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Mixed-norm H_2/H_∞ regulation and estimation: The discrete-time case

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Abstract A discrete-time H_2 static output feedback design problem involving a constraint on H_∞ disturbance attenuation is addressed and state space formulae are derived The dual problem of discrete-time dynamic estimation with an H_∞ error bound is also addressed. These results are analogous to results obtained previously for the continuous-time problem

Keywords. H_2/H_∞ design, static output feedback; H_∞-estimation.

1. Introduction

It has recently been shown [1,3,5,7,10,12,17] that H_∞-constrained controllers can be characterized by means of algebraic Riccati equations These results comprise a fairly extensive theory encompassing both static and dynamic controllers. In particular, the results in [1,7] address both H_2 and H_∞ design aspects simultaneously within the context of the standard problem for full- and reduced-order controllers. This mixed-norm problem thus permits design tradeoffs between loop shaping, unstructured uncertainty, and rms performance.

Although the results cited above have been developed for continuous-time plants controlled by analog controllers, there has also been some effort directed toward developing a discrete-time version of the H_∞ Riccati equation theory [6,9,11,13–15]. The purpose of the present paper is to extend the mixed-norm H_2/H_∞ Riccati equation approach of [1,7] to the discrete-time case.

For sampled-data systems that involve continuous-time plants controlled by discrete-time controllers with A/D and D/A interfaces it is often possible to first design analog controllers which can subsequently be discretized for digital implementation. This *indirect* method has the advantage that the sample rate can be changed without redesigning the original analog control law. However, there are several disadvantages to this approach. For example, if the sampling rate is ultimately limited (as it usually is in practice), then the original continuous-time design must have correspondingly limited bandwidth. Furthermore, the

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discretization (i.e., digitalization) process itself is nontrivial since there are many alternative discretization procedures exhibiting different characteristics.

The goal of the discrete-time H_2/H_∞ problem is to minimize an H_2 performance criterion subject to a prespecified H_∞ constraint on the closed-loop transfer function. As in the continuous-time case, the H_∞ constraint is embedded within the optimization process by replacing the closed-loop covariance Lyapunov equation by an appropriate discrete-time Riccati equation whose solution leads to an upper bound on the H_2 performance. The key idea to this approach is to view the upper bound as an auxiliary cost and, for a fixed controller structure, seek feedback gains that minimize the H_2 bound and guarantee that the disturbance attenuation constraint is enforced. The principal result is a sufficient condition involving coupled Riccati/Lyapunov equations whose solutions, when they exist, are used to explicitly construct feedback gains for characterizing full-state and static output feedback controllers with bounded H_2 and H_∞ costs Note that, strictly speaking, the problem addressed is suboptimal in both the H_2 sense and the H_∞ sense. However, solving the design equations for successively smaller H_∞ disturbance attenuation constraints should, in the limit, yield an H_∞-optimal controller over the class of fixed-structure stabilizing controllers Although our main result gives sufficient conditions, these conditions will also be necessary as long as the mixed-norm optimization problem possesses at least one extremal over the class of fixed-structure controllers (see Lemma 2 2)

Finally, we also consider the discrete-time H_2/H_∞ dynamic estimation problem. Specifically, we extend the least squares discrete-time formulation to include a frequency-domain bound (i e., H_∞ norm) on the state-estimation error For details on continuous-time H_2/H_∞ estimation see [2,8]. As a special case of the results given in the present paper we obtain the H_2 static output feedback solution, discrete-time LQR problem, and the discrete-time steady-state Kalman filter problem.

Notation

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{E}$	real numbers, $r\times s$ real matrices, expected value,
I_r, $(\ )^{\mathrm{T}}$, $(\ )^*$	$r\times r$ identity matrix, transpose, complex conjugate transpose,
tr	trace,
$\sigma_{\max}(X)$	largest singular value of matrix X,
$\mathcal{RH}_\infty$	real-rational subspace of H_∞,
n, m, l, d, r, q, d_∞	positive integers,
x, y, x_e, y_e, u	n, l, n, q, m-dimensional vectors,
A, B, C	$n\times n$, $n\times m$, $l\times n$ matrices,
A_e, B_e, C_e, K	$n\times n$, $n\times l$, $q\times n$, $m\times l$ matrices,
D, D_∞, D_1, D_2	$n\times d$, $n\times d_\infty$, $n\times d$, $l\times d$ matrices,
E, E_1, E_2	$r\times q$, $q\times m$, $q\times m$ matrices,
R, R_1, R_{12}, R_2	$E^{\mathrm{T}}E$, $E_1^{\mathrm{T}}E_1$, $E_1^{\mathrm{T}}E_2$, $E_2^{\mathrm{T}}E_2$, $R>0$, $R_2>0$,
$w(k)$	d-dimensional discrete-time white noise process,
$w_\infty(k)$	l_2 disturbance signal,
V, V_1, V_2	covariance of $Dw(\cdot)$, $D_1w(\cdot)$, $D_2w(\cdot)$; $V=DD^{\mathrm{T}}$, $V_1=D_1D_1^{\mathrm{T}}$, $V_2=D_2D_2^{\mathrm{T}}>0$,
V_{12}	cross covariance of $D_1w(\cdot)$, $D_2w(\cdot)$; $V_{12}=D_1D_2^{\mathrm{T}}$

2. Problem statement

In this section we introduce the discrete-time static output-feedback control problem with constrained H_∞ disturbance attenuation Without the H_2 performance criterion the problem considered here corresponds to a standard discrete-time H_∞ control problem.

