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Optimal projection equations for discrete-time fixed-order dynamic compensation of linear systems with multiplicative white noise

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The optimal projection equations for discrete-time reduced-order dynamic compensation are generalized to include the effects of state-, control- and measurement-dependent noise. In addition, the discrete-time static output feedback problem with multiplicative disturbances is considered. For both problems, the design equations are presented in a concise, unified manner to facilitate their accessibility for developing numerical algorithms for practical applications.

Notation and definitions

$\mathbf{R}$, $\mathbf{R}^{r\times s}$, $\mathbf{R}^r$, $\mathbf{E}$	real numbers, $r \times s$ real matrices, $\mathbf{R}^{r\times 1}$, expectation
I_n, $(\ )^{\mathrm{T}}$	$n \times n$ identity, transpose
$\otimes$	Kronecker product
$\tau_\perp$	$I_n - \tau$, $\tau \in \mathbf{R}^{n\times n}$
asymptotically stable matrix	matrix with eigenvalues in the open unit disc
non-negative-semisimple matrix	semisimple (diagonalizable) matrix with non-negative eigenvalues
non-negative-definite matrix	symmetric matrix with non-negative eigenvalues
positive-definite matrix	symmetric matrix with positive eigenvalues
n, m, l, n_c, p	positive integers, $1 \leqslant n_c \leqslant n$
x, x_c	n-, n_c-dimensional vectors
u, y	m-, l-dimensional vectors
$A, A_i; B, B_i; C, C_i$	$n \times n$ matrices, $n \times m$ matrices, $l \times n$ matrices, $i = 1, \ldots, p$
A_c, B_c, C_c, D_c	$n_c \times n_c$, $n_c \times l$, $m \times n_c$, $m \times l$ matrices
k	discrete-time index $1, 2, \ldots$
$v_i(k)$	unit variance white noise, $i = 1, \ldots, p$
$w_1(k)$, $w_2(k)$	n-dimensional, l-dimensional white noise processes
V_1, V_2	$n \times n$ covariance of w_1, $l \times l$ covariance of w_2; $V_1 \geqslant 0$, $V_2 \geqslant 0$
V_{12}	$n \times l$ cross-covariance of w_1, w_2
R_1, R_2	state and control weightings; $R_1 \geqslant 0$, $R_2 \geqslant 0$
R_{12}	$n \times m$ cross weighting; $R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}} \geqslant 0$
$\tilde{A}, \tilde{A}_i$	$A + BD_cC$, $A_i + B_iD_cC + BD_cC_i$, $i = 1, \ldots, p$
$\tilde{w}$	$w_1 + BD_cw_2 + \sum_{i=1}^{p} B_iD_cw_2$

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$\tilde{V}$	$V_1 + V_{12}D_c^{\mathrm{T}}B^{\mathrm{T}} + BD_cV_{12}^{\mathrm{T}} + BD_cV_2D_c^{\mathrm{T}}B^{\mathrm{T}} + \sum_{i=1}^{p} B_iD_cV_2D_c^{\mathrm{T}}B_i^{\mathrm{T}}$
$\tilde{R}$	$R_1 + R_{12}D_cC + C^{\mathrm{T}}D_c^{\mathrm{T}}R_{12}^{\mathrm{T}} + C^{\mathrm{T}}D_c^{\mathrm{T}}R_2D_cC + \sum_{i=1}^{p} C_i^{\mathrm{T}}D_c^{\mathrm{T}}R_2D_cC_i$
$\tilde{\tilde{A}}, \tilde{\tilde{A}}_i$	$\begin{bmatrix} \tilde{A} & BC_c \\ B_cC & A_c \end{bmatrix}, \begin{bmatrix} \tilde{A}_i & B_iC_c \\ B_cC_i & 0 \end{bmatrix}$
$\tilde{\tilde{w}}$	$\begin{bmatrix} \tilde{w} \\ B_cw_2 \end{bmatrix}$
$\tilde{\tilde{V}}$	$\begin{bmatrix} \tilde{V} & V_{12}B_c^{\mathrm{T}} + BD_cV_2B_c^{\mathrm{T}} \\ B_cV_{12}^{\mathrm{T}} + B_cV_2D_c^{\mathrm{T}}B^{\mathrm{T}} & B_cV_2B_c^{\mathrm{T}} \end{bmatrix}$
$\tilde{\tilde{R}}$	$\begin{bmatrix} \tilde{R} & R_{12}C_c + C^{\mathrm{T}}D_c^{\mathrm{T}}R_2C_c \\ C_c^{\mathrm{T}}R_{12}^{\mathrm{T}} + C_c^{\mathrm{T}}R_2D_cC & C_c^{\mathrm{T}}R_2C_c \end{bmatrix}$
$Z_{(i,j)}$	(i, j) element of matrix Z
$\operatorname{tr} Z$	trace of square matrix Z
$\rho(Z)$	rank of matrix Z
E_i	matrix with unity in the (i, i) position and zeros elsewhere
$\Pi_i(\psi)$	$\psi E_i \psi^{-1}$ (unit-rank eigenprojection)

