0.2208 & 0 & 0 \\ -0.2208 & -0.0002 & 0 & 0 \\ 0 & 0 & -0.0103 & 1.4320 \\ 0 & 0 & -1.4320 & -0.0103 \end{bmatrix} x + \begin{bmatrix} -0.1439 \\ 0.2168 \\ -0.0426 \\ 1.1890 \end{bmatrix} u,$$

$$y_{\mathrm{col}} = [-0.0545 \quad 0.0819 \quad -0.0352 \quad 0.8181]x.$$

As in [17], the matrices $D_1$, $D_2$, $E_1$, and $E_2$ are chosen to be

$$D_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

so that the LQG compensator places a notch at the second modal frequency. Uncertainty in the damped natural frequency of the second mode $\omega_{\mathrm{d2}} = 1.432$ is modeled by choosing

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

A quasi-Newton optimization algorithm was used to compute full-order controllers ($n_{\mathrm{c}} = n$) that minimize the cost bound $\mathscr{J}$ for several values of $\gamma$. In particular, the algorithm is a continuation algorithm with correction steps performed using quasi-Newton corrections with the BFGS inverse Hessian update. The line-search portions

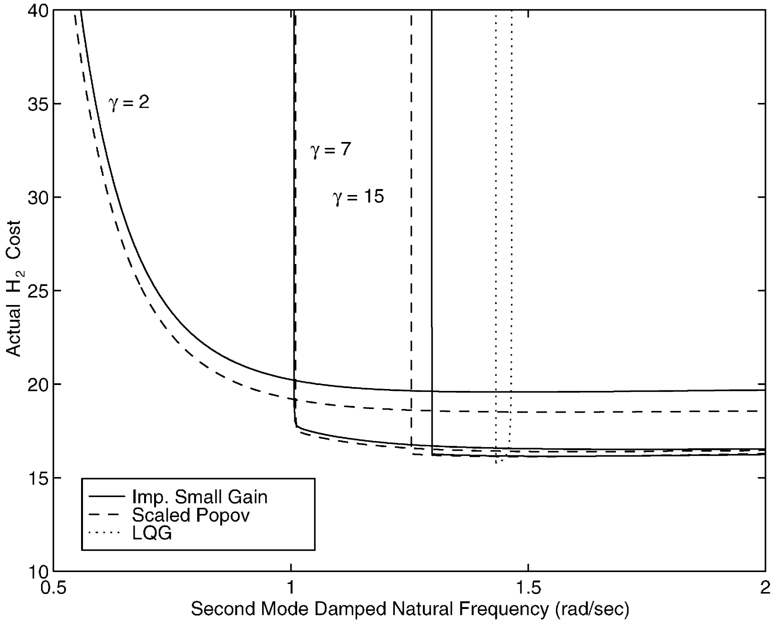


Fig. 2. Dependence of the $\mathscr{H}_2$ cost on the damped natural frequency of the second mode.

of the algorithm were modified to include a constraint-checking subroutine which decreases the length of the search direction vector until it lies entirely within the set of parameters that yield a stable closed-loop system. This modification ensures that the cost function $\mathscr{J}$ remains defined at every point in the line search process. For details of the algorithm, see [12]. The actual $\mathscr{H}_2$ cost was computed for a range of values of the damped natural frequency of the second mode for the LQG controller and for the implicit small gain (with $S_1 = A_1$ and $\alpha_1$ and $\beta_1$ obtained by $\partial\mathscr{J}/\partial\alpha_i = 0$ and $\partial\mathscr{J}/\partial\beta_i = 0$) and scaled Popov controllers [6] corresponding to $\gamma = 15$, 7, and 2. The cost dependence is shown in Fig. 2. As $\gamma$ decreases, the $\mathscr{H}_2$ cost of the nominal closed-loop system increases while the $\mathscr{H}_2$ cost of the perturbed closed-loop system remains near the nominal value for a larger range of perturbations. The LQG controller stabilizes the closed-loop system for only small perturbations in the damped natural frequency of the second mode, while the implicit small gain controllers stabilize the closed-loop system and provide performance close to the optimal level even for large perturbations. Hence, robust performance over a large range of the uncertain parameter is achieved for only a small increase in the $\mathscr{H}_2$ cost above the optimal. Also note that the robustness/performance tradeoffs of the implicit small gain controllers are comparable to those of the scaled Popov controllers which are obtained using frequency-dependent multipliers [6].

The frequency responses of the LQG controller and the implicit small-gain controllers with $\gamma = 15$, 7, and 2 are shown in Fig. 3. The LQG controller is unstable and achieves closed-loop stability and nominal performance by placing a notch at the

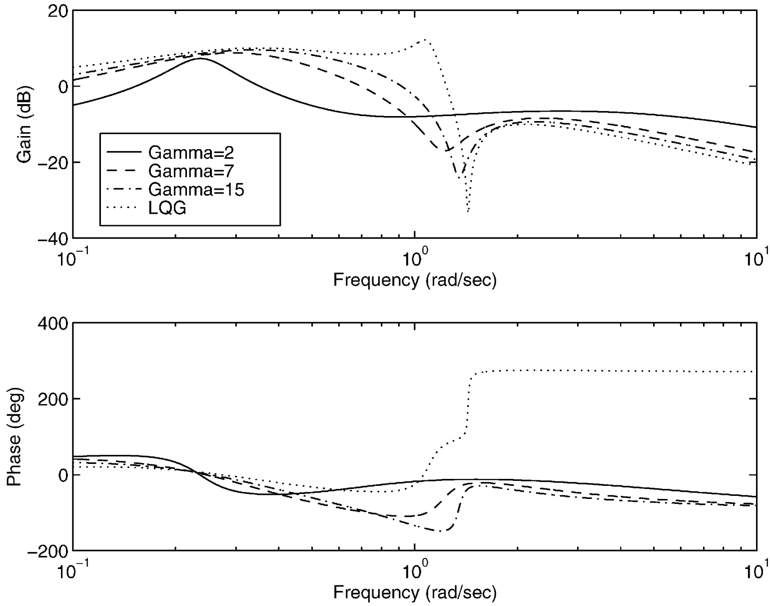


Fig. 3. Frequency responses of implicit small gain controllers.

nominal damped natural frequency $\omega_{d2}$ of the uncertain second mode. Hence, closed-loop performance degrades considerably when the damped natural frequency of the second mode is perturbed. The implicit small-gain controller with $\gamma = 7$ has only a shallow notch near the damped natural frequency of the second mode, while the controller with $\gamma = 2$ has no notch near that frequency. Hence, these controllers sacrifice nominal performance for improved robust performance over a larger range of the uncertain damped natural frequency. As $\gamma$ decreases, the controllers guarantee robust performance over a larger range of $\delta$. Note that the controller obtained with $\gamma = 2$ is positive real. Since the plant is a model of a flexible structure with a colocated sensor and actuator pair, it is also positive real, and thus the closed-loop system is asymptotically stable for all values of the uncertain damped natural frequency.

## 6. Conclusion

This paper extended the implicit small-gain guaranteed cost bound [7] to controller synthesis. Specifically, the implicit small-gain guaranteed cost bound was used to address the problem of robust stability and $\mathscr{H}_2$ performance via fixed-order dynamic compensation. A quasi-Newton optimization algorithm was used to obtain robust controllers for an illustrative example. The design example considered demonstrated the effectiveness of the implicit small-gain guaranteed cost bound. Finally, we note that the conservatism of the proposed implicit small-gain guaranteed cost bound is difficult to predict and will depend upon the actual value of $\tilde{P}$ determined by solving (9).

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