H_∞-constrained static output feedback control problem. Given the nth-order plant

$$x(k+1) = Ax(k) + Bu(k) + Dw(k), \quad k = 0, 1, 2, \ldots, \tag{2.1}$$

$$y(k) = Cx(k), \tag{2.2}$$

determine a static output feedback law

$$u(k) = Ky(k) \tag{2.3}$$

that satisfies the following design criteria

(i) the closed-loop system (2.1)–(2.3) is asymptotically stable, i.e., $\tilde{A} \triangleq A + BKC$ is asymptotically stable;

(ii) the $q \times d_\infty$ transfer function

$$G(z) \triangleq (E_1 + E_2KC)(zI_n - \tilde{A})^{-1}D_\infty \tag{2.4}$$

from disturbances $w_\infty(k)$ to performance variables $z(k) = E_1x(k) + E_2u(k)$ satisfies the constraint

$$\| G(z) \|_\infty \le \gamma, \tag{2.5}$$

where

$$\| G(z) \|_\infty = \sup_{w_\infty \in l_2} \frac{\| z \|_2}{\| w_\infty \|_2} = \sup_{\theta \in [0, 2\pi]} \sigma_{\max} \| G(e^{j\theta}) \|,$$

and $\gamma > 0$ is a given constant; and

(iii) the performance functional

$$J(K) \triangleq \lim_{k \to \infty} \mathbb{E}\left[x^{\mathrm{T}}(k)R_1x(k) + 2x^{\mathrm{T}}(k)R_{12}u(k) + u^{\mathrm{T}}(k)R_2u(k)\right] \tag{2.6}$$

is minimized.

Note that the closed-loop system (2.1)–(2.3) can be written as

$$x(k+1) = (A + BKC)x(k) + Dw(k) \tag{2.7}$$

and that (2.6) becomes

$$\begin{aligned} J(K) &= \lim_{k \to \infty} \mathbb{E}\left[(E_1 + E_2KC)x(k)\right]^{\mathrm{T}}\left[(E_1 + E_2KC)x(k)\right] \\ &= \lim_{k \to \infty} \mathbb{E}\left[x^{\mathrm{T}}(k)\tilde{R}x(k)\right], \end{aligned} \tag{2.8}$$

where

$$\tilde{R} \triangleq R_1 + R_{12}KC + C^{\mathrm{T}}K^{\mathrm{T}}R_{12}^{\mathrm{T}} + C^{\mathrm{T}}K^{\mathrm{T}}R_2KC. \tag{2.9}$$

For convenience we have defined $R_1 \triangleq E_1^{\mathrm{T}}E_1$ and $R_2 \triangleq E_2^{\mathrm{T}}E_2$ which appear in subsequent expressions. Note that $R_{12} \triangleq E_1^{\mathrm{T}}E_2$ is a cross-weighting term which is included for greater design flexibility. Furthermore, by defining the transfer function

$$\tilde{G}(z) \triangleq (E_1 + E_2KC)(zI_n - \tilde{A})^{-1}D, \tag{2.10}$$

it can be shown that when $\tilde{A}$ is asymptotically stable, (2.8) is given by

$$J(K) = \| \tilde{G}(z) \|_2^2. \tag{2.11}$$

Note that for both the H_2 and H_∞ designs, the respective transfer functions (2.10) and (2.4) involve the same weighting matrices E_1 and E_2 for the state and control variables However, the disturbances $w(k)$ and $w_\infty(k)$ are different. Specifically, $w(k)$ is a discrete-time white noise process, while $w_\infty(k)$ is an l_2 disturbance signal.

Next, we note that if $\tilde{A}$ is asymptotically stable for a given feedback law K, then the H_2 performance (2 8) is given by

$$J(K) = \operatorname{tr} \tilde{Q}\tilde{R}, \tag{2.12}$$

where the steady-state closed-loop state covariance defined by

$$\tilde{Q} \triangleq \lim_{k\to\infty} \mathbb{E}\left[x(k)x^{\mathrm{T}}(k)\right] \tag{2 13}$$

exists and satisfies the $n \times n$ algebraic Lyapunov equation

$$\tilde{Q} = \tilde{A}\tilde{Q}\tilde{A}^{\mathrm{T}} + V, \tag{2 14}$$

where $V \triangleq DD^{\mathrm{T}}$. Finally, we need the following proposition for the statement of the main result of this section.

Proposition 2.1. *Suppose $\tilde{A}$ is asymptotically stable for a given $K \in \mathbb{R}^{m\times l}$. Then*

$$J(K) = \operatorname{tr} \tilde{P}V, \tag{2 15}$$

where $\tilde{P}$ is the unique, $n \times n$ nonnegative-definite solution to

$$\tilde{P} = \tilde{A}^{\mathrm{T}}\tilde{P}\tilde{A} + \tilde{R}. \tag{2.16}$$

Proof. It need only be noted that

$$\operatorname{tr} \tilde{Q}\tilde{R} = \operatorname{tr} \sum_{i=0}^{\infty} \tilde{A}^i V \tilde{A}^{i\mathrm{T}} \tilde{R} = \operatorname{tr} V \sum_{i=0}^{\infty} \tilde{A}^{i\mathrm{T}} \tilde{R} \tilde{A}^i = \operatorname{tr} \tilde{P}V. \qquad \square$$

The key step in enforcing the disturbance attenuation constraint (2 5) is to replace the algebraic Lyapunov equation (2.16) by an algebraic Riccati equation that overbounds $\tilde{P}$ given by (2.16). Justification of this technique is provided by the following result

Lemma 2.1. *Let $K \in \mathbb{R}^{m\times l}$ be given and assume there exists an $n \times n$ nonnegative-definite matrix P satisfying*

$$P = \tilde{A}^{\mathrm{T}}P\tilde{A} + \tilde{A}^{\mathrm{T}}PD_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)^{-1}D_\infty^{\mathrm{T}}P\tilde{A} + \tilde{R} \tag{2.17}$$