For arbitrary $n \times n$ Q, P, $\hat{Q}$, $\hat{P}$, τ define:

$$V_{2s} \triangleq V_2 + CQC^{\mathrm{T}} + \sum_{i=1}^{p} C_iQC_i^{\mathrm{T}}, \qquad R_{2s} \triangleq R_2 + B^{\mathrm{T}}PB + \sum_{i=1}^{p} B_i^{\mathrm{T}}PB_i$$

$$Q_s \triangleq AQC^{\mathrm{T}} + V_{12} + \sum_{i=1}^{p} A_iQC_i^{\mathrm{T}}, \qquad P_s \triangleq B^{\mathrm{T}}PA + R_{12}^{\mathrm{T}} + \sum_{i=1}^{p} B_i^{\mathrm{T}}PA_i$$

$$Q_{s1} \triangleq V_{12} + \sum_{i=1}^{p} A_iQC_i^{\mathrm{T}}, \qquad P_{s1} \triangleq R_{12}^{\mathrm{T}} + \sum_{i=1}^{p} B_i^{\mathrm{T}}PA_i$$

$$\hat{V}_{2s} \triangleq V_2 + CQC^{\mathrm{T}} + \sum_{i=1}^{p} C_i(Q + \tau\hat{Q}\tau^{\mathrm{T}})C_i^{\mathrm{T}}, \qquad \hat{R}_{2s} \triangleq R_2 + B^{\mathrm{T}}PB + \sum_{i=1}^{p} B_i^{\mathrm{T}}(P + \tau^{\mathrm{T}}\hat{P}\tau)B_i$$

$$\hat{Q}_s \triangleq AQC^{\mathrm{T}} + V_{12} + \sum_{i=1}^{p} A_i(Q + \tau\hat{Q}\tau^{\mathrm{T}})C_i^{\mathrm{T}}, \qquad \hat{P}_s \triangleq B^{\mathrm{T}}PA + R_{12}^{\mathrm{T}} + \sum_{i=1}^{p} B_i^{\mathrm{T}}(P + \tau^{\mathrm{T}}\hat{P}\tau)A_i$$

$$\hat{Q}_{s1} \triangleq V_{12} + \sum_{i=1}^{p} A_i(Q + \tau\hat{Q}\tau^{\mathrm{T}})C_i^{\mathrm{T}}, \qquad \hat{P}_{s1} \triangleq R_{12}^{\mathrm{T}} + \sum_{i=1}^{p} B_i^{\mathrm{T}}(P + \tau^{\mathrm{T}}\hat{P}\tau)A_i$$

1. Introduction

Hyland and Bernstein (1984) showed that the first-order necessary conditions for quadratically optimal continuous-time fixed-order dynamic compensation can be transformed into a coupled system of four matrix equations (two modified Riccati equations and two modified Lyapunov equations). The coupling is due to the presence of an oblique projection (idempotent matrix) which arises as a rigorous consequence of optimality. This formulation provides a generalization of classical LQG control theory, since in the full-order case the projection becomes the identity matrix, the modified Lyapunov equations drop out, and the modified Riccati equations reduce to the usual LQG equations. Coupling via the optimal projection implies that sequential reduced-order design procedures consisting of either model reduction followed by controller design or controller design followed by controller reduction are generally suboptimal. Furthermore, the coupled structure of the equations yields the insight that in the reduced-order case there is no longer separation between the operations of state estimation and state-estimate feedback, i.e. the certainty equivalence principle breaks down.

The above developments for the continuous-time problem have, moreover, been carried out by Bernstein, Davis and Hyland (1986) in a discrete-time setting. As in the continuous-time case, the optimal reduced-order compensator is characterized by a pair of modified Riccati equations and a pair of modified Lyapunov equations coupled by an oblique projection. Furthermore, because of the discrete-time setting it is now possible to permit static feedthrough gains in both the full- and reduced-order controller designs. As pointed out by Hyland and Bernstein (1984), non-singular control weighting and measurement noise in the continuous-time case permit only a purely dynamic (strictly proper) controller. Note that this is precisely the case in continuous-time LQG theory, which yields strictly proper feedback controllers.