Then

$$(\tilde{A}, \tilde{R}) \text{ is detectable} \tag{2.18}$$

if and only if

$$\tilde{A} \text{ is asymptotically stable}. \tag{2.19}$$

In this case,

$$\| G(z) \|_\infty \le \gamma \tag{2.20}$$

and

$$\tilde{P} \le P \tag{2 21}$$

Consequently,

$$J(K) \le \mathcal{J}(K, P), \tag{2.22}$$

where

$$\mathscr{J}(K, P) \triangleq \operatorname{tr} PV \tag{2.23}$$

Proof. It follows from [16, Theorem 3 6] that (2.18) implies that

$$\left(\tilde{A},\ \tilde{A}^{\mathrm{T}}PD_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)^{-1}D_\infty^{\mathrm{T}}P\tilde{A} + \tilde{R}\right)$$

is also detectable. Using the assumed existence of a nonnegative solution to (2 17) and [16, Lemma 12.2′, p. 282] it now follows that $\tilde{A}$ is asymptotically stable. The converse is immediate. To prove (2.20) replace $\tilde{R}$ by $\tilde{E}^{\mathrm{T}}\tilde{E}$ where $\tilde{E} \triangleq E_1 + E_2KC$ and define $M \triangleq \gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty$ so that (2.17) becomes

$$0 = -\tilde{E}^{\mathrm{T}}\tilde{E} - \tilde{A}^{\mathrm{T}}P\tilde{A} + P - \tilde{A}^{\mathrm{T}}PD_\infty M^{-1}D_\infty^{\mathrm{T}}\tilde{A}, \tag{2 24}$$

or, equivalently,

$$\tilde{E}^{\mathrm{T}}\tilde{E} = -\tilde{A}^{\mathrm{T}}P\tilde{A} + e^{j\theta}P\,e^{-j\theta} - \tilde{A}^{\mathrm{T}}PD_\infty M^{-1}D_\infty^{\mathrm{T}}P\tilde{A}, \tag{2 25}$$

where $\theta \in [0, 2\pi]$. Next, define $z \triangleq e^{j\theta}$ and add and subtract $\tilde{A}^{\mathrm{T}}P\tilde{A}$, $\tilde{A}^{\mathrm{T}}Pz$, and $\bar{z}P\tilde{A}$ to (2 25) so that (2 25) becomes

$$\tilde{E}^{\mathrm{T}}\tilde{E} = -\tilde{A}^{\mathrm{T}}P\tilde{A} + zP\bar{z} - \tilde{A}^{\mathrm{T}}PD_\infty M^{-1}D_\infty^{\mathrm{T}}P\tilde{A} + \tilde{A}^{\mathrm{T}}P\tilde{A} - \tilde{A}^{\mathrm{T}}P\tilde{A} + \bar{z}P\tilde{A} - \bar{z}P\tilde{A} + \tilde{A}Pz - \tilde{A}^{\mathrm{T}}Pz, \tag{2 26}$$

or, equivalently

$$\tilde{E}^{\mathrm{T}}\tilde{E} = \left(\bar{z}I_n - \tilde{A}\right)^{\mathrm{T}}P\left(zI_n - \tilde{A}\right) + \left(\bar{z}I_n - \tilde{A}\right)^{\mathrm{T}}P\tilde{A} + \tilde{A}^{\mathrm{T}}P\left(zI_n - \tilde{A}\right) - \tilde{A}^{\mathrm{T}}PD_\infty M^{-1}D_\infty^{\mathrm{T}}P\tilde{A}. \tag{2.27}$$

Next, forming $D_\infty^{\mathrm{T}}(\bar{z}I_n - \tilde{A})^{-\mathrm{T}}$ (2.27) $(zI_n - \tilde{A})^{-1}D_\infty$ yields

$$\begin{aligned} D_\infty^{\mathrm{T}}\left(\bar{z}I_n - \tilde{A}\right)^{-T}\tilde{E}^{\mathrm{T}}\tilde{E}\left(zI_n - \tilde{A}\right)^{-1}D_\infty &= D_\infty^{\mathrm{T}}PD_\infty + D_\infty^{\mathrm{T}}P\tilde{A}\left(zI_n - \tilde{A}\right)^{-1}D_\infty + D_\infty^{\mathrm{T}}\left(\bar{z}I_n - \tilde{A}\right)^{-\mathrm{T}}\tilde{A}^{\mathrm{T}}PD_\infty \\ &\quad - D_\infty^{\mathrm{T}}\left(\bar{z}I_n - \tilde{A}\right)^{-T}\tilde{A}^{\mathrm{T}}PD_\infty M^{-1}D_\infty^{\mathrm{T}}P\tilde{A}\left(zI_n - \tilde{A}\right)^{-1}D_\infty. \end{aligned} \tag{2.28}$$

Multiplying (2.28) by -1 and adding $\gamma^2 I_{d_\infty}$ to both sides of (2.28) yields

$$\begin{aligned} \gamma^2 I_{d_\infty} - G^*(z)G(z) &= \left[M^{1/2} - M^{-1/2}D_\infty^{\mathrm{T}}P\tilde{A}\left(zI_n - \tilde{A}\right)^{-1}D_\infty\right]^* \\ &\quad \cdot\left[M^{1/2} - M^{-1/2}D_\infty^{\mathrm{T}}P\tilde{A}\left(zI_n - \tilde{A}\right)^{-1}D_\infty\right] \geq 0, \end{aligned}$$

which implies $G^*(z)G(z) \leq \gamma^2 I_{d_\infty}$ This proves (2.20). To prove (2 21), subtract (2.16) from (2.17) to obtain

$$P - \tilde{P} = \tilde{A}^{\mathrm{T}}(P - \tilde{P})\tilde{A} + \tilde{A}^{\mathrm{T}}PD_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)^{-1}D_\infty^{\mathrm{T}}P\tilde{A}$$

which, since $\tilde{A}$ is asymptotically stable, is equivalent to

$$P - \tilde{P} = \sum_{i=0}^{\infty}(\tilde{A}^i)^{\mathrm{T}}\left[\tilde{A}^{\mathrm{T}}PD_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)^{-1}D_\infty^{\mathrm{T}}P\tilde{A}\right]\tilde{A}^i \geq 0.$$

Finally, (2.23) follows immediately from (2.21). □

Lemma 2.1 shows that H_∞ disturbance attenuation is automatically enforced when a nonnegative-definite solution to (2 17) is known to exist and $\tilde{A}$ is asymptotically stable Furthermore, all such solutions provide upper bounds on the H_2 performance criterion (2.6). Next, we present a partial converse of

Lemma 2 1 that guarantees the existence of a unique nonnegative-definite solution to (2.17) when (2.20) is satisfied.