An immediate application of the discrete-time results is a rigorous treatment of the linear-quadratic sampled-data reduced-order dynamic-compensation problem (Bernstein, Davis and Greeley 1986). By explicitly accounting for real-time computational delay in the feedback loop, the sampled-data control-design problem can be transformed into an equivalent discrete-time problem. The dimension of the equivalent discrete-time system, however, is augmented by the available measurements which are treated as delay states. The optimal projection equations for discrete-time fixed-order dynamic compensation can thus be used to obtain controllers of tractably low dimension in spite of dimension augmentation.

Design considerations concerning stability and performance robustness with respect to unknown parameter variations can also be incorporated into the fixed-order dynamic-compensation design process. This can be accomplished by introducing white noise into the plant via the imperfectly known parameters (Bernstein and Hyland 1985, Bernstein and Greeley 1986 a). Intuitively speaking, the quadratically optimal feedback controller designed in the presence of such disturbances is automatically desensitized to actual parameter variations. As shown by Bernstein and Greeley (1986 b), the modification of the closed-loop covariance equation due to multiplicative noise can be used to guarantee robust stability and performance by means of a Lyapunov function and a performance bound.

An interesting aspect of the design equations for the multiplicative noise model is the breakdown of the separation principle even in the full-order case. That is, even when coupling due to the oblique projection is absent, coupling due to stochastic effects remains. This is a graphic portrayal of observations made previously, e.g. by Gustafson and Speyer (1975). An alternative, apparently suboptimal approach involving certainty equivalent controllers for guaranteeing stochastic stability was developed by Yaz (1986).

The purpose of the present paper is to extend the optimal projection equations for fixed-order discrete-time dynamic compensation given by Bernstein, Davis and Hyland (1986) to include the effects of state-, control- and measurement-dependent white noise. The main result (Theorem 3.1) presents the necessary conditions for optimality as a system of four matrix equations (two modified discrete-time Riccati equations and two modified discrete-time Lyapunov equations) coupled by both the optimal projection and stochastic effects. For the sake of completeness, the optimality conditions for discrete-time static output feedback are given by Theorem 2.1.

2. Static output feedback

2.1. *Discrete-time static output-feedback problem*

Given the controlled system

$$x(k+1)=\left(A+\sum_{i=1}^{p} v_i(k)A_i\right)x(k)+\left(B+\sum_{i=1}^{p} v_i(k)B_i\right)u(k)+w_1(k) \tag{1}$$

$$y(k)=\left(C+\sum_{i=1}^{p} v_i(k)C_i\right)x(k)+w_2(k) \tag{2}$$

where $k=1,2,\ldots$, determine D_c such that the static output feedback law

$$u(k)=D_c y(k) \tag{3}$$

minimizes the performance criterion

$$J \triangleq \lim_{k\to\infty} \mathbf{E}[x^{\mathrm{T}}(k)R_1x(k)+2x^{\mathrm{T}}(k)R_{12}u(k)+u^{\mathrm{T}}(k)R_2u(k)] \tag{4}$$

Using the notation given at the beginning of this paper, the closed-loop system (1)–(3) can be written as

$$x(k+1)=\left(\tilde{A}+\sum_{i=1}^{p} v_i(k)\tilde{A}_i\right)x(k)+\tilde{w}(k) \tag{5}$$

Define the second-moment matrix

$$Q(k)=\mathbf{E}[x(k)x^{\mathrm{T}}(k)] \tag{6}$$

satisfying

$$Q(k+1)=\tilde{A}Q(k)\tilde{A}^{\mathrm{T}}+\sum_{i=1}^{p}\tilde{A}_iQ(k)\tilde{A}_i^{\mathrm{T}}+\tilde{V} \tag{7}$$

To consider the steady state, we restrict our consideration to the set of second-moment stabilizing gains

$$\mathbf{S}\triangleq\left\{D_c:\tilde{A}\otimes\tilde{A}+\sum_{i=1}^{p}\tilde{A}_i\otimes\tilde{A}_i \text{ is asymptotically stable}\right\}$$