Lemma 2.2. *Let $K \in \mathbb{R}^{m \times l}$ be given, suppose $\tilde{A}$ is asymptotically stable, and let $G(z) \in \mathcal{RH}_\infty$ with $\| G(z) \|_\infty < \gamma$ Then there exists a unique nonnegative-definite solution P satisfying* (2.17) *and such that the eigenvalues of $\tilde{A} + D_\infty(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}} P D_\infty) D_\infty^{\mathrm{T}} P \tilde{A}$ lie in the open unit disk.*

Proof. The assumptions that $G(z) \in \mathcal{RH}_\infty$ and $\| G(z) \|_\infty < \gamma$ imply that

$$I_{d_\infty} - \gamma^{-2} G^*(z) G(z) > 0,$$

for all z such that $|z| = 1$ This guarantees the existence of a spectral factor $N(z)$ such that

$$I_{d_\infty} - \gamma^{-2} G^*(z) G(z) = N^*(z) N(z),$$

for all z, where $N^{\pm 1}(z) \in \mathcal{RH}_\infty$. It is easily verified that (see [4, Theorem 4.1])

$$N(z) = \gamma^{-1} M^{1/2} - \gamma^{-1} M^{-1/2} D_\infty^{\mathrm{T}} P \tilde{A} \left(z I_n - \tilde{A} \right)^{-1} D_\infty$$

where $P = P^{\mathrm{T}}$ satisfies (2 17). The proof that the eigenvalues of $\tilde{A} + D_\infty(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}} P D_\infty) D_\infty^{\mathrm{T}} P \tilde{A}$ lie in the open unit disk and the uniqueness of P is given in [6]. □

3. The auxiliary minimization problem

As shown in the previous section, replacing (2.16) by (2.17) enforces the H_∞ disturbance attenuation constraint (2.20) and yields an upper bound for the H_2 performance criterion. That is, given a controller K for which there exists a nonnegative-definite solution to (2.17), the *actual* H_2 performance of the controller is guaranteed to be no worse than the bound given by $\mathcal{J}(K, P)$. Hence, $\mathcal{J}(K, P)$ can be interpreted as an *auxiliary* cost which leads to the following optimization problem

Auxiliary minimization problem. Determine $K \in \mathbb{R}^{m \times l}$ that minimizes $\mathcal{J}(K, P)$ subject to (2.17).

It follows from Lemma 2.1 that the satisfaction of (2.17) along with the generic condition (2.18) leads to: (1) closed-loop stability, (2) prespecified H_∞ performance attenuation; and (3) an upper bound for the H_2 performance criterion Hence, it remains to determine (K, P) that minimizes $\mathcal{J}(K, P)$ and thus provides an optimized bound for the actual H_2 performance $J(K)$

4. Sufficient conditions for H_∞ disturbance attenuation

In this section we state sufficient conditions for characterizing static output feedback controllers guaranteeing closed-loop stability, constrained H_∞ disturbance attenuation, and an optimized H_2 performance bound. For arbitrary $P, Q \in \mathbb{R}^{n \times n}$ define the notation

$$R_{2a} \triangleq R_2 + B^{\mathrm{T}} P B + B^{\mathrm{T}} P D_\infty \left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}} P D_\infty \right)^{-1} D_\infty^{\mathrm{T}} P B,$$

$$P_a \triangleq B^{\mathrm{T}} P A + R_{12}^{\mathrm{T}} + B^{\mathrm{T}} P D_\infty \left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}} P D_\infty \right)^{-1} D_\infty^{\mathrm{T}} P A,$$

$$\nu \triangleq Q C^{\mathrm{T}} \left(C Q C^{\mathrm{T}} \right)^{-1} C, \qquad \nu_\perp \triangleq I_n - \nu,$$

when the indicated inverses exist.

Theorem 4.1. *Suppose there exist $n \times n$ nonnegative-definite matrices P, Q such that $CQC^{\mathrm{T}} > 0$ and*

$$P = A^{\mathrm{T}}PA + R_1 + A^{\mathrm{T}}PD_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)^{-1}D_\infty^{\mathrm{T}}PA - P_a^{\mathrm{T}}R_{2a}^{-1}P_a + \nu_\perp^{\mathrm{T}}P_a^{\mathrm{T}}R_{2a}^{-1}P_a\nu_\perp, \tag{4.1}$$

$$Q = \left[I_n + D_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)^{-1}D_\infty^{\mathrm{T}}P\right]\left(A - BR_{2a}^{-1}P_a\nu\right)Q\left(A - BR_{2a}^{-1}P_a\nu\right)^{\mathrm{T}} \cdot \left[I_n + D_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)^{-1}D_\infty^{\mathrm{T}}P\right]^{\mathrm{T}} + V, \tag{4.2}$$

and let K be given by

$$K = -R_{2a}^{-1}P_aQC^{\mathrm{T}}\left(CQC^{\mathrm{T}}\right)^{-1} \tag{4.3}$$