The requirement $D_c\in\mathbf{S}$ implies the existence of the steady-state closed-loop state covariance $Q\triangleq\lim_{k\to\infty}Q(k)$. Furthermore, Q and its non-negative-definite dual P are unique solutions of the modified discrete-time Lyapunov equations

$$Q=\tilde{A}Q\tilde{A}^{\mathrm{T}}+\sum_{i=1}^{p}\tilde{A}_iQ\tilde{A}_i^{\mathrm{T}}+\tilde{V} \tag{8}$$

$$P=\tilde{A}^{\mathrm{T}}P\tilde{A}+\sum_{i=1}^{p}\tilde{A}_i^{\mathrm{T}}P\tilde{A}_i+\tilde{R} \tag{9}$$

An additional technical requirement is that D_c be confined to the set

$$\mathbf{S}^+ \triangleq \{D_c \in \mathbf{S}\colon V_{2s} > 0 \quad \text{and} \quad R_{2s} > 0\}$$

In order to obtain closed-form expressions for extremal values of the closed-loop control gains, the static- and dynamic-compensation problems require the technical assumption

$$[B_i \neq 0 \Rightarrow C_i = 0], \quad i = 1, \ldots, p \tag{10}$$

i.e. for each $i \in \{1, \ldots, p\}$, B_i and C_i are not both non-zero. Essentially, (10) expresses the condition that the control-dependent and measurement-dependent noises are independent. There are no constraints, however, on correlation with the state-dependent noise. By optimizing (4) with respect to D_c and manipulating (8) and (9), we obtain the following result.

Theorem 2.1

Suppose $D_c \in \mathbf{S}^+$ solves the discrete-time static output-feedback problem. Then there exist $n \times n$ $Q, P \geqslant 0$ such that

$$D_c = -R_{2s}^{-1}[B^T PAQC^T + P_{s1}QC^T + B^T PQ_{s1}]V_{2s}^{-1} \tag{11}$$

and such that Q and P satisfy

$$Q = AQA^T + V_1 + \sum_{i=1}^{p} [(A_i + B_i D_c C)Q(A_i + B_i D_c C)^T + B_i D_c V_2 D_c^T B_i^T] + (Q_s + BD_c V_{2s})V_{2s}^{-1}(Q_s + BD_c V_{2s})^T - Q_s V_{2s} Q_s^T \tag{12}$$

$$P = A^T PA + R_1 + \sum_{i=1}^{p} [(A_i + BD_c C_i)^T P(A_i + BD_c C_i) + C_i^T D_c^T R_2 D_c C_i] + (P_s + R_{2s} D_c C)^T R_{2s}^{-1}(P_s + R_{2s} D_c C) - P_s^T R_{2s} P_s \tag{13}$$

3. Dynamic output feedback

We now expand the formulation of the static problem to include a dynamic compensator.

3.1. *Discrete-time dynamic output-feedback problem*

Given the controlled system (1), (2), determine A_c, B_c, C_c, D_c such that the dynamic output-feedback law

$$x_c(k+1) = A_c x_c(k) + B_c y(k) \tag{14}$$

$$u(k) = C_c x_c(k) + D_c y(k) \tag{15}$$

minimizes the performance criterion (4).

We restrict our attention to the second-moment-stabilizing controllers

$$\hat{\mathbf{S}} \triangleq \left\{(A_c, B_c, C_c, D_c)\colon \tilde{\tilde{A}} \otimes \tilde{\tilde{A}} + \sum_{i=1}^{p} \tilde{\tilde{A}}_i \otimes \tilde{\tilde{A}}_i \text{ is asymptotically stable and } (A_c, B_c, C_c) \text{ is minimal}\right\}$$

which implies the existence of $\tilde{Q} \triangleq \lim_{k\to\infty} \mathbf{E}[\tilde{x}(k)\tilde{x}^T(k)]$, where $\tilde{x}(k) \triangleq [x^T(k), x_c^T(k)]^T$.

Furthermore, $\tilde{Q}$ and its non-negative-definite dual $\tilde{P}$ are the unique solutions to the modified discrete-time Lyapunov equations

$$\tilde{Q} = \tilde{\tilde{A}}\tilde{Q}\tilde{\tilde{A}}^{\mathrm{T}} + \sum_{i=1}^{p} \tilde{\tilde{A}}_i\tilde{Q}\tilde{\tilde{A}}_i^{\mathrm{T}} + \tilde{\tilde{V}} \tag{16}$$

$$\tilde{P} = \tilde{\tilde{A}}^{\mathrm{T}}\tilde{P}\tilde{\tilde{A}} + \sum_{i=1}^{p} \tilde{\tilde{A}}_i^{\mathrm{T}}\tilde{P}\tilde{\tilde{A}}_i + \tilde{\tilde{R}} \tag{17}$$

An additional technical assumption is that (A_c, B_c, C_c, D_c) be confined to the set

$$\mathbf{\hat{S}}^+ \triangleq \{(A_c, B_c, C_c, D_c) \in \mathbf{\hat{S}}: \hat{R}_{2s} > 0 \quad \text{and} \quad \hat{V}_{2s} > 0\}$$

The following lemma is required.

Lemma 3.1

Let $\tau \in \mathbf{R}^{n \times n}$. Then

$$\tau^2 = \tau \tag{18}$$

$$\rho(\tau) = n_c \tag{19}$$

if and only if there exist $G, \Gamma \in \mathbf{R}^{n_c \times n}$ such that

$$G^{\mathrm{T}}\Gamma = \tau \tag{20}$$

$$\Gamma G^{\mathrm{T}} = I_{n_c} \tag{21}$$

Proof

See Bernstein, Davis and Hyland (1986).

For convenience call G and Γ satisfying (20) and (21) a *projective factorization* of τ. Furthermore, for $n \times n$ non-negative-definite matrices $\hat{Q}$, $\hat{P}$, define the set of *contragrediently diagonalizing* transformations (see Rao and Mitra 1971, p. 123)

$$\mathbf{D}(\hat{Q}, \hat{P}) \triangleq \{\psi \in \mathbf{R}^{n \times m}: \psi^{-1}\hat{Q}\psi^{-\mathrm{T}} \quad \text{and} \quad \psi^{\mathrm{T}}\hat{P}\psi \text{ are diagonal}\}$$

Theorem 3.1

Suppose $(A_c, B_c, C_c, D_c) \in \mathbf{\hat{S}}^+$ solves the discrete-time dynamic output-feedback problem. Then there exist $n \times n$ Q, P, $\hat{Q}$, $\hat{P} \geqslant 0$ such that

$$A_c = \Gamma[A - B\hat{R}_{2s}^{-1}\hat{P}_s - \hat{Q}_s\hat{V}_{2s}^{-1}C - BD_cC]G^{\mathrm{T}} \tag{22}$$

$$B_c = \Gamma[\hat{Q}_s\hat{V}_{2s}^{-1} + BD_c] \tag{23}$$

$$C_c = -[\hat{R}_{2s}^{-1}\hat{P}_s + D_cC]G^{\mathrm{T}} \tag{24}$$

$$D_c = -\hat{R}_{2s}^{-1}[B^{\mathrm{T}}PAQC^{\mathrm{T}} + \hat{P}_{s1}QC^{\mathrm{T}} + B^{\mathrm{T}}P\hat{Q}_{s1}]\hat{V}_{2s}^{-1} \tag{25}$$

and such that Q, P, $\hat{Q}$, $\hat{P}$ satisfy

$$Q = AQA^{\mathrm{T}} + V_1 + \tau_\perp\hat{Q}\tau_\perp^{\mathrm{T}} + \sum_{i=1}^{p} [(A_i - B_i\hat{R}_{2s}^{-1}\hat{P}_s)\tau\hat{Q}\tau^{\mathrm{T}}(A_i - B_i\hat{R}_{2s}^{-1}\hat{P}_s)^{\mathrm{T}}$$
$$+ (A_i + B_iD_cC)Q(A_i + B_iD_cC)^{\mathrm{T}} + B_iD_cV_2D_c^{\mathrm{T}}B_i^{\mathrm{T}}] - \hat{Q}_s\hat{V}_{2s}^{-1}\hat{Q}_s^{\mathrm{T}} \tag{26}$$

$$P = A^{\mathrm{T}}PA + R_1 + \tau_{\perp}^{\mathrm{T}}\hat{P}\tau_{\perp} + \sum_{i=1}^{p} [(A_i - \hat{Q}_s\hat{V}_{2s}^{-1}C_i)^{\mathrm{T}}\tau^{\mathrm{T}}\hat{P}\tau(A_i - \hat{Q}_s\hat{V}_{2s}^{-1}C_i)$$
$$+ (A_i + BD_cC_i)^{\mathrm{T}}P(A_i + BD_cC_i) + C_i^{\mathrm{T}}D_c^{\mathrm{T}}R_2D_cC_i] - \hat{P}_s^{\mathrm{T}}\hat{R}_{2s}^{-1}\hat{P}_s \tag{27}$$