Then $(\tilde{A}, \tilde{R})$ is detectable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $G(z)$ satisfies the H_∞ disturbance attenuation constraint (2.20) *and the H_2 performance criterion* (2.6) *satisfies the bound*

$$J(K) = \|G(z)\|_2^2 \leq \operatorname{tr} PV \tag{4.4}$$

Proof. First we obtain necessary conditions for the auxiliary minimization problem and then show by construction that these conditions serve as sufficient conditions for closed-loop stability and prespecified disturbance attenuation. Thus, to optimize (2.23) subject to (2.17) over the open set

$$\mathcal{S} \triangleq \left\{(K, P)\quad P > 0 \text{ and } \tilde{A} + D_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)D_\infty^{\mathrm{T}}P\tilde{A} \text{ is asymptotically stable}\right\},$$

form the Lagrangian

$$\mathcal{L}(K, P, Q, \lambda) \triangleq \operatorname{tr}\left[\lambda PV + \left(\tilde{A}^{\mathrm{T}}P\tilde{A} + \tilde{A}^{\mathrm{T}}PD_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)^{-1}D_\infty^{\mathrm{T}}P\tilde{A} + \tilde{R} - P\right)Q\right], \tag{4.5}$$

where the Lagrange multipliers $\lambda \geq 0$ and $Q \in \mathbb{R}^{n\times n}$ are not both zero. By viewing K, P as independent variables and using the identity $\partial/\partial Y \operatorname{tr}(XY^{-1}Z) = -(Y^{-1}ZXY^{-1})^{\mathrm{T}}$, we obtain

$$\frac{\partial\mathcal{L}}{\partial P} = \left[I_n + D_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)^{-1}D_\infty^{\mathrm{T}}P\right]\tilde{A}Q\tilde{A}^{\mathrm{T}}\left[I_n + D_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)^{-1}D_\infty^{\mathrm{T}}P\right]^{\mathrm{T}} - Q + \lambda V. \tag{4.6}$$

Since $\tilde{A} + D_\infty(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty)^{-1}D_\infty^{\mathrm{T}}P\tilde{A}$ is assumed to be asymptotically stable, $\lambda = 0$ implies $Q = 0$ Hence it can be assumed without loss of generality that $\lambda = 1$ Furthermore, note that Q is nonnegative-definite. Thus the stationary conditions with $\lambda = 1$ are given by

$$\frac{\partial\mathcal{L}}{\partial P} = \left[I_n + D_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)^{-1}D_\infty^{\mathrm{T}}P\right]\tilde{A}Q\tilde{A}^{\mathrm{T}} \cdot \left[I_n + D_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)^{-1}D_\infty^{\mathrm{T}}P\right]^{\mathrm{T}} + V - Q = 0, \tag{4.7}$$

$$\frac{\partial\mathcal{L}}{\partial K} = R_2KCQC^{\mathrm{T}} + B^{\mathrm{T}}PBKCQC^{\mathrm{T}} + B^{\mathrm{T}}PD_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)^{-1}D_\infty^{\mathrm{T}}BKCQC^{\mathrm{T}} + B^{\mathrm{T}}PAQC^{\mathrm{T}} + R_{12}^{\mathrm{T}}QC^{\mathrm{T}} + B^{\mathrm{T}}PD_\infty\left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}}PD_\infty\right)^{-1}D_\infty^{\mathrm{T}}PAQC^{\mathrm{T}} = 0 \tag{4.8}$$

Since CQC^{T} is invertible (see Remark 4.5), (4 8) implies (4.3). Next, with K given by (4.3), equations (4.1) and (4.2) are equivalent to (2.17) and (4 7), respectively. It now follows from Lemma 2.1 that the detectability condition (2.18) is equivalent to the stability of $\tilde{A}$. In this case the H_∞ disturbance attenuation constraint (2.20) holds, and the H_2 cost is bounded from above and is given by (2.23) or, equivalently, (4.4) □

Remark 4.1. Theorem 4.1 presents sufficient conditions for designing discrete-time static output feedback controllers with a prespecified H_∞ constraint on the closed-loop transfer function. These sufficient conditions comprise a system of two algebraic equations, one modified discrete-time Riccati equation and one modified discrete-time Lyapunov equation If the H_∞ disturbance attenuation constraint is sufficiently relaxed, i e , $\gamma \to \infty$, then $(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}} P D_\infty)^{-1} \to 0$ and thus equations (4 1), (4.2) collapse to the standard H_2 discrete-time output feedback result.

Remark 4.2. In applying Theorem 4.1 it is not actually necessary to check (2.18) which holds generically Rather, it suffices to check the stability of $\tilde{A}$ directly which is guaranteed to be equivalent to (2.18).

Remark 4.3. In applying Theorem 4.1 the principal issue concerns conditions on the problem data under which equations (4 1) and (4 2) possess nonnegative-definite solutions. Thus, if $\| G(z) \|_\infty < \gamma$ can be satisfied for a given $\gamma > 0$, it is of interest to know whether one such controller can be obtained by solving (4.1), (4 2) Lemma 2.2 guarantees that (2.17) possesses a solution for any controller satisfying $\| G(z) \|_\infty < \gamma$. Thus, our sufficient condition will also be necessary as long as the auxiliary minimization problem possesses at least one extremal over $\mathcal{S}$.

Remark 4.4. The set $\mathcal{S}$ in the proof of Theorem 4.1 constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the auxiliary minimization problem. Specifically, the requirement that $P > 0$ replaces $P \geq 0$ by an open set constraint, while the stability of $\tilde{A} + D_\infty(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}} P D_\infty)^{-1} D_\infty^{\mathrm{T}} P \tilde{A}$ serves as a normality condition

Remark 4.5. The definiteness condition $CQC^{\mathrm{T}} > 0$ holds if C has full row rank and Q is positive definite. Conversely, if $CQC^{\mathrm{T}} > 0$, then C must have full row rank but Q need not necessarily be positive definite. As shown in the proof of Theorem 4 1, this condition implies the existence of the static-gain projection ν.

Remark 4.6. As shown in the proof of Theorem 4 1, equations (4.1) and (4 2) are obtained by minimizing $\mathcal{J}(K, P) = \operatorname{tr} PV$ thus providing a minimized upper bound for the actual H_2 cost $J(K) = \operatorname{tr} \tilde{P} V$ since $P \geq \tilde{P}$

Next, we specialize Theorem 4.1 to the full-state feedback case. When the full state is available $C = I_n$ so that the projection $\nu = I_n$ and $\nu_\perp = 0$. In this case (4 3) becomes

$$K = -R_{2a}^{-1} P_a \tag{4.9}$$

and (4 1) specializes to

$$P = A^{\mathrm{T}} P A + R_1 + A^{\mathrm{T}} P D_\infty \left(\gamma^2 I_{d_\infty} - D_\infty^{\mathrm{T}} P D_\infty\right)^{-1} D_\infty^{\mathrm{T}} P A - P_a^{\mathrm{T}} R_{2a}^{-1} P_a \tag{4.10}$$

while (4.2) is superfluous and can be omitted Furthermore, the H_2 cost is bounded by

$$J(K) \leq \operatorname{tr} PV \tag{4.11}$$

Finally, to recover the standard discrete-time LQR result let $\gamma \to \infty$ so that (4.10) corresponds to the standard discrete-time regulator Riccati equation

5. Dynamic estimation with an H_∞ error constraint

In this section we introduce the discrete-time mixed norm H_2/H_∞ estimation problem with an H_∞ constraint on the H_∞ norm of the state-estimation error Specifically, we constrain the transfer function between disturbances and error states to have H_∞ norm less than γ.

H_∞-constrained Kalman filter problem. Given the nth-order observable dynamic system

$$x(k+1) = Ax(k) + D_1 w(k), \quad k = 0, 1, 2, \ldots, \tag{5.1}$$

$$y(k) = Cx(k) + D_2 w(k), \tag{5.2}$$

determine an nth-order state estimator

$$x_e(k+1) = A_e x_e(k) + B_e y(k), \tag{5.3}$$

$$y_e(k) = C_e x_e(k), \tag{5.4}$$

that satisfies the following design criteria:

(i) A_e is asymptotically stable,

(ii) the $r \times d$ transfer function

$$H(z) \triangleq EL(zI_n - A_e)^{-1}(D_1 - B_e D_2)$$

from disturbances $w(k)$ to error states $E[Lx(k) - y_e(k)]$ satisfies the constraint

$$\| H(z) \|_\infty \le \gamma, \tag{5.5}$$

where $\gamma > 0$ is a given constant; and

(iii) the H_2 state-estimation error criterion

$$J(A_e, B_e, C_e) \triangleq \lim_{k \to \infty} \mathbb{E}[Lx(k) - y_e(k)]^{\mathrm{T}} R [Lx(k) - y_e(k)] \tag{5.6}$$

is minimized and

$$\lim_{k \to \infty} [x(k) - x_e(k)] = 0, \tag{5.7}$$

for all $x(0)$ and $x_e(0)$ when $D_1 = 0$ and $D_2 = 0$.

Note that (5.6) is the usual least-squares estimation criterion whereas (5.7) implies that perfect observation is achieved at steady state for the plant and observer dynamics under zero external disturbances and arbitrary initial conditions

To satisfy the observation constraint (5.7), define the error states $e(k) = x(k) - x_e(k)$ satisfying

$$\begin{aligned} e(k+1) &= x(k+1) - x_e(k+1) \\ &= (A - B_e C)x(k) - A_e x(k) + (D_1 - B_e D_2) w(k) \end{aligned} \tag{5.8}$$

Next, note that the explicit dependence of the error states $e(k)$ on the states $x(k)$ can be eliminated by constraining

$$A_e = A - B_e C, \tag{5.9}$$

so that (5.8) becomes

$$e(k+1) = (A - B_e C) e(k) + (D_1 - B_e D_2) w(k). \tag{5.10}$$

This formulation permits the state $x(k)$ to contain unstable modes, i.e., A can be unstable. Analogously, note that the H_2 least-squares state-estimation error criterion can be written as

$$J(A_e, B_e, C_e) = \lim_{k \to \infty} \mathbb{E}[Lx(k) - C_e x_e(k)]^{\mathrm{T}} R [Lx(k) - C_e x_e(k)] \tag{5.11}$$

so that the explicit dependence of the estimation error criterion on the state $x(k)$ can be eliminated by constraining

$$C_e = L. \tag{5.12}$$

Henceforth, we assume that A_e and C_e are given by (5 9) and (5.12). In this case, (5.11) becomes

$$J(A_e, B_e, C_e) = \lim_{k\to\infty} \operatorname{tr} \mathbb{E}\left[L^{\mathrm{T}}RLe(k)e^{\mathrm{T}}(k)\right]. \tag{5.13}$$

Next, if A_e is asymptotically stable, then the H_2 estimation-error criterion (5.13) is given by

$$J(A_e, B_e, C_e) = \| H(z) \|_2^2 = \operatorname{tr} \tilde{Q}L^{\mathrm{T}}RL, \tag{5.14}$$

where the steady-state error covariance defined by

$$\tilde{Q} = \lim_{k\to\infty} \mathbb{E}\left[e(k)e^{\mathrm{T}}(k)\right], \tag{5.15}$$

exists and satisfies the $n \times n$ algebraic Lyapunov equation

$$\tilde{Q} = (A - B_eC)\tilde{Q}(A - B_eC)^{\mathrm{T}} + \tilde{V}, \tag{5.16}$$

where $\tilde{V}$ is the $n \times n$ nonnegative-definite intensity of $(D_1 - B_eD_2)w(k)$ given by

$$\tilde{V} \triangleq V - V_{12}B_e^{\mathrm{T}} - B_eV_{12}^{\mathrm{T}} + B_eV_2B_e^{\mathrm{T}} \tag{5.17}$$

Finally, note that by defining

$$\tilde{y}(k) = E\left[Lx(k) - C_ex_e(k)\right] = ELe(k), \tag{5.18}$$

it follows from (5.10) and (5 18) that the transfer function from disturbances $w(k)$ to error states $ELe(k)$ is given by $H(z)$ A novel feature of this mathematical formulation is the dual interpretation of the disturbance $w(k)$. Specifically, within the context of H_2 optimality the disturbances are interpreted as white noise signals while, simultaneously, for the purpose of H_∞ error estimation the very same disturbance signals have the alternative interpretation of l_2 disturbance signals.