$$\hat{Q} = (A - B\hat{R}_{2s}^{-1}\hat{P}_s)\tau\hat{Q}\tau^{\mathrm{T}}(A - B\hat{R}_{2s}^{-1}\hat{P}_s)^{\mathrm{T}} + (\hat{Q}_s + BD_c\hat{V}_{2s})\hat{V}_{2s}^{-1}(\hat{Q}_s + BD_c\hat{V}_{2s})^{\mathrm{T}} \tag{28}$$

$$\hat{P} = (A - \hat{Q}_s\hat{V}_{2s}^{-1}C)^{\mathrm{T}}\tau^{\mathrm{T}}\hat{P}\tau(A - \hat{Q}_s\hat{V}_{2s}^{-1}C) + (\hat{P}_s + \hat{R}_{2s}D_cC)^{\mathrm{T}}\hat{R}_{2s}^{-1}(\hat{P}_s + \hat{R}_{2s}D_cC) \tag{29}$$

where

$$\tau \triangleq \sum_{i=1}^{n_c} \Pi_i(\psi) = G^{\mathrm{T}}\Gamma \tag{30}$$

for some $\psi \in \mathbf{D}(\hat{Q}, \hat{P})$ such that $(\psi^{-1}\hat{Q}\hat{P}\psi)_{(i,i)} \neq 0$, $i = 1, \ldots, n_c$, and some projective factorization G, Γ of τ.

Remark 3.1

To specialize the result to the strictly proper (no feedthrough) case, merely ignore (25) and set $D_c = 0$ wherever it appears.

Remark 3.2

As previously pointed out by Bernstein, Davis and Hyland (1986), the indeterminacy in specifying the projective factorization G, Γ satisfying (20) and (21) corresponds to an arbitrary choice of internal state-space basis for the design system (A_c, B_c, C_c).

Remark 3.3

In the full-order case $n_c = n$, the projection τ becomes the identity and (28) and (29) play no role. In this case $G^{\mathrm{T}}\Gamma = \Gamma G^{\mathrm{T}} = I_n$ and thus G and Γ can be chosen to be the identity. Deleting all multiplicative white noise terms corresponding to state-, control- and measurement-dependent disturbances, i.e. $A_i, B_i, C_i = 0, i = 1, \ldots, p$, and specializing further to the purely dynamic case ($D_c = 0$) yields the standard LQG result. Alternatively, setting $n_c < n$ and deleting the multiplicative noise terms yields the results of Bernstein, Davis and Hyland (1986).

4. Proof of Theorem 3.1

Partition $(n + n_c) \times (n + n_c)$ $\tilde{Q}$, $\tilde{P}$ into $n \times n$, $n \times n_c$, and $n_c \times n_c$ sub-blocks as

$$\tilde{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_2 \end{bmatrix}, \quad \tilde{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\mathrm{T}} & P_2 \end{bmatrix}$$

and define the $n \times n$ non-negative-definite matrices

$$Q \triangleq Q_1 - Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}}, \quad P \triangleq P_1 - P_{12}P_2^{-1}P_{12}^{\mathrm{T}}$$
$$\hat{\hat{Q}} \triangleq Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}}, \quad \hat{\hat{P}} \triangleq P_{12}P_2^{-1}P_{12}^{\mathrm{T}}$$
$$\hat{Q} \triangleq (A - B\hat{R}_{2s}^{-1}\hat{P}_s)\hat{\hat{Q}}(A - B\hat{R}_{2s}^{-1}\hat{P}_s)^{\mathrm{T}} + (\hat{Q}_s + BD_c\hat{V}_{2s})\hat{V}_{2s}^{-1}(\hat{Q}_s + BD_c\hat{V}_{2s})^{\mathrm{T}}$$
$$\hat{P} \triangleq (A - Q_s\hat{V}_{2s}^{-1}C)^{\mathrm{T}}\hat{\hat{P}}(A - \hat{Q}_s\hat{V}_{2s}^{-1}C) + (\hat{P}_s + \hat{R}_{2s}D_cC)^{\mathrm{T}}\hat{R}_{2s}^{-1}(\hat{P}_s + \hat{R}_{2s}D_cC)$$

where $\tau\hat{Q}\tau^{\mathrm{T}}$ and $\tau^{\mathrm{T}}\hat{P}\tau$ in $\hat{Q}_s$, $\hat{P}_s$, $\hat{V}_{2s}$, and $\hat{R}_{2s}$ are replaced by $\hat{\hat{Q}}$ and $\hat{\hat{P}}$, and the $n_c \times n$,

$n_c \times n_c$, and $n_c \times n$ matrices

$$G \triangleq Q_2^{-1} Q_{12}^{\mathrm{T}}, \quad M \triangleq Q_2 P_2, \quad \Gamma \triangleq -P_2^{-1} P_{12}^{\mathrm{T}}$$