Once again, the key step in enforcing (5.5) is to replace the error covariance equation (5.16) by an algebraic Riccati equation that enforces the H_∞ norm constraint and overbounds the error covariance $\tilde{Q}$.

Lemma 5.1. *Let $B_e \in \mathbb{R}^{n\times l}$ be given and assume there exists an $n \times n$ nonnegative-definite matrix Q satisfying*

$$\begin{aligned} Q = {} & (A - B_eC)Q(A - B_eC)^{\mathrm{T}} \\ & + (A - B_eC)QL^{\mathrm{T}}E^{\mathrm{T}}\left(\gamma^2 I_r - ELQL^{\mathrm{T}}E^{\mathrm{T}}\right)^{-1}ELQ(A - B_eC)^{\mathrm{T}} + \tilde{V} \end{aligned} \tag{5.19}$$

Then

$$\left(A - B_eC, \tilde{V}\right) \text{ is stabilizable} \tag{5.20}$$

if and only if

$$A_e \text{ is asymptotically stable.} \tag{5.21}$$

In this case,

$$\| H(z) \|_\infty \le \gamma \tag{5.22}$$

and

$$\tilde{Q} \le Q. \tag{5.23}$$

Consequently,

$$J(A_e, B_e, C_e) \le \mathscr{J}(A_e, B_e, C_e, Q), \tag{5.24}$$

where

$$\mathscr{J}(A_e, B_e, C_e, Q) = \operatorname{tr} QL^{\mathrm{T}}RL \tag{5.25}$$

Finally, if $H(z) \in \mathcal{RH}_\infty$ and $\| H(z) \|_\infty < \gamma$, then there exists a unique nonnegative-definite solution Q satisfying (5.19) *and such that the eigenvalues of*

$$(A - B_eC) + (A - B_eC)QL^TE^T(\gamma^2 I_r - ELQL^TE^T)EL$$

lie in the open unit disk.

Proof. The proof is completely analogous to the proofs of Lemma 2 1 and Lemma 2.2. □

Lemma 5.1 shows that the H_∞ estimation error constraint is enforced when a nonnegative-definite solution to (5.19) is known to exist and A_e is asymptotically stable Furthermore, all such solutions provide upper bounds for the H_2 estimation error $\| H(z) \|_2^2$. Thus, as in the first part of the paper, the combined H_2/H_∞ estimation problem can be recast as an auxiliary minimization problem.

6. Sufficient conditions for combined least-squares and frequency-domain error estimation

In this section we state sufficient conditions for characterizing discrete-time dynamic estimators guaranteeing H_∞-constrained estimation with an optimized bound on the least-squares state-estimation error criterion. For convenience in stating this result define the additional notation

$$V_{2a} \triangleq V_2 + CQC^T + CQL^TE^T(\gamma^2 I_r - ELQL^TE^T)^{-1} ELQC^T,$$

$$Q_a \triangleq AQC^T + V_{12} + AQL^TE^T(\gamma^2 I_r - ELQL^TE^T)^{-1} ELQC^T,$$

for arbitrary $Q \in \mathbb{R}^{n \times n}$

Theorem 6.1. *Suppose there exists an $n \times n$ nonnegative-definite matrix Q satisfying*

$$Q = AQA^T + V_1 + AQL^TE^T(\gamma^2 I_r - ELQL^TE^T)^{-1} ELQA^T - Q_a V_{2a}^{-1} Q_a^T, \tag{6 1}$$

and let (A_e, B_e, C_e) be given by

$$A_e = A - Q_a V_{2a}^{-1} C, \tag{6 2}$$

$$B_e = Q_a V_{2a}^{-1}, \tag{6.3}$$

$$C_e = L. \tag{6.4}$$

Then $(A_e, \tilde{V})$ is stabilizable if and only if A_e is asymptotically stable. In this case, the transfer function $H(z)$ satisfies the H_∞ estimation error constraint (5.22) *and the H_2 least-squares state-estimation error criterion* (5.6) *satisfies the bound*

$$J(A_e, B_e, C_e) = \| H(z) \|_2^2 \le \operatorname{tr} QL^TRL \tag{6.5}$$

Proof. As in the proof of Theorem 4.1 we first obtain necessary conditions for the auxiliary minimization problem and then show by construction that these conditions serve as sufficient conditions for stability of the estimator dynamics and a prespecified H_∞-constraint on the state-estimation error. Thus, to optimize (5 19) subject to (5.25) over the open set

$$\mathcal{S}_e \triangleq \Big\{ (A_e, B_e, C_e, Q) \colon (A - B_eC) + (A - B_eC)QL^TE^T(\gamma^2 I_r - ELQL^TE^T)EL$$
$$\text{is asymptotically stable and}$$
$$\Big(A_e + A_eQL^TE^T(\gamma^2 I_r - ELQL^TE^T)^{-1} EL, C_e\Big) \text{ is observable} \Big\},$$

form the Lagrangian

$$\mathscr{L}(B_e, Q, P, \lambda) \triangleq \operatorname{tr}\Big\{\lambda QL^{\mathrm T}RL + \Big[(A - B_eC)Q(A - B_eC)^{\mathrm T} + (A - B_eC)QL^{\mathrm T}E^{\mathrm T}(\gamma^2 I_r - ELQL^{\mathrm T}E^{\mathrm T})^{-1}ELQ(A - B_eC)^{\mathrm T} + \tilde V - Q\Big]P\Big\}, \tag{6 6}$$

where the Lagrange multipliers $\lambda \geq 0$ and $P \in \mathbb{R}^{n \times n}$ are not both zero By viewing B_e, Q as independent variables we obtain

$$\frac{\partial \mathscr{L}}{\partial Q} = \big[I_n + QL^{\mathrm T}E^{\mathrm T}(\gamma^2 I_r - ELQLE^{\mathrm T})EL\big]^{\mathrm T}(A - B_eC)^{\mathrm T}P(A - B_eC) \big[I_n + QL^{\mathrm T}E^{\mathrm T}(\gamma^2 I_r - ELQL^{\mathrm T}E^{\mathrm T})^{-1}EL\big] - P + \lambda L^{\mathrm T}RL. \tag{6.7}$$