Define the lagrangian

$$L(A_c, B_c, C_c, D_c, \tilde{Q}, \tilde{P}, \lambda) \triangleq \operatorname{tr}\left[\lambda J(A_c, B_c, C_c, D_c) + \left(\tilde{\tilde{A}}\tilde{Q}\tilde{\tilde{A}}^{\mathrm{T}} + \sum_{i=1}^{p} \tilde{\tilde{A}}_i \tilde{Q} \tilde{\tilde{A}}_i^{\mathrm{T}} + \tilde{\tilde{V}} - \tilde{Q}\right)\tilde{P}\right]$$

where the Lagrange multipliers $\lambda \geqslant 0$ and $\tilde{P} \in \mathbf{R}^{(n+n_c)\times(n+n_c)}$ are not both zero. Setting $\partial L/\partial \tilde{Q} = 0$, $\lambda = 0$ implies $\tilde{P} = 0$ since $(A_c, B_c, C_c, D_c) \in \hat{\mathbf{S}}^+$. Hence, without loss of generality, set $\lambda = 1$. Thus the stationarity conditions are given by

$$\frac{\partial L}{\partial \tilde{P}} = \tilde{\tilde{A}}\tilde{Q}\tilde{\tilde{A}}^{\mathrm{T}} + \sum_{i=1}^{p} \tilde{\tilde{A}}_i \tilde{Q} \tilde{\tilde{A}}_i^{\mathrm{T}} + \tilde{\tilde{V}} - \tilde{Q} = 0 \tag{31}$$

$$\frac{\partial L}{\partial \tilde{Q}} = \tilde{\tilde{A}}^{\mathrm{T}}\tilde{P}\tilde{\tilde{A}} + \sum_{i=1}^{p} \tilde{\tilde{A}}^{\mathrm{T}}\tilde{P}\tilde{\tilde{A}} + \tilde{\tilde{R}} - \tilde{P} = 0 \tag{32}$$

$$\frac{\partial L}{\partial A_c} = P_{12}^{\mathrm{T}} A Q_{12} + P_{12}^{\mathrm{T}} B D_c C Q_{12} + P_{12}^{\mathrm{T}} B C_c Q_2 + P_2 A_c Q_2 + P_2 B_c C Q_{12} = 0 \tag{33}$$

$$\frac{\partial L}{\partial B_c} = P_2 B_c \hat{V}_{2s} + P_{12}^{\mathrm{T}} \hat{Q}_s + P_{12}^{\mathrm{T}} B D_c \hat{V}_{2s} = 0 \tag{34}$$

$$\frac{\partial L}{\partial C_c} = \hat{P}_s Q_{12} + \hat{R}_{2s} C_c Q_2 + \hat{R}_{2s} D_c C Q_{12} = 0 \tag{35}$$

$$\frac{\partial L}{\partial D_c} = \hat{R}_{2s} D_c \hat{V}_{2s} + B^{\mathrm{T}} P A Q C^{\mathrm{T}} + \hat{P}_{s1} Q C^{\mathrm{T}} + B^{\mathrm{T}} P \hat{Q}_{s1} = 0 \tag{36}$$

Expanding (31) and (32) yields

$$\begin{aligned} &AQA^{\mathrm{T}} + (\hat{Q}_s + BD_c\hat{V}_{2s})\hat{V}_{2s}^{-1}(\hat{Q}_s + BD_c\hat{V}_{2s})^{\mathrm{T}} - \hat{Q}_s\hat{V}_{2s}^{-1}\hat{Q}_s^{\mathrm{T}} \\ &\quad + (A - B\hat{R}_{2s}^{-1}\hat{P}_s)\hat{Q}(A - B\hat{R}_{2s}^{-1}\hat{P}_s)^{\mathrm{T}} \\ &\quad + V_1 - Q - \hat{Q} + \sum_{i=1}^{p} [(A_i - B_i\hat{R}_{2s}^{-1}\hat{P}_s)\hat{Q}(A_i - B_i\hat{R}_{2s}^{-1}\hat{P}_s)^{\mathrm{T}} \\ &\quad + (A_i + B_iD_cC)Q(A_i + B_iD_cC)^{\mathrm{T}} + B_iD_cV_2D_c^{\mathrm{T}}B_i^{\mathrm{T}}] = 0 \end{aligned} \tag{37}$$

$$[(\hat{Q}_s + BD_c\hat{V}_{2s})\hat{V}_{2s}^{-1}(\hat{Q}_s + BD_c\hat{V}_{2s})^{\mathrm{T}} + (A - B\hat{R}_{2s}^{-1}\hat{P}_s)\hat{Q}(A - B\hat{R}_{2s}^{-1}\hat{P}_s)^{\mathrm{T}} - \hat{Q}]\Gamma^{\mathrm{T}} = 0 \tag{38}$$

$$\begin{aligned} &\Gamma[(\hat{Q}_s + BD_c\hat{V}_{2s})\hat{V}_{2s}^{-1}(\hat{Q}_s + BD_c + \hat{V}_{2s})^{\mathrm{T}} \\ &\qquad + (A - B\hat{R}_{2s}^{-1}\hat{P}_s)\hat{Q}(A - B\hat{R}_{2s}^{-1}\hat{P}_s)^{\mathrm{T}} - \hat{Q}]\Gamma^{\mathrm{T}} = 0 \end{aligned} \tag{39}$$

$$\begin{aligned} &A^{\mathrm{T}}PA + (\hat{P}_s + \hat{R}_{2s}D_cC)^{\mathrm{T}}\hat{R}_{2s}^{-1}(\hat{P}_s + \hat{R}_{2s}D_cC) - \hat{P}_s^{\mathrm{T}}\hat{R}_{2s}^{-1}\hat{P}_s \\ &\quad + (A - \hat{Q}_s\hat{V}_{2s}^{-1}C)^{\mathrm{T}}\hat{P}(A - \hat{Q}_s\hat{V}_{2s}^{-1}C) \\ &\quad + R_1 - P - \hat{P} + \sum_{i=1}^{p} [(A_i - \hat{Q}_s\hat{V}_{2s}^{-1}C_i)^{\mathrm{T}}\hat{P}(A_i - \hat{Q}_s\hat{V}_{2s}^{-1}C_i) \\ &\quad + (A_i + BD_cC_i)^{\mathrm{T}}P(A_i + BD_cC_i) + C_i^{\mathrm{T}}D_c^{\mathrm{T}}R_2D_cC_i] = 0 \end{aligned} \tag{40}$$

$$[(\hat{P}_s + \hat{R}_{2s}D_cC)^{\mathrm{T}}\hat{R}_{2s}^{-1}(\hat{P}_s + \hat{R}_{2s}D_cC) + (A - \hat{Q}_s\hat{V}_{2s}^{-1}C)^{\mathrm{T}}\hat{P}(A - \hat{Q}_s\hat{V}_{2s}^{-1}C) - \hat{P}]G^{\mathrm{T}} = 0 \tag{41}$$

$$G[(\hat{P}_s + \hat{R}_{2s}D_cC)^T\hat{R}_{2s}^{-1}(\hat{P}_s + \hat{R}_{2s}D_cC) + (A - \hat{Q}_s\hat{V}_{2s}^{-1}C)^T\hat{P}(A - \hat{Q}_s\hat{V}_{2s}^{-1}C) - \hat{\hat{P}}]G^T = 0 \tag{42}$$

Using (33)–(36) we obtain (22)–(25). Using $(37) + G^T\Gamma(38)G - (38)G - (38G)^T$ and $G^T\Gamma(38)G - (38)G - (38G)^T$ yields (26) and (28). Similarly, using $(40) + \Gamma^TG(41)\Gamma - (41)\Gamma - (41\Gamma)^T$ and $\Gamma^TG(41)\Gamma - (41)\Gamma - (41\Gamma)^T$ we obtain (27) and (29). Also, $\Gamma(38) - (39)$ or $G(41) - (42)$ yields $\Gamma G^T = I_{n_c}$ so that $\tau = G^T\Gamma = \tau^2$. Finally, (39) and (42) imply $\hat{\hat{Q}} = \tau\hat{Q}\tau^T$ and $\hat{\hat{P}} = \tau^T\hat{P}\tau$. □

Remark 4.1

An interesting difference between the above discrete-time derivation and the continuous-time derivation of Hyland and Bernstein (1984) is that the explicit gain expressions and the definition of the optimal projection arise in the reverse order.

5. Directions for further research

The principal application of Theorems 2.1 and 3.1 is the sampled-data problem with parameter uncertainties. Although generalization of the results of Bernstein *et al.* (1986) is possible, there appear to be a number of mathematical issues which arise. A related development appears in Tiedemann and De Koning (1984). A more extensive treatment of the results of the present paper can be found in Haddad (1987).

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