Since $(A - B_eC) + (A - B_eC)QL^{\mathrm T}E^{\mathrm T}(\gamma^2 I_r - ELQL^{\mathrm T}E^{\mathrm T})^{-1}EL$ is assumed to be stable, $\lambda = 0$ implies $P = 0$. Hence, it can be assumed without loss of generality that $\lambda = 1$. Thus the stationary conditions with $\lambda = 1$ are given by

$$\frac{\partial \mathscr{L}}{\partial Q} = \big[I_n + QL^{\mathrm T}E^{\mathrm T}(\gamma^2 I_r - ELQL^{\mathrm T}E^{\mathrm T})^{-1}EL\big]^{\mathrm T}(A - B_eC)^{\mathrm T}P(A - B_eC) \big[I_n + QL^{\mathrm T}E^{\mathrm T}(\gamma^2 I_r - ELQL^{\mathrm T}E^{\mathrm T})^{-1}EL\big] + L^{\mathrm T}RL - P = 0, \tag{6.8}$$

$$\frac{\partial \mathscr{L}}{\partial B_e} = PB_eV_2 + PB_eCQC^{\mathrm T} + PB_eCQL^{\mathrm T}E^{\mathrm T}(\gamma^2 I_r - ELQL^{\mathrm T}E^{\mathrm T})^{-1}ELQC^{\mathrm T} - PAQC^{\mathrm T} - PV_{12}QC^{\mathrm T} - PAQL^{\mathrm T}E^{\mathrm T}(\gamma^2 I_r - ELQL^{\mathrm T}E^{\mathrm T})^{-1}ELQC^{\mathrm T} = 0 \tag{6.9}$$

Next, note that (6 8) is equivalent to

$$P = \big[A_e + A_eQL^{\mathrm T}E^{\mathrm T}(\gamma^2 I_r - ELQLE^{\mathrm T})^{-1}EL\big]^{\mathrm T}P \cdot\big[A_e + A_eQL^{\mathrm T}E^{\mathrm T}(\gamma^2 I_r - ELQL^{\mathrm T}E^{\mathrm T})^{-1}EL\big] + C_e^{\mathrm T}RC_e. \tag{6.10}$$

Now, since $(A_e + A_eQL^{\mathrm T}E^{\mathrm T}(\gamma^2 I_r - ELQL^{\mathrm T}E^{\mathrm T})^{-1}EL,\ C_e)$ is observable it follows from standard discrete-time Lyapunov theory that P is positive-definite.

Thus, since P is invertible (6.9) implies (6.3). Now (6.2) is a restatement of (5.9) with B_e given by (6 3) Next, with B_e given by (6.2), equation (6.1) is equivalent to (5 19). It now follows from Lemma 5 1 that the stabilizability condition (5.20) is equivalent to the stability of A_e. In this case the H_∞-norm constraint on the state-estimation error (5 22) holds, and the H_2 state-estimation criterion is bounded from above and is given by (5.25) □

Remark 6.1. Theorem 6.1 presents sufficient conditions for designing full-order discrete-time dynamic estimators with a prespecified H_∞-norm constraint on the state-estimation error These conditions involve one modified Riccati equation similar to the discrete-time observer Riccati equation with additional quadratic terms of the form $AQL^{\mathrm T}E^{\mathrm T}(\gamma^2 I_r - ELQL^{\mathrm T}E^{\mathrm T})^{-1}ELQA^{\mathrm T}$ which enforce the H_∞ constraint. Note that if the H_∞ estimation constraint is sufficiently relaxed, i.e., $\gamma \to \infty$, (6 1) reduces to the standard observer Riccati equation of steady-state discrete-time Kalman filter theory.

Remark 6.2. The principal issue in applying Theorem 6 1 concerns conditions on the problem data under which the modified observer Riccati equation possesses nonnegative-definite solutions. For γ sufficiently large, (6.1) approximates the standard discrete-time Kalman filter result so that existence is assured. However, the important case of interest involves small γ so that significant H_∞ estimation is enforced The last part of Lemma 5.1 implies that (6.1) possesses a solution for any dynamic estimator satisfying $\| H(z) \|_\infty < \gamma$. Thus, it follows that the sufficient conditions of Theorem 6.1 are also necessary as long as the auxiliary minimization problem possesses at least one extremal over $\mathscr{S}_e$.

Remark 6.3. Equations (4 10) and (6 1) give the solutions to the mixed norm H_2/H_∞ regulation and estimation problems. Although the form of (4.10), (6.1) would lead one to surmise that the mixed-norm H_2/H_∞ output feedback dynamic compensation result would involve equations (4.10) and (6.1), this is not generally the case since separation between regulation and estimation does not hold for the mixed-norm H_2/H_∞ problem as was pointed out in [1] for the continuous-time case Of course, under certain simplifying assumptions (see [1], Section 5) one may obtain separation. However, the resulting structure of the compensator gains differs markedly from the standard LQG gains As shown in [15], the H_∞ dynamic compensation problem without any H_2 contribution involves two decoupled equations given by (4.10) and (6.1).